Generating Functions

ITT9131 Konkreetne Matemaatika

Chapter Seven

- Domino Theory and Change
- Basic Maneuvers
- Solving Recurrences
- Special Generating Functions
- Convolutions
- Exponential Generating Functions
- Dirichlet Generating Functions

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Solving recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n in terms of n.

"Algorithm"

1. Write down a single equation that expresses g_n in terms of other elements of the sequence. This equation should be valid for all integers n, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by z^n and sum over all n. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generating function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of z^n; this is a closed form for g_n.

Next subsection

Example: Fibonacci numbers revisited

Step 1 The recurrence

$$g_n = \begin{cases} 0, & \text{if } n \leqslant 0; \\ 1, & \text{if } n = 1; \\ g_{n-1} + g_{n-2} & \text{if } n > 1; \end{cases}$$

can be represented by the single equation

$$g_n = g_{n-1} + g_{n-2} + [n = 1],$$

where $n \in (-\infty, +\infty)$.
This is because the "simple" Fibonacci recurrence $g_n = g_{n-1} + g_{n-2}$ holds for every $n \geqslant 2$ by construction, and for every $n \leqslant 0$ as by hypothesis $g_n = 0$ if $n < 0$; but for $n = 1$ the left-hand side is 1 and the right-hand side is 0, so we need the correction summand $[n = 1]$.

Example: Fibonacci numbers revisited (2)

Step 2 For any n, multiply both sides of the equation by z^n …

$$
\begin{aligned}
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
g_{-2}z^{-2} &= g_{-3}z^{-2} + g_{-4}z^{-2} + [-2=1]z^{-2} \\
g_{-1}z^{-1} &= g_{-2}z^{-1} + g_{-3}z^{-1} + [-1=1]z^{-1} \\
g_0 &= g_{-1} + g_{-2} + [0=1] \\
g_1 z &= g_0 z + g_{-1} z + [1=1]z \\
g_2 z^2 &= g_1 z^2 + g_0 z^2 + [2=1]z^2 \\
g_3 z^3 &= g_2 z^3 + g_1 z^3 + [3=1]z^3 \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

… and sum over all n.

$$\sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n=1] z^n$$

Example: Fibonacci numbers revisited (3)

Step 3 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n [n=1] z^n =$$
$$= \sum_n g_n z^{n+1} + \sum_n g_n z^{n+2} + z =$$
$$= zG(z) + z^2 G(z) + z$$

Solving the equation yields

$$G(z) = \frac{z}{1 - z - z^2}$$

Step 4 Expansion the equation into power series $G(z) = \sum g_n z^n$ gives us the solution (see next slides):

$$g_n = \frac{\Phi^n - \widehat{\Phi}^n}{\sqrt{5}}$$

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Next section

Motivation

- A generating function is often in the form of a rational function

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials.

- Our goal is to find "partial fraction expansion" of $R(z)$, i.e. represent $R(z)$ in the form

$$R(z) = S(z) + T(z),$$

where $S(z)$ has known expansion into the power series, and $T(z)$ is a polynomial.

- A good candidate for $S(z)$ is a finite sum of functions like

$$S(z) = \frac{a_1}{(1-\rho_1 z)^{m_1+1}} + \frac{a_2}{(1-\rho_2 z)^{m_2+1}} + \cdots + \frac{a_\ell}{(1-\rho_\ell z)^{m_\ell+1}}$$

- We have proven the relation

$$\frac{a}{(1-\rho z)^{m+1}} = \sum_{n\geqslant 0} \binom{m+n}{m} a\rho^n z^n$$

- Hence, the coefficient of z^n in expansion of $S(z)$ is

$$s_n = a_1 \binom{m_1+n}{m_1} \rho_1^n + a_2 \binom{m_2+n}{m_2} \rho_2^n + \cdots a_\ell \binom{m_\ell+n}{m_\ell} \rho_\ell^n.$$

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Step 1: Finding $\rho_1, \rho_2, \ldots, \rho_m$

- Suppose $Q(z)$ has the form

$$Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_m z^m, \qquad \text{where } q_m \neq 0.$$

- The "reflected" polynomial Q^R has a relation to Q:

$$\begin{aligned} Q^R(z) &= z^m + q_1 z^{m-1} + q_2 z^{m-2} + \cdots + q_{m-1} z + q_m \\ &= z^m \left(1 + q_1 \frac{1}{z} + q_2 \frac{1}{z^2} + \cdots + q_{m-1} \frac{1}{z^{m-1}} + q_m \frac{1}{z^m}\right) \\ &= z^m Q\left(\frac{1}{z}\right) \end{aligned}$$

