A Functional Correspondence between
Normalization Functions
and Abstract Machines

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Plan

- The functional correspondence.
- The case of the untyped lambda-calculus.
- From one-step reduction to evaluation and to normalization.
- $X$ by evaluation.
- NBE for what else?
The functional correspondence

abstract machine

evaluator

closure conversion (to make it first order)

CPS transformation (to make it sequential)

defunctionalization (to make it first order)
structure S
= struct
    datatype term = VAR of int
    | LAM of term
    | APP of term * term
end
Target terms (de Bruijn levels)

structure T
= struct
    datatype nf = LAM of nf
    | VAL of at
    and at = VAR of int
    | APP of at * nf
end
Example: $\lambda x.\lambda k.k \, x$

Source term:

$S.\text{LAM} \ (S.\text{LAM} \ (S.\text{APP} \ (S.\text{VAR} \ 0, \ S.\text{VAR} \ 1)))$

Target term:

$T.\text{LAM} \ (T.\text{LAM} \ (T.\text{VAL} \ (T.\text{APP} \ (T.\text{VAR} \ 1, \ T.\text{VAL} \ (T.\text{VAR} \ 0))))}$
datatype expval = FUN of denval -> expval
  | RES of int -> T.at
with type denval = unit -> expval
(* reify : expval -> int -> T.nf *)
fun reify (RES r)
   = (fn n => T.VAL (r n))
| reify (FUN f)
   = (fn n
       => let fun d ()
           = RES (fn n' => T.VAR n)
        in T.LAM (reify (f d) (n+1))
       end)
(* eval : S.term * denval Env.env -> expval *)
fun eval (S.VAR i, e)
    = List.nth (e, i) ()
| eval (S.LAM t, e)
    = FUN (fn a => eval (t, a ::: e))
\[
\text{eval (S.APP (t0, t1), e)}
\]
\[
= (\text{case eval (t0, e)}
\]
\[
\text{of (FUN f)}
\]
\[
\quad \Rightarrow f (\text{fn ()} \Rightarrow \text{eval (t1, e)})
\]
\[
\mid (\text{RES r})
\]
\[
\quad \Rightarrow \text{RES (fn n} \Rightarrow \text{T.APP (r n,}
\]
\[
\quad \quad \text{reify (eval (t1, e)) n}))
\]
fun main t
    = reify (eval (t, nil)) 0
The derivation

- uncurry
- closure convert expressible values
- defunctionalize residual abstract syntax
- CPS transform
- defunctionalize
datatype expval = CLO of S.term * expval list
| RES of target
and target = VAR of int
| APP of target * expval
datatype econt
    = ECONT0
    | ECONT1 of target * econt
    | ECONT2 of S.term * expval list * econt
    | ECONT3 of int * rcont

and rcont
    = RCONT0
    | RCONT1 of T.at * tcont
    | RCONT2 of rcont

and tcont
    = TCONT0 of rcont
    | TCONT1 of expval * int * tcont
(* reify : expval * int * rcont -> T.nf *)
fun reify (RES r, n, k)
    = apply_target (r, n, TCONT0 k)
  | reify (CLO (t, e), n, k)
    = eval (t,
            (RES (VAR n)) :: e,
            ECONT3 (n+1, k))
(* eval : S.term * denval Env.env * econt
   -> T.nf *)

and eval (S.VAR i, e, k)
   = (case List.nth (e, i)
       of (RES r)
           => apply_econt (k, RES r)
           | (CLO (t', e'))
           => eval (t', e', k))

| eval (S.LAM t, e, k)
   = apply_econt (k, CLO (t, e))

| eval (S.APP (t0, t1), e, k)
   = eval (t0, e, ECONT2 (t1, e, k))
(* apply_target : target * int * tcont
   -> T.nf *)

and apply_target (VAR n, n', k)
   = apply_tcont (k, T.VAR n)

| apply_target (APP (r0, v1), n, k)
   = apply_target (r0, n, TCONT1 (v1, n, k))
(* apply_tcont : T.at -> T.nf *)

and apply_tcont (TCONT0 k, at)
    = apply_rcont (k, T.VAL at)

| apply_tcont (TCONT1 (v1, n, k), at0)
    = reify (v1, n, RCONT1 (at0, k))
(* apply_econt : econt * expval -> T.nf *)
and apply_econt (ECONT0, v)
  = reify (v, 0, RCONT0)
  | apply_econt (ECONT1 (r0, k), v1)
  = apply_econt (k, RES (APP (r0, v1)))
  | apply_econt (ECONT2 (t1, e, k), CLO (t', e'))
  = eval (t', (CLO (t1, e)) :: e', k)
  | apply_econt (ECONT2 (t1, e, k), RES r0)
  = eval (t1, e, ECONT1 (r0, k))
  | apply_econt (ECONT3 (n, k), v)
  = reify (v, n, RCONT2 k)
(*) apply_rcont : rcont * T.nf -> T.nf *)
and apply_rcont (RCONT0, r)
  = r
  | apply_rcont (RCONT1 (at0, k), r1)
  = apply_tcont (k, T.APP (at0, r1))
  | apply_rcont (RCONT2 k, r)
  = apply_rcont (k, T.LAM r)
fun main t = eval (t, nil, ECONT0)
Key idea:
Reuse the evaluation mechanism of the implementation language.
NBE for what else?

