Concrete Mathematics
Exercises from 4 October 2016

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Warmups

Exercise 2.12

Show that the function $p(k) = k + (-1)^k c$ is a permutation of the set of all integers, whenever c is an integer.
Solution. A way to solve the exercise is to prove that $p(k)$ has an *inverse function* $q(n)$, defined for every integer n, such that $p(k) = n$ if and only if $q(n) = k$.

So let $p(k) = k + (-1)^k c = n$. Then $n + c = k + (1 + (-1)^k)c$. But $1 + (-1)^k$ is 2 if k is even and 0 if k is odd, which means that k and $n + c$ are either both even or both odd: hence, $(-1)^k = (-1)^{n+c}$. We can thus rewrite $k = n + c - (1 + (-1)^k)c = n - (-1)^{n+c}c$: this is the inverse function $q(n)$ we were looking for.

Exercise 2.13

Use the repertoire method to find a closed form for $\sum_{k=0}^{n}(-1)^k k^2$.
Solution.

The sequence $S_n = \sum_{k=0}^{n}(-1)^k k^2$ is a special solution of the recurrence equation

$$\begin{aligned} R_0 &= \alpha \, , \\ R_n &= R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2) \quad \text{for } n \geq 1 \end{aligned}$$

for the special values $\alpha = \beta = \gamma = 0, \delta = 1$. As we know that we can express $R_n = \alpha A(n) + \beta B(n) + \gamma C(n) + \delta D(n)$ for special functions $A(n)$, $B(n)$,

$C(n)$ and $D(n)$, if we manage to find $D(n)$ in closed form, then that will be the closed form of S_n.

Let us use the repertoire method. First of all, for $\alpha = 1, \beta = \gamma = \delta = 0$ we find $A(n) = 1$ for every $n \geq 0$. The next step should *not* be to put $R_n = 1$ for every $n \geq 0$, as we already know that this is associate to the special values $\alpha = 1$, $\beta = \gamma = \delta = 0$. Instead, we put $R_n = (-1)^n$, which corresponds to $\alpha = 1$, $\beta = 2$, $\gamma = \delta = 0$, and tackles with the issue of the $(-1)^n$ factor in the summand: from this we get $A(n) + 2B(n) = (-1)^n$. As we know that $A(n) = 1$ for every $n \geq 0$, this means $2B(n) = (-1)^n - 1$, thus $B