Exercise 1.15

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Problem: Josephus had a friend who was saved by getting into the next-to-last position. What is $I(n)$, the number of the penultimate survivor when every second person is executed?

Solution: Let's first compute $I(n)$ for some small n.

n	Order of elimination	$I(n)$
1	1	$\perp$
2	2, 1	2
3	2, 1, 3	1
4	2, 4, 3, 1	3
5	2, 4, 1, 5, 3	5

We say that $I(1)$ is not defined since we need at least two participants to have a penultimate survivor.

We can observe that for all $n > 2$, we start the elimination process (counting) from position 1, we eliminate position 2 and then need to solve the same problem, but with size one smaller and where the position 1 is the current position 3. This gives us the following equations for solving the problem:

$$I(2) = 2 \tag{1}$$

$$I(n) = ((I(n-1) + 1) \ mod \ n) + 1, \quad \forall n > 2 \tag{2}$$

In the step case we rename the resulting position by adding 2, but we do it in two steps to compensate for the fact that we start counting from 1 instead of 0.

Similarily to the Josephus problem, we can observe that if we start with $2n$ people, the first pass through the circle eliminates half of the participants and the next to be eliminated is 3 (we are left with $1, 3, 5, \ldots, 2n-3, 2n-1$). This is the same as solving the problem with n people, but the number of each position has been doubled and decreased by one. Starting with $2n+1$ people and after the first pass the position 1 is the next to be eliminated and after that we are left with the same problem, but with n people $(3, 5, 7, \ldots, 2n-1, 2n+1)$ and their numbers are doubled and increased by one. This gives us the more efficient recurrence:

$$I(2) = 2 \tag{3}$$

$$I(3) = 1 \tag{4}$$

$$I(2n) = 2 * I(n) - 1, \quad \forall n > 2 \tag{5}$$

$$I(2n+1) = 2 * I(n) + 1, \quad \forall n > 2 \tag{6}$$

Here we do not need to do modular arithmetic because the problem is avoided, doing a recursive call to n can return at most n and this means that the step cases cannot overflow.

Using either of the two recurrences we can now compute a bigger table of $I(n)$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$I(n)$	2	1	3	5	1	3	5	7	9	11	1	3	5	7	9
n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$I(n)$	11	13	15	17	19	21	23	1	3	5	7	9	11	13	15

From the table we can immediatly spot a pattern, $I(n)$ is counting increasing sequences of odd numbers (except when $n = 2$). The starting position of each of the sequences is shaded in the table. We can also observe that the length of every such sequence is the value of n at the starting point of that sequence, which means that we are counting to 3, 6, 12, 24 and so on. This is a good place to observe that these are sums of two consecutive powers of 2: $1+2, 2+4, 4+8, 8+16$ and also that $2^m + 2^{m-1} + 2^m + 2^{m-1} = 2 * 2^m + 2 * 2^{m-1} = 2^{m+1} + 2^m$. This gives us the following equation for computing $I(n)$:

$$I(2^m + 2^{m-1} + l) = 2l + 1, \quad m \geq 0 \ \wedge \ 0 \leq l < 2^m + 2^{m-1} \tag{7}$$

We will now prove this by induction on m. When $m = 0$ we must take $l = 1/2$ which gives us:

$$I(2^0 + 2^{-1} + 1/2) = 2 * 1/2 + 1 = 1 + 1 = 2$$

and we see that the base case is satisfied. The step case is in two parts, depending on whether l is even or odd. If $2^{m+1} + 2^m + l = 2n$ then l is even and

$$\begin{aligned} I(2^{m+1} + 2^m + l) &= 2 * I(2^m + 2^{m-1} + l/2) - 1 \\ &= 2 * (2(l/2) + 1) - 1 \\ &= 2l + 1 \end{aligned}$$

by (5) and the induction hypothesis. If $2^{m+1} + 2^m + l = 2n + 1$ then l is odd and

$$\begin{aligned} I(2^{m+1} + 2^m + l) &= 2 * I(2^m + 2^{m-1} + (l-1)/2) + 1 \\ &= 2 * (2((l-1)/2) + 1) + 1 \\ &= 2l + 1 \end{aligned}$$

by (6) and the induction hypothesis. This completes the proof of (7).