\[ \mathcal{M} = \langle E, \ast, \varepsilon \rangle \]

\[ E = \text{set of elements} \]

\[ \ast : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \]

\[ m \in \mathcal{M} \]

\[ e \in E \]

\[ m ::= \varepsilon \mid m_1 \ast m_2 \mid e \]
Conversion rules

\[ m_1 \star (m_2 \star m_3) \leftrightarrow (m_1 \star m_2) \star m_3 \]

\[ m \star \varepsilon \leftrightarrow m \]

\[ \varepsilon \star m \leftrightarrow m \]
Notion of normal form

\[ m \in \mathcal{M}_{nf} \]
\[ e \in E \]
\[ m ::= \varepsilon_{nf} \mid e \ast_{nf} m \]
Reduction-based normalization

1. We orient the conversion rules into reduction rules:

\[ m_1 \star (m_2 \star m_3) \leftarrow (m_1 \star m_2) \star m_3 \]

\[ \varepsilon \star m \rightarrow m \]

2. We apply the reduction rules until a normal form is obtained.
Incidentally

“Reduction rules” is a wonderful title.
Reduction-free normalization (1/2)

Notion of evaluation:

\[ \text{eval} : \mathcal{M} \rightarrow (\mathcal{M}_{nf} \rightarrow \mathcal{M}_{nf}) \]

\[ \text{eval}(\varepsilon) = \lambda m. m \]

\[ \text{eval}(m_1 \ast m_2) = (\text{eval}(m_1)) \circ (\text{eval}(m_2)) \]

\[ \text{eval}(e) = \lambda m. e \ast_{nf} m \]
Reduction-free normalization (2/2)

Notion of reification:

\[
\text{reify} : (\mathcal{M}_{nf} \rightarrow \mathcal{M}_{nf}) \rightarrow \mathcal{M}_{nf} \\
\text{reify}(f) = f(\varepsilon_{nf})
\]
Reduction-free normalization (2/2)

Notion of reification:

\[ \text{reify} : (\mathcal{M}_{nf} \to \mathcal{M}_{nf}) \to \mathcal{M}_{nf} \]
\[ \text{reify}(f) = f(\varepsilon_{nf}) \]

Normalization:

\[ \text{normalize} : \mathcal{M} \to \mathcal{M}_{nf} \]
\[ \text{normalize}(m) = \text{reify}(\text{eval}(m)) \]
Pragmatic look

From a programming standpoint:

- Reduction-based: iterative flattening by reordering.

- Reduction-free: flatten function with an accumulator.
Pragmatic look

From a programming standpoint:

• Reduction-based: iterative flattening by reordering.

• Reduction-free: flatten function with an accumulator.

Also: Correctness issues.
Cool thing: NBE scales

- The λ-calculus.
- The computational λ-calculus.
- Other algebraic structures.
In short

NBE uses the expressive power of functional programming.
In particular

For the $\lambda$-calculus and the computational $\lambda$-calculus, i.e., for proof theorists and functional programmers:

- Normalization steps involve substitutions (due to parameter passing).
- Why not reuse 30 years of compiler expertise to carry out these substitutions?
Joint work with Vincent Balat
(GPCE’03)

- Setting: The typed $\lambda$-calculus.
- Issue: Type isomorphisms.
- Application: Data conversion, library search, language interoperability.
In particular

The type-theoretical counterpart of Tarski’s high school algebra problem (1950):

Can one prove all arithmetic propositions using the equations taught in high school?
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cf. Balat, Di Cosmo, Fiore, LICS’02, POPL’04.
The problem

- Verify that two families of functions are inverses of each other, pointwise.
- NBE approach: Write them in ML, compose them, normalize them, and verify textual identity.
- The problem: Huge normalization time, gigantic normal forms.
The root of the problem

Duplication of work across conditional expressions due to use of continuations.

Our (pragmatic) solution: Dynamic memoization of intermediate results.
Results: Size of normalized programs

- Size of normalized programs:
  - 4750
  - 18984
  - 49290
  - 101716
  - 297120

- Size of programs for different n:
  - n = 3: 4750
  - n = 5: 18984
  - n = 7: 49290
  - n = 9: 101716
  - n = 11: 182310
  - n = 13: 297120
  - n = 15: 8984
  - n = 20: 12002
Results: Normalization time

Normalization time (ms)

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<th>n</th>
<th>22</th>
<th>2</th>
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<td>11</td>
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<td>15</td>
<td></td>
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</tr>
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</table>
Mission accomplished

NBE is now usable
for more and bigger examples
of type isomorphisms.
And furthermore

Using set/cupto instead of shift/reset, Balat met Filinski’s challenge of identifying once and thrice (POPL’04).