A Functional Correspondence between Normalization Functions and Abstract Machines

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Plan

- The functional correspondence.
- The case of the untyped lambda-calculus.
- From one-step reduction to evaluation and to normalization.
- X by evaluation.
- NBE for what else?

The functional correspondence

evaluator

↓

closure conversion (to make it first order)

CPS transformation (to make it sequential)

defunctionalization (to make it first order)

abstract
machine

Source terms (de Bruijn indices)

```
structure S
= struct
    datatype term = VAR of int
                  | LAM of term
                  | APP of term * term
  end
```

Target terms (de Bruijn levels)

```
structure T
= struct
    datatype nf = LAM of nf
                | VAL of at
         and at = VAR of int
                | APP of at * nf
  end
```

Example: $\lambda x.\lambda k.k\ x$

Source term:

```
S.LAM (S.LAM (S.APP (S.VAR 0,
                     S.VAR 1)))
```

Target term:

```
T.LAM (T.LAM (T.VAL
    (T.APP (T.VAR 1,
            T.VAL (T.VAR 0)))))
```

```
datatype expval = FUN of denval -> expval
                | RES of int -> T.at
withtype denval = unit -> expval
```

```
(*  reify : expval -> int -> T.nf  *)
fun reify (RES r)
    = (fn n => T.VAL (r n))
  | reify (FUN f)
    = (fn n
       => let fun d ()
                  = RES (fn n' => T.VAR n)
          in T.LAM (reify (f d) (n+1))
          end)
```

```
(*  eval : S.term * denval Env.env -> expval  *)
fun eval (S.VAR i, e)
    = List.nth (e, i) ()
  | eval (S.LAM t, e)
    = FUN (fn a => eval (t, a :: e))
```

```
| eval (S.APP (t0, t1), e)
  = (case eval (t0, e)
       of (FUN f)
          => f (fn () => eval (t1, e))
        | (RES r)
          => RES (fn n => T.APP (r n,
                      reify (eval (t1, e)) n)))
```

```
(*  main : S.term -> T.nf  *)
fun main t
    = reify (eval (t, nil)) 0
```

The derivation

- uncurry
- closure convert expressible values
- defunctionalize residual abstract syntax
- CPS transform
- defunctionalize

```
datatype expval = CLO of S.term * expval list
                | RES of target
     and target = VAR of int
                | APP of target * expval
```

```
datatype econt
    = ECONT0
    | ECONT1 of target * econt
    | ECONT2 of S.term * expval list * econt
    | ECONT3 of int * rcont
and rcont
    = RCONT0
    | RCONT1 of T.at * tcont
    | RCONT2 of rcont
and tcont
    = TCONT0 of rcont
    | TCONT1 of expval * int * tcont
```

```
(*  reify : expval * int * rcont -> T.nf  *)
fun reify (RES r, n, k)
    = apply_target (r, n, TCONT0 k)
  | reify (CLO (t, e), n, k)
    = eval (t,
            (RES (VAR n)) :: e,
            ECONT3 (n+1, k))
```

```
(*  eval : S.term * denval Env.env * econt
           -> T.nf  *)
and eval (S.VAR i, e, k)
    = (case List.nth (e, i)
         of (RES r)
            => apply_econt (k, RES r)
          | (CLO (t', e'))
            => eval (t', e', k))
  | eval (S.LAM t, e, k)
    = apply_econt (k, CLO (t, e))
  | eval (S.APP (t0, t1), e, k)
    = eval (t0, e, ECONT2 (t1, e, k))
```

```
(*  apply_target : target * int * tcont
                       -> T.nf  *)
and apply_target (VAR n, n', k)
    = apply_tcont (k, T.VAR n)
  | apply_target (APP (r0, v1), n, k)
    = apply_target (r0, n, TCONT1 (v1, n, k))
```

```
(*  apply_tcont : T.at -> T.nf  *)
and apply_tcont (TCONT0 k, at)
    = apply_rcont (k, T.VAL at)
  | apply_tcont (TCONT1 (v1, n, k), at0)
    = reify (v1, n, RCONT1 (at0, k))
```

```
(*  apply_econt : econt * expval -> T.nf  *)
and apply_econt (ECONT0, v)
    = reify (v, 0, RCONT0)
  | apply_econt (ECONT1 (r0, k), v1)
    = apply_econt (k, RES (APP (r0, v1)))
  | apply_econt (ECONT2 (t1, e, k), CLO (t', e'))
    = eval (t', (CLO (t1, e)) :: e', k)
  | apply_econt (ECONT2 (t1, e, k), RES r0)
    = eval (t1, e, ECONT1 (r0, k))
  | apply_econt (ECONT3 (n, k), v)
    = reify (v, n, RCONT2 k)
```

```
(*  apply_rcont :  rcont * T.nf -> T.nf  *)
and apply_rcont (RCONT0, r)
    = r
  | apply_rcont (RCONT1 (at0, k), r1)
    = apply_tcont (k, T.APP (at0, r1))
  | apply_rcont (RCONT2 k, r)
    = apply_rcont (k, T.LAM r)
```

```
(*  main : S.term * (T.nf -> T.nf) -> T.nf  *)
fun main t
    = eval (t, nil, ECONT0)
```

