Sums of Riemann's zeta function
Homework for ITT9131 Concrete Mathematics

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1 Exercise 31

31 Riemann's zeta function $\zeta(k)$ is defined to be the infinite sum

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots = \sum_{j \geq 1} \frac{1}{j^k} \tag{1}$$

Prove that

$$\sum_{k \geq 2} (\zeta(k) - 1) = 1. \tag{2}$$

What is the value of

$$\sum_{k \geq 1} (\zeta(2k) - 1)? \tag{3}$$

2 Proof of equation (2)

$$\mathbf{U} = \sum \begin{pmatrix} 1/2^2 & 1/3^2 & 1/4^2 & 1/5^2 & 1/6^2 & \dots \\ 1/2^3 & 1/3^3 & 1/4^3 & 1/5^3 & 1/6^3 & \dots \\ 1/2^4 & 1/3^4 & 1/4^4 & 1/5^4 & 1/6^4 & \dots \\ 1/2^5 & 1/3^5 & 1/4^5 & 1/5^5 & 1/6^5 & \dots \\ 1/2^6 & 1/3^6 & 1/4^6 & 1/5^6 & 1/6^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

Proof. Let denote the sum of matrix with U. We know that all summands are positive and we can change the order of summing.

$$U = \sum_{k \geq 2} (\zeta(k) - 1) = \sum_{k \geq 2} \sum_{j \geq 2} \frac{1}{j^k} = \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} \tag{5}$$

For calculating U, we first try to find closed form formula for second sum. Let

$$T_j = \sum_{i \geq 2} \frac{1}{j^i} \tag{6}$$

$$S_j = \sum_{i \geq 1} \frac{1}{j^i} \tag{7}$$

then

$$T_j = S_j - \frac{1}{j} \tag{8}$$

$$T_j = \frac{S_j}{j} \tag{9}$$

$$S_j = j * T_j \tag{10}$$

Now we calculate the closed form

$$S_j = T_j + \frac{1}{j} \tag{11}$$

$$S_j - \frac{S_j}{j} = \frac{1}{j} \tag{12}$$

$$(j-1) * S_j = 1 \tag{13}$$

$$S_j = \frac{1}{j} - 1 \tag{14}$$

$$T_j = \frac{1}{j-1} - \frac{1}{j} \tag{15}$$

Now

$$U = \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} \tag{16}$$

$$= \sum_{j \geq 2} T_j = \sum_{j \geq 2} (\frac{1}{j-1} - \frac{1}{j}) \tag{17}$$

Let's write the sequence up

$$U = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) \ldots \tag{18}$$

We would like to regroup, but in infinite sums it's not allowed when we are summing sequence of elements with alternating sign. Therefore we represent the sum as limit of finite sums sequence.

$$U = \lim_{P\to\infty} \sum_{j=2}^{P} (\frac{1}{j-1} - \frac{1}{j}) \tag{19}$$

$$= \lim_{P\to\infty} [(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{P-1} - \frac{1}{P})] \tag{20}$$

$$= \lim_{P\to\infty} [\frac{1}{1} + (-\frac{1}{2} + \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{3}) + (-\frac{1}{4} + \ldots + \frac{1}{P-1}) - \frac{1}{P}] \tag{21}$$

$$= \lim_{P\to\infty} [1 + 0 + 0 + \ldots + 0 - \frac{1}{P}] \tag{22}$$

$$= \lim_{P\to\infty} [1 - \frac{1}{P}] = 1 \tag{23}$$

So we got

$$U = \sum_{k\geq 2} (\zeta(k) - 1) = 1 \tag{24}$$

□

3 Calculating value of formula (3)

We have to calculate $V = \sum_{k\geq 1}(\zeta(2k) - 1)$, let's look at the matrix

$$\mathbf{V} = \sum \begin{pmatrix} 1/2^2 & 1/3^2 & 1/4^2 & 1/5^2 & 1/6^2 & \ldots \\ 1/2^4 & 1/3^4 & 1/4^4 & 1/5^4 & 1/6^4 & \ldots \\ 1/2^6 & 1/3^6 & 1/4^6 & 1/5^6 & 1/6^6 & \ldots \\ 1/2^8 & 1/3^8 & 1/4^8 & 1/5^8 & 1/6^8 & \ldots \\ 1/2^{10} & 1/3^{10} & 1/4^{10} & 1/5^{10} & 1/6^{10} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{25}$$

As all summands are positive, we can change the order, thus the sum V is

$$V = \sum_{k\geq 1} (\zeta(k) - 1) = \sum_{k\geq 1} \sum_{j\geq 2} \frac{1}{j^{2k}} = \sum_{j\geq 2} \sum_{k\geq 1} \frac{1}{j^{2k}} \tag{26}$$

Let's calculate the closed form formula of internal sum:

$$M_j = \sum_{k\geq 1} \frac{1}{j^{2k}} \tag{27}$$

$$M_j = \frac{1}{j^2} + \frac{M_j}{j^2} \tag{28}$$

$$j^2 M_j = 1 + M_j \tag{29}$$

$$(j^2 - 1)M_j = 1 \tag{30}$$

$$M_j = \frac{1}{j^2 - 1} \tag{31}$$

$$= \frac{2}{2(j^2 - 1)} \tag{32}$$

$$= \frac{(j - j) + (1 + 1)}{2(j^2 - 1)} \tag{33}$$

$$= \frac{(j + 1) - (j - 1)}{2(j + 1)(j - 1)} \tag{34}$$

$$= \frac{1}{2(j - 1)} - \frac{1}{2(j + 1)} \tag{35}$$

Let's calculate the sum V as limit of finite sums sequence.

$$V = \lim_{P\to\infty} \sum_{j=2}^{P} (\frac{1}{2(j-1)} - \frac{1}{2(j+1)}) \tag{36}$$

$$= \lim_{P\to\infty} \frac{1}{2}[(\frac{1}{1} - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \ldots + (\frac{1}{P-1} - \frac{1}{P+1})] \tag{37}$$

$$= \lim_{P\to\infty} \frac{1}{2}[1 + \frac{1}{2} + (-\frac{1}{3} + \frac{1}{3}) + (-\frac{1}{4} + \frac{1}{4}) + \ldots - \frac{1}{P} - \frac{1}{P+1}] \tag{38}$$

$$= \lim_{P\to\infty} \frac{1}{2}[1 + \frac{1}{2} - \frac{1}{P} - \frac{1}{P+1}] \tag{39}$$

$$= \frac{1}{2}[1 + \frac{1}{2} - 0 - 0] \tag{40}$$

$$= \frac{3}{4} \tag{41}$$

Answer: $\sum_{k\geq 1}(\zeta(2k) - 1) = 3/4$.