Two Sums of Stirling Subset Numbers
Homework for ITT9131 Concrete Mathematics

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1 The Problem

Exercise 6.32 in the course textbook: we obtained the formulas

$$\sum_{k \leq m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

and

$$\sum_{0 \leq k \leq m} \binom{k}{n} = \binom{m+1}{n+1}$$

by unfolding the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

in two ways. What identities appear when the analogous recurrence

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

is unwound?

2 Unwinding to the Right

With such a clear hint in the problem statement, it's natural to start from just checking what we get by unwinding (assuming $m, n \geq 0$):

$$\begin{aligned}
\left\{ {n+m+1 \atop m} \right\} &= m\left\{ {n+m \atop m} \right\} + \left\{ {n+m \atop m-1} \right\} \\
&= m\left\{ {n+m \atop m} \right\} + (m-1)\left\{ {n+m-1 \atop m-1} \right\} + \left\{ {n+m-1 \atop m-2} \right\} \\
&\dots \\
&= m\left\{ {n+m \atop m} \right\} + \dots + 2\left\{ {n+2 \atop 2} \right\} + 1\left\{ {n+1 \atop 1} \right\} + \left\{ {n+1 \atop 0} \right\}.
\end{aligned}$$

Noting that $\left\{ {n+1 \atop 0} \right\} = 0$ for $n \geq 0$ and assuming the original combinatorial interpretation of $\left\{ {n \atop m} \right\}$ (where $\left\{ {n \atop m} \right\} = 0$ for all $m < 0$ irrespective of n), we get the hypothesis

$$\sum_{k \leq m} k\left\{ {n+k \atop k} \right\} = \left\{ {n+m+1 \atop m} \right\},$$

which we can prove by induction:

Base: For $m = 0$, we have

$$\sum_{k \leq 0} k\left\{ {n+k \atop k} \right\} = 0 = \left\{ {n+1 \atop 0} \right\},$$

as $\left\{ {n+k \atop k} \right\} = 0$ for all $k < 0$ and $k\left\{ {n+k \atop k} \right\} = 0$ for $k = 0$ on the left side, and also $\left\{ {n+1 \atop 0} \right\} = 0$ for $n \geq 0$ on the right side, so indeed both sides vanish.

Step: Assume $\sum_{k \leq m} k\left\{ {n+k \atop k} \right\} = \left\{ {n+m+1 \atop m} \right\}$. Then

$$\begin{aligned}
\sum_{k \leq m+1} k\left\{ {n+k \atop k} \right\} &= \sum_{k \leq m} k\left\{ {n+k \atop k} \right\} + (m+1)\left\{ {n+m+1 \atop m+1} \right\} \\
&= \left\{ {n+m+1 \atop m} \right\} + (m+1)\left\{ {n+m+1 \atop m+1} \right\} \\
&= \left\{ {n+m+2 \atop m+1} \right\},
\end{aligned}$$

which again matches, and so we have proven our hypothesis (for $n, m \geq 0$, and assuming the combinatorial interpretation of $\left\{ {n \atop m} \right\}$).

3 Unwinding to the Left

Like in the previous case, we start by following the hint in the problem statement and unwinding (assuming $m, n \geq 0$ again):

$$\begin{aligned}
\left\{ {m+1 \atop n+1} \right\} &= (n+1)\left\{ {m \atop n+1} \right\} + \left\{ {m \atop n} \right\} \\
&= (n+1)\left((n+1)\left\{ {m-1 \atop n+1} \right\} + \left\{ {m-1 \atop n} \right\} \right) + \left\{ {m \atop n} \right\} \\
&= (n+1)^2\left\{ {m-1 \atop n+1} \right\} + (n+1)\left\{ {m-1 \atop n} \right\} + \left\{ {m \atop n} \right\} \\
&\dots \\
&= (n+1)^{m+1}\left\{ {0 \atop n+1} \right\} + (n+1)^m\left\{ {0 \atop n} \right\} + \dots + (n+1)\left\{ {m-1 \atop n} \right\} + \left\{ {m \atop n} \right\}.
\end{aligned}$$

Noting that $\left\{ {0 \atop n+1} \right\} = 0$ for $n \geq 0$, we get the hypothesis

$$\sum_{0 \leq k \leq m} (n+1)^{m-k}\left\{ {k \atop n} \right\} = \left\{ {m+1 \atop n+1} \right\},$$

which we can prove by induction:

Base: For $m = 0$, we have

$$\sum_{0 \leq k \leq 0} (n+1)^{-k}\left\{ {k \atop n} \right\} = \left\{ {0 \atop n} \right\} = [n = 0] = \left\{ {1 \atop n+1} \right\},$$

as it should be.

Step: Assume $\sum_{0 \leq k \leq m}(n+1)^{m-k}\left\{ {k \atop n} \right\} = \left\{ {m+1 \atop n+1} \right\}$. Then

$$\begin{aligned}
\sum_{0 \leq k \leq m+1} (n+1)^{(m+1)-k}\left\{ {k \atop n} \right\} &= \sum_{0 \leq k \leq m} (n+1)^{(m-k)+1}\left\{ {k \atop n} \right\} + (n+1)^0\left\{ {m+1 \atop n} \right\} \\
&= (n+1) \sum_{0 \leq k \leq m} (n+1)^{m-k}\left\{ {k \atop n} \right\} + \left\{ {m+1 \atop n} \right\} \\
&= (n+1)\left\{ {m+1 \atop n+1} \right\} + \left\{ {m+1 \atop n} \right\} \\
&= \left\{ {m+2 \atop n+2} \right\},
\end{aligned}$$

which again matches, and so we have proven our hypothesis (for $n, m \geq 0$).