Exercise 6.43

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Task: Prove that the infinite sum:

0.1
+ 0.01
+ 0.002
+ 0.0003
+ 0.00005
+ 0.000008
+ 0.0000013

converges to a rational number.

Solution: We are looking for a sum S (1) where f_k is a Fibonacci number (2).

$$S=\sum_{k\geq 1}\frac{f_k}{10^k} \tag{1}$$

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 1 \\ f_k &= f_{k-1}+f_{k-2}, \forall k>2 \end{aligned} \tag{2}$$

$$f_k=\frac{\Phi^k-\hat{\Phi}^k}{\sqrt{5}} \tag{3}$$

Let us work on a general case where we replace 10 with z and try to find the finite sum S_n (4):

$$S_n=\sum_{k=1}^{n}\frac{f_k}{z^k} \tag{4}$$

Since Fibonacci numbers have the recurrent property we can look at the following 3 sums:

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\frac{f_k}{z^k} = \frac{f_1}{z} + \sum_{k=2}^{n-2}\frac{f_k}{z^k} + \frac{f_{n-1}}{z^{n-1}} + \frac{f_n}{z^n} \\ z S_n &= \sum_{k=0}^{n-1}\frac{f_{k+1}}{z^k} = \frac{f_1}{1} + \frac{f_2}{z} + \sum_{k=2}^{n-2}\frac{f_{k+1}}{z^k} + \frac{f_n}{z^{n-1}} \\ z^2 S_n &= \sum_{k=-1}^{n-2}\frac{f_{k+2}}{z^k} = \frac{z f_1}{1} + \frac{f_2}{1} + \frac{f_3}{z} + \sum_{k=2}^{n-2}\frac{f_{k+2}}{z^k} \end{aligned} \tag{5}$$

Starting from f_3 we can rewrite the third sum as:

$$z^2 S_n = \frac{z f_1}{1} + \frac{f_2}{1} + \frac{f_1+f_2}{z} + \sum_{k=2}^{n-2}\frac{f_k+f_{k+1}}{z^k} \tag{6}$$

When we subtract the first two sums from the third sum most summands will be eliminated and we're left with:

$$\begin{aligned} z^2 S_n - z S_n - S_n &= \frac{z f_1}{1} + \frac{f_2}{1} - \frac{f_1}{1} - \frac{f_n}{z^{n-1}} - \frac{f_{n-1}}{z^{n-1}} - \frac{f_n}{z^n} \\ (z^2 - z - 1) S_n &= \frac{z 1}{1} + \frac{1}{1} - \frac{1}{1} - \frac{z f_n}{z^n} - \frac{z f_{n-1}}{z^n} - \frac{f_n}{z^n} \end{aligned} \tag{7}$$

$$S_n = \frac{z}{z^2 - z - 1} - \frac{z f_{n+1} + f_n}{z^n (z^2 - z - 1)} \tag{8}$$

The sum (8) can only be calculated when the denominators aren't 0:

$$\begin{gathered} z^2 - z - 1 = 0 \\ z = \frac{1 \pm \sqrt{1+4}}{2} \\ z_1 = \Phi \\ z_2 = \hat{\Phi} \end{gathered} \tag{9}$$

$$S_n = \frac{z}{z^2 - z - 1} - \frac{z f_{n+1} + f_n}{z^n (z^2 - z - 1)} \quad , z \notin \{\hat{\Phi}, 0, \Phi\} \tag{10}$$

When n goes to infinity we want the subtrahend to go to 0. When z equals 1 then this is clearly not the case. Let us replace f_k in (10) with (3) and examine the subtrahend:

$$\begin{gathered} \frac{z \dfrac{(\Phi^{n+1} - \hat{\Phi}^{n+1})}{\sqrt{5}} + \dfrac{(\Phi^n - \hat{\Phi}^n)}{\sqrt{5}}}{z^n (z^2 - z - 1)} \\ \frac{z (\Phi^{n+1} - \hat{\Phi}^{n+1}) + (\Phi^n - \hat{\Phi}^n)}{z^n (z^2 - z - 1) \sqrt{5}} \end{gathered} \tag{11}$$

We want the denominator to be greater than numerator:

$$z (\Phi^{n+1} - \hat{\Phi}^{n+1}) + (\Phi^n - \hat{\Phi}^n) < z^n (z^2 - z - 1) \sqrt{5} \tag{12}$$

Since $|\hat{\Phi}|$ is less than 1, $\hat{\Phi}^n$ will go to 0 when n goes to infinity :

$$\begin{gathered} z \Phi^{n+1} + \Phi^n < \sqrt{5} (z^n - z - 1) z^n , n \to \infty \\ (z \Phi + 1) \Phi^n < z^n (z^2 - z - 1) \sqrt{5} \end{gathered} \tag{13}$$

$$\Phi^n < z^n \frac{(z^2 - z - 1) \sqrt{5}}{z \Phi + 1} \tag{14}$$

$$-\frac{1}{\Phi} = -\frac{2}{1+\sqrt{5}} = -\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{-2(1-\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2} = \hat{\Phi} \tag{15}$$

In (14) on the left side Φ is positive, therefore Φ^n is always positive. On the right side the numerator is negative when $\hat{\Phi} < z < \Phi$ and denominator is negative when $z < -1/\Phi = \hat{\Phi}$ (15). So we are interested only in the case when $z > \Phi$ because then the fraction is a finite positive value and z^n will outperform Φ^n when n goes to infinity. The infinite sum will be:

$$S = \frac{z}{z^2 - z - 1} , z > \Phi \tag{16}$$

If $z = 10 > \Phi$ we get:

$$S = \frac{10}{100 - 10 - 1} = \frac{10}{89} \tag{17}$$