X by evaluation

Key idea:

Reuse the evaluation mechanism

of the implementation language.

NBE for what else?

$$
\begin{aligned}
\mathcal{M} &= \langle E, \star, \varepsilon \rangle \\
E &= \text{set of elements} \\
\star &: \mathcal{M} \times \mathcal{M} \to \mathcal{M}
\end{aligned}
$$

$$
\begin{aligned}
m &\in \mathcal{M} \\
e &\in E \\
m &::= \varepsilon \mid m_1 \star m_2 \mid e
\end{aligned}
$$

Conversion rules

$$\mathfrak{m}_1 \star (\mathfrak{m}_2 \star \mathfrak{m}_3) \leftrightarrow (\mathfrak{m}_1 \star \mathfrak{m}_2) \star \mathfrak{m}_3$$

$$\mathfrak{m} \star \varepsilon \leftrightarrow \mathfrak{m}$$

$$\varepsilon \star \mathfrak{m} \leftrightarrow \mathfrak{m}$$

Notion of normal form

$$
\begin{array}{rcl}
m & \in & \mathcal{M}_{nf} \\
e & \in & E \\
m & ::= & \varepsilon_{nf} \mid e \star_{nf} m
\end{array}
$$

Reduction-based normalization

1. We orient the conversion rules

   into <u>reduction rules</u>:

$$m_1 \star (m_2 \star m_3) \;\leftarrow\; (m_1 \star m_2) \star m_3$$

$$\varepsilon \star m \;\rightarrow\; m$$

2. We apply the reduction rules

   until a normal form is obtained.

Incidentally

“Reduction rules” is a wonderful title.

Reduction-free normalization (1/2)

Notion of evaluation:

$$eval \; : \; \mathcal{M} \to (\mathcal{M}_{nf} \to \mathcal{M}_{nf})$$

$$\begin{aligned} eval(\varepsilon) &= \lambda m.m \\ eval(m_1 \star m_2) &= (eval(m_1)) \circ (eval(m_2)) \\ eval(e) &= \lambda m.e \star_{nf} m \end{aligned}$$

Reduction-free normalization (2/2)

Notion of reification:

$$\begin{array}{rcl} \mathsf{reify} & : & (\mathcal{M}_{\mathsf{nf}} \to \mathcal{M}_{\mathsf{nf}}) \to \mathcal{M}_{\mathsf{nf}} \\ \mathsf{reify}(f) & = & f(\varepsilon_{\mathsf{nf}}) \end{array}$$

Reduction-free normalization (2/2)

Notion of reification:

$$\begin{aligned} \mathsf{reify} &: (\mathcal{M}_{\mathsf{nf}} \to \mathcal{M}_{\mathsf{nf}}) \to \mathcal{M}_{\mathsf{nf}} \\ \mathsf{reify}(f) &= f(\varepsilon_{\mathsf{nf}}) \end{aligned}$$

Normalization:

$$\begin{aligned} \mathsf{normalize} &: \mathcal{M} \to \mathcal{M}_{\mathsf{nf}} \\ \mathsf{normalize}(m) &= \mathsf{reify}(\mathsf{eval}(m)) \end{aligned}$$

Pragmatic look

From a programming standpoint:

- Reduction-based: iterative flattening
  by reordering.

- Reduction-free: flatten function
  with an accumulator.

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From a programming standpoint:

- Reduction-based: iterative flattening
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- Reduction-free: flatten function
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Also: Correctness issues.

Cool thing: NBE scales

- The λ-calculus.
- The computational λ-calculus.
- Other algebraic structures.

In short

NBE uses the expressive power

of functional programming.