- If $\rho_1, \rho_2, \ldots, \rho_m$ are roots of Q^R, then $(z - \rho_i) | Q^R(z)$:

$$Q^R(z) = (z - \rho_1)(z - \rho_2) \cdots (z - \rho_m)$$

- Then $(1 - \rho_i z) | Q(z)$:

$$Q(z) = z^m (\frac{1}{z} - \rho_1)(\frac{1}{z} - \rho_2) \cdots (\frac{1}{z} - \rho_m) = (1 - \rho_1 z)(1 - \rho_2 z) \cdots (1 - \rho_m z)$$

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Step 1: Finding $\rho_1, \rho_2, \ldots, \rho_m$ (2)

In all, we have proven

Lemma

$Q^R(z) = (z-\rho_1)(z-\rho_2)\cdots(z-\rho_m)$ iff $Q(z) = (1-\rho_1 z)(1-\rho_2 z)\cdots(1-\rho_m z)$

Example: $Q(z) = 1 - z - z^2$

$$Q^R(z) = z^2 - z - 1$$

This $Q^R(z)$ has roots

$$z_1 = \frac{1+\sqrt{5}}{2} = \Phi \qquad \text{and} \qquad z_2 = \frac{1-\sqrt{5}}{2} = \widehat{\Phi}$$

Therefore $Q^R(z) = (z-\Phi)(z-\widehat{\Phi})$ and $Q(z) = (1-\Phi z)(1-\widehat{\Phi} z)$.

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Next subsection

Step 2: Decomposition into Partial Fractions

If the following conditions are valid for the fraction $\frac{P(z)}{Q(z)}$:

- all roots of $Q^R(z)$ are distinct (we denote these roots as $\rho_1, \rho_2, \ldots$),
- $\deg P(z) < \deg Q(z) = \ell$,

then the denominator is factorizable as $Q(z) = a_0(1 - z\rho_1) \cdots (1 - z\rho_\ell)$ and the fraction can be expanded as

$$\frac{P(z)}{Q(z)} = \frac{A_1}{1 - \rho_1 z} + \frac{A_2}{1 - \rho_1 z} + \cdots + \frac{A_\ell}{1 - \rho_\ell z}. \tag{1}$$

where $A_1, A_2, \ldots, A_\ell$ are constants.

The constants $A_1, A_2, \ldots, A_\ell$ can be found as a solution of the system of linear equations defined by the equality (1).

Example: Decomposition of $\frac{z^2-3z+28}{6z^3-5z^2-2z+1}$

- We have here $P(z) = z^2 - 3z + 28$ and $Q(z) = 6z^3 - 5z^2 - 2z + 1$;
- Reflected polynomial $Q^R(z) = z^3 - 2z^3 - 5z + 6 = (z-1)(z+2)(z-3)$ and $Q(z) = (1-z)(1+2z)(1-3z)$.

Hence,

$$\frac{P_1(z)}{Q(z)} = \frac{A}{1-z} + \frac{B}{1+2z} + \frac{C}{1-3z} =$$

$$= \frac{A(1+2z)(1-3z) + B(1-z)(1-3z) + C(1-z)(1+2z)}{Q(z)} =$$

$$= \frac{(-6A+3B-2C)z^2 + (-A-4B+C)z + (A+B+C)}{Q(z)}$$

Comparing the numerator of this fraction with the polynomial $P_1(z)$ leads to the system of equations:

$$\begin{cases} -6A+3B-2C & = 1 \\ -A-4B+C & = -3 \\ A+B+C & = 28 \end{cases}$$

Example $\frac{z^2-3z+28}{6z^3-5z^2-2z+1}$ (continuation)

The solution of the system is

$$A = -\frac{13}{3} \qquad B = \frac{119}{15} \qquad C = \frac{122}{5}.$$

So, we have

$$S(z) = \frac{-13}{3(1-z)} + \frac{119}{15(1+2z)} + \frac{122}{5(1-3z)}.$$

and the power series $S(z) = \sum_{n \geqslant 0} s_n z^n$, where the coefficient

$$s_n = -\frac{13}{3} + \frac{119}{15}(-2)^n + \frac{122}{5}3^n.$$

Next subsection

Step 2 (alternative): Partial Rational Expansion

Theorem 1 (for Distinct Roots)

If $R(z) = P(z)/Q(z)$ is the generating function for the sequence $\langle r_n \rangle$,
where $Q(z) = (1-\rho_1 z)(1-\rho_2 z)\cdots(1-\rho_\ell z)$,
and the numbers $(\rho_1, \ldots, \rho_\ell)$ are distinct,
and if $P(z)$ is a polynomial of degree less than ℓ, then

$$r_n = a_1\rho_1^n + a_2\rho_2^n + \cdots + a_\ell\rho_\ell^n, \qquad \text{where} \qquad a_k = \frac{-\rho_k P(1/\rho_k)}{Q'(1/\rho_k)}$$

Sketch of proof.