In particular

For the λ-calculus and the computational λ-calculus, i.e., for proof theorists and functional programmers:

- Normalization steps involve substitutions (due to parameter passing).
- Why not reuse 30 years of compiler expertise to carry out these substitutions?

Joint work with Vincent Balat (GPCE'03)

- Setting: The typed λ-calculus.
- Issue: Type isomorphisms.
- Application: Data conversion, library search, language interoperability.

In particular

The type-theoretical counterpart of Tarski's

high school algebra problem (1950):

Can one prove all arithmetic propositions using the equations taught in high school?

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cf. Balat, Di Cosmo, Fiore, LICS’02, POPL’04.

The problem

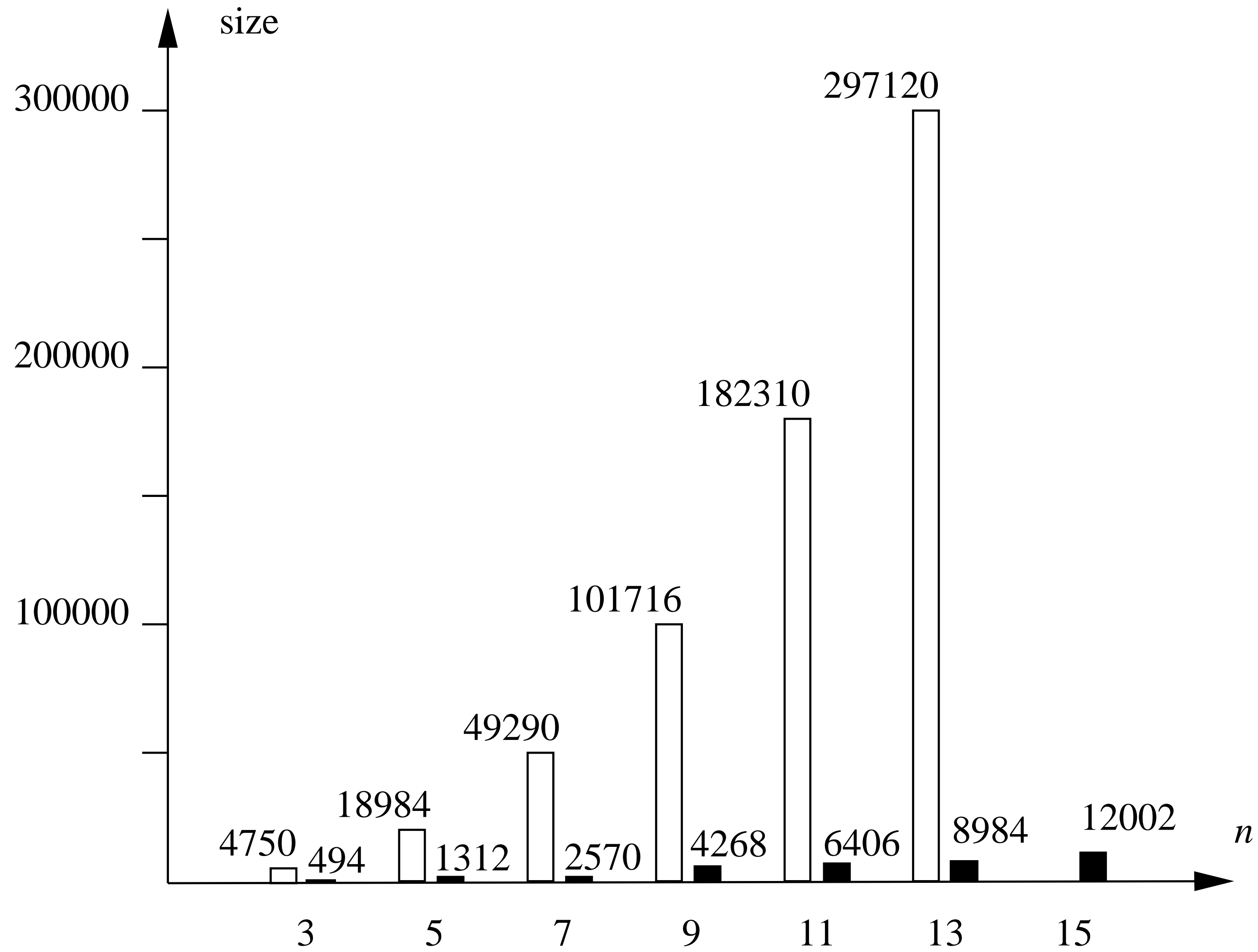
- Verify that two families of functions are inverses of each other, pointwise.
- NBE approach: Write them in ML, compose them, normalize them, and verify textual identity.
- The problem: Huge normalization time, gigantic normal forms.

The root of the problem

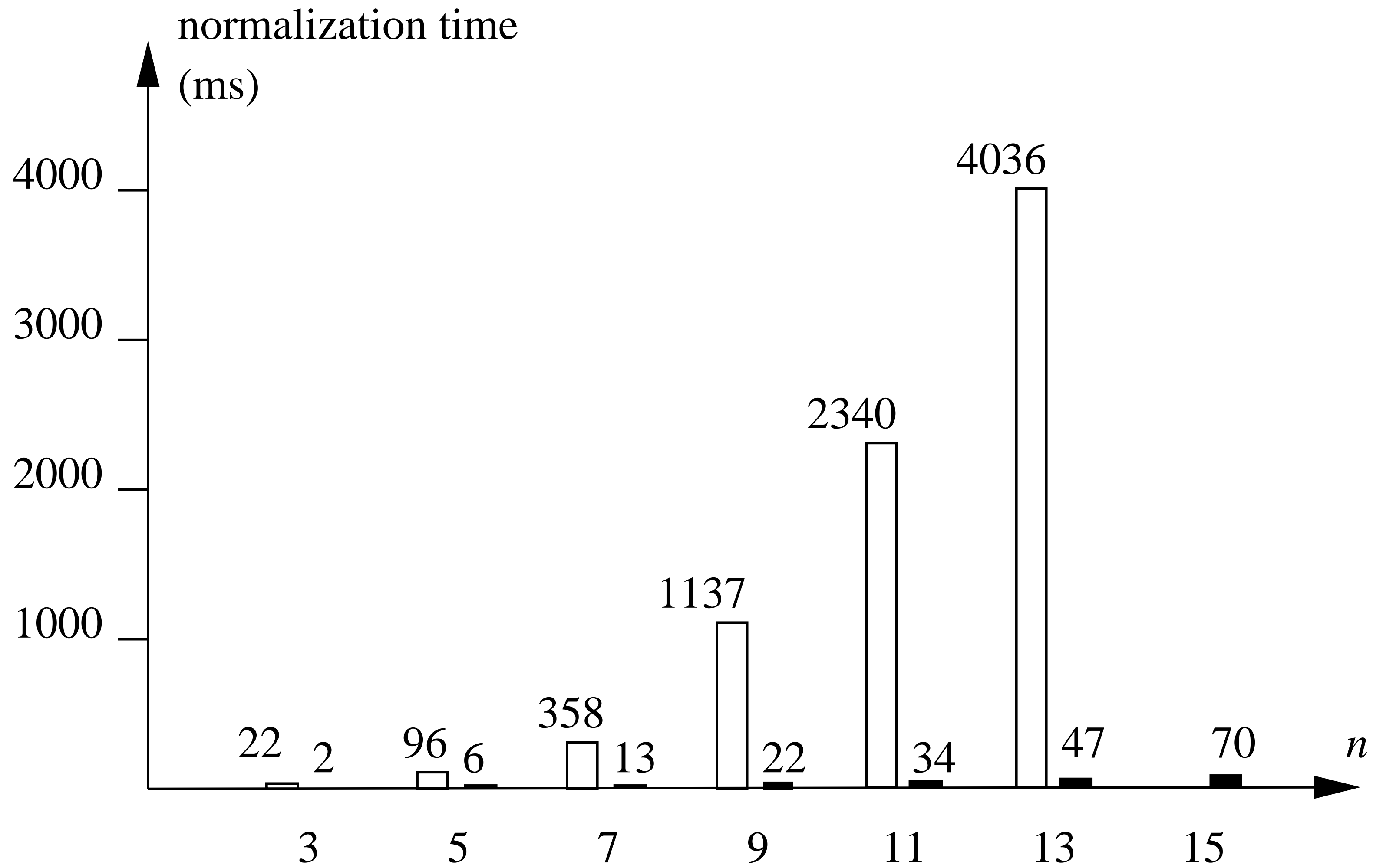
Duplication of work across conditional expressions due to use of continuations.

Our (pragmatic) solution: Dynamic memoization of intermediate results.

Results: Size of normalized programs



Results: Normalization time



Mission accomplished

NBE is now usable

for more and bigger examples

of type isomorphisms.

And furthermore

Using set/cupto instead of shift/reset,

Balat met Filinski's challenge of identifying

once and thrice (POPL'04).