- We show that $R(z) = S(z)$ for $S(z) = \frac{a_1}{1-\rho_1 z} + \cdots + \frac{a_\ell}{1-\rho_\ell z}$ and any $z \neq \alpha_k = 1/\rho_k$ (only the points where $R(z)$ might be equal to infinity).
- L'Hôpital's Rule is used

continues …

Recalling l'Hôpital's Rule

If either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$
and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Step 2: Partial Rational Expansion (2)

Continuation of the proof.

- $T(z) = R(z) - S(z)$ is a rational function of z and it suffices to show that $\lim_{z\to\alpha_k}(z-\alpha_k)T(z) = 0$.
- Thus we need to prove the following equality

$$\lim_{z\to\alpha_k}(z-\alpha_k)R(z) = \lim_{z\to\alpha_k}(z-\alpha_k)S(z).$$

- Due to

$$\frac{a_k(z-\alpha_k)}{1-\rho_j z} = \frac{a_k(z-\frac{1}{\rho_k})}{1-\rho_j z} = \frac{-a_k(1-\rho_k z)}{\rho_k(1-\rho_j z)} \to 0, \text{ if } k \neq j \text{ and } z \to \alpha_k$$

  the right-hand side is

$$\lim_{z\to\alpha_k}(z-\alpha_k)S(z) = \lim_{z\to\alpha_k}(z-\alpha_k)\frac{a_k(z-\alpha_k)}{1-\rho_k z} = \frac{-a_k}{\rho_k} = \frac{P(1/\rho_k)}{Q'(1/\rho_k)}$$

continues ...

Step 2: Partial Rational Expansion (3)

Continuation of the proof.

- The left-hand side limit is

$$\lim_{z\to\alpha_k}(z-\alpha_k)R(z)=\lim_{z\to\alpha_k}(z-\alpha_k)\frac{P(z)}{Q(z)}=P(\alpha_k)\lim_{z\to\alpha_k}\frac{z-\alpha_k}{Q(z)}=\frac{P(\alpha_k)}{Q'(\alpha_k)}=\frac{P(1/\rho_k)}{Q'(1/\rho_k)}$$

by l'Hôpital's rule

Q.E.D.

General Expansion Theorem for Rational Generating Functions.

Theorem 2 (for possibly Multiple Roots)

If $R(z) = P(z)/Q(z)$ is the generating function for the sequence $\langle r_n \rangle$, where $Q(z) = (1-\rho_1 z)^{d_1} \cdots (1-\rho_\ell z)^{d_\ell}$ and the numbers $\rho_1, \ldots, \rho_\ell$ are distinct, and if $P(z)$ is a polynomial of degree less than $d = d_1 + \ldots + d_\ell$, then

$$r_n = f_1(n)\rho_1^n + \cdots + f_\ell(n)\rho_\ell^n, \qquad \text{for all} \qquad n \geqslant 0,$$

where each $f_k(n)$ is a polynomial of degree $d_k - 1$ with leading coefficient

$$a_k = \frac{(-\rho_k)^{d_k} P(1/\rho_k) d_k}{Q^{(d_k)}(1/\rho_k)} = \frac{P(1/\rho_k)}{(d_k - 1)! \prod_{j \neq k} (1 - \rho_j/\rho_k)^{d_j}}$$

Proof: (omitted) by induction on $d = d_1 + \ldots + d_\ell$.

Warmup: What if $\deg P \geqslant \deg Q$?

The problem

- The hypotheses of the Rational Expansion Theorem include that the degree of the numerator be smaller than that of the denominator.
- What if it is not so?

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The problem

- The hypotheses of the Rational Expansion Theorem include that the degree of the numerator be smaller than that of the denominator.
- What if it is not so?

Answer: It is a false problem!

If $\deg P \geqslant \deg Q$, then we can do *polynomial division* and uniquely determine two polynomials $S(z)$, $R(z)$ such that:

- $\deg R < \deg Q$;
- $P(z) = Q(z) \cdot S(z) + R(z)$.

Then

$$\frac{P(z)}{Q(z)} = S(z) + \frac{R(z)}{Q(z)} :$$

the first summand only influences finitely many coefficients, and on the second one the Rational Expansion Theorem can be applied.

Example: Fibonacci numbers revisited once more(2)

Step 3 Solving the equation

$$G(z) = \frac{z}{1 - z - z^2}$$

Step 4 Expand the (rational) equation $G(z) = P(z)/Q(z)$ for $P(z) = z$ and $Q(z) = 1 - z - z^2$:

- From the example above we know that $Q(z) = (1 - \Phi z)(1 - \widehat{\Phi} z)$
- As $Q'(z) = -1 - 2z$, we have

$$\frac{-\Phi P(1/\Phi)}{Q'(1/\Phi)} = \frac{-1}{-1 - 2/\Phi} = \frac{\Phi}{\Phi + 2} = \frac{1}{\sqrt{5}}$$

and

$$\frac{-\widehat{\Phi} P(1/\widehat{\Phi})}{Q'(1/\widehat{\Phi})} = \frac{\widehat{\Phi}}{\widehat{\Phi} + 2} = -\frac{1}{\sqrt{5}}$$

- Theorem 1 gives us

$$g_n = \frac{\Phi^n - \widehat{\Phi}^n}{\sqrt{5}}$$

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Next section

Next subsection

Example: A more-or-less random recurrence.

Step 1 Given recurrence

$$g_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } 0 \leqslant n < 2; \\ g_{n-1} + 2g_{n-2} + (-1)^n & \text{if } 2 \leqslant n; \end{cases}$$

can be represented by the single equation

$$g_n = g_{n-1} + 2g_{n-2} + (-1)^n [n \geqslant 0] + [n = 1].$$

Some values:

n	0	1	2	3	4	5	6	7
g_n	1	1	4	5	14	23	52	97

Example: A more-or-less random recurrence (2)

Step 2 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + 2\sum_n g_{n-2} z^n + \sum_{n \geqslant 0} (-1)^n z^n + \sum_n [n=1] z^n =$$

$$= \sum_n g_n z^{n+1} + 2\sum_n g_n z^{n+2} + \frac{1}{1+z} + z =$$

$$= zG(z) + 2z^2 G(z) + \frac{1+z+z^2}{1+z}$$

Step 3 Solving the equation

$$G(z) = \frac{1+z+z^2}{(1-z-2z^2)(1+z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$$

Example: A more-or-less random recurrence (2)

Step 2 Write down $G(z) = \sum_n g_n z^n$ and transform

$$G(z) = \sum_n g_n z^n = \sum_n g_{n-1} z^n + 2 \sum_n g_{n-2} z^n + \sum_{n \geqslant 0} (-1)^n z^n + \sum_n [n=1] z^n =$$

$$= \sum_n g_n z^{n+1} + 2 \sum_n g_n z^{n+2} + \frac{1}{1+z} + z =$$

$$= zG(z) + 2z^2 G(z) + \frac{1+z+z^2}{1+z}$$

Step 3 Solving the equation

$$G(z) = \frac{1+z+z^2}{(1-z-2z^2)(1+z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$$

Example: A more-or-less random recurrence (3)

Step 4 Expand the (rational) equation $G(z) = P(z)/Q(z)$ for $P(z) = 1 + z + z^2$ and $Q(z) = (1-2z)(1+z)^2$:

- Theorem 2 gives us for some constant c:

$$g_n = a_1 2^n + (a_2 n + c)(-1)^n,$$

where

$$a_1 = \frac{P(1/2)}{0!(1+1/2)^2} = \frac{4(1+1/2+1/4)}{9} = \frac{7}{9}$$

and

$$a_2 = \frac{P(-1)}{1!(1+2)} = \frac{1+1-1}{3} = \frac{1}{3}$$

- Special case $n = 0$ implies $1 = g_0 = \frac{7}{9} + c$ that gives $c = 1 - \frac{7}{9} = \frac{2}{9}$.
- The answer is

$$g_n = \frac{7}{9} 2^n + \left(\frac{1}{3} n + \frac{2}{9}\right)(-1)^n.$$

If $R(z) = P(z)/Q(z)$ the generating function for the sequence $\langle r_n \rangle$, where $Q(z) = (1-\rho_1 z)^{d_1} \cdots (1-\rho_\ell z)^{d_\ell}$ and the numbers $(\rho_1, \ldots, \rho_\ell)$ are distinct, and if $P(z)$ is a polynomial of degree less than $d_1 + \ldots + d_\ell$, then

$$r_n = f_1(n)\rho_1^n + \cdots + f_\ell(n)\rho_\ell^n, \qquad \text{for all} \qquad n \geqslant 0,$$

where each $f_k(n)$ is a polynomial of degree $d_k - 1$ with a leading coefficient

$$a_k = \frac{(-\rho_k)^{d_k} P(1/\rho_k) d_k}{Q^{(d_k)}(1/\rho_k)} = \frac{P(1/\rho_k)}{(d_k-1)! \prod_{j \neq k}(1-\rho_j/\rho_k)^{d_j}}$$

Decomposition into Partial Fractions

The same function: $G(z) = \frac{P(z)}{Q(z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$

- Decompose it as

$$G(z) = \frac{A}{1-2z} + \frac{B}{1+z} + \frac{C}{(1+z)^2}$$

- Expand

$$\begin{aligned} G(z) &= \frac{A}{1-2z} + \frac{B}{1+z} + \frac{C}{(1+z)^2} = \\ &= \frac{A(1+z)^2 + B(1-2z)(1+z) + C(1-2z)}{(1-2z)(1+z)^2} = \\ &= \frac{(A-2B)z^2 + (2A-B-2C)z + A+B+C}{(1-2z)(1+z)^2} \end{aligned}$$

continues ...

Decomposition into Partial Fractions (2)

The function: $G(z) = \frac{P(z)}{Q(z)} = \frac{1+z+z^2}{(1-2z)(1+z)^2}$

- System of equations:

$$\begin{cases} A-2B & =1 \\ 2A-B-2C & =1 \\ A+B+C & =1 \end{cases}$$

- The solution: $A = \frac{7}{9}, B = -\frac{1}{9}, C = \frac{1}{3}$
- The result of decomposition $G(z) = \frac{7}{9(1-2z)} - \frac{1}{9(1+z)} + \frac{1}{3(1+z)^2}$
- using the basic identity

$$\frac{a}{(1-\rho z)^k} = \sum_{n\geqslant 0} \binom{n+k-1}{k-1} a\rho^n z^n,$$

we get the power series

$$G(z) = \sum_{n\geqslant 0} \left[\frac{7}{9}2^n - \frac{1}{9}(-1)^n + \frac{n+1}{3}(-1)^n\right] z^n = \sum_{n\geqslant 0} g_n z^n,$$

where

$$g_n = \frac{7}{9}2^n + \left(\frac{1}{3}n + \frac{2}{9}\right)(-1)^n.$$

Example 3: Usage of derivatives

Step 1 Given recurrence

$$g_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \frac{2}{n} g_{n-2}, & \text{if } n > 0; \end{cases}$$

can be represented by the single equation

$$g_n = \frac{2}{n} g_{n-2} + [n = 0].$$

Some values:

n	0	1	2	3	4	5	6	7	8	9	10
g_n	1	0	1	0	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{24}$	0	$\frac{1}{120}$

Step 2 Write down $G(z) = \sum_n g_n z^n$ and its first derivative:

$$G(z) = \sum_n g_n z^n = \sum_n [n = 0] z^n + 2 \sum_n \frac{g_{n-2}}{n} z^n = 1 + 2 \sum_n \frac{g_{n-2}}{n} z^n$$

$$G'(z) = 2 \sum_n \frac{g_{n-2} \cdot n}{n} z^{n-1} = 2z \sum_n g_{n-2} z^{n-2} = 2zG(z)$$

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Example 3: Usage of derivatives (2)

Step 3 We need to solve the differential equation $G'(z) = 2zG(z)$.

- We rewrite the equation as

$$\frac{dG(z)}{dz} = 2zG(z)$$

- By treating $G(z)$ as it was another variable, we further rewrite:

$$\frac{dG(z)}{G(z)} = 2zdz$$

  (Such differential equations are called separable, because they can be solved by "separating the variables".)

- By equating the indefinite integrals, we get:

$$\ln G(z) = z^2 + \overline{C}$$

- By taking exponentials, we obtain:

$$G(z) = C\mathrm{e}^{z^2}, \text{ where } C = \mathrm{e}^{\overline{C}}$$

- By applying $G(0) = g_0 = 1$ we get $C = 1$.

In conclusion: $G(z) = \mathrm{e}^{z^2}$.

Example 3: Usage of derivatives (3)

Step 4 Considering that $e^z = \sum_{n \geqslant 0} \frac{1}{n!} z^n$,

- and denoting $u = z^2$, we get

$$G(z) = e^{z^2} = e^u = \sum \frac{1}{n!} u^n$$
$$= \sum \frac{1}{n!} (z^2)^n = \sum \frac{1}{n!} z^{2n}$$
$$= \sum \frac{1}{(\frac{n}{2})!} [n \text{ is even}] z^n$$

- To conclude:

$$g_n = \begin{cases} \frac{1}{k!}, & \text{if } n = 2k, k \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Q.E.D.

Example 3: Usage of derivatives (3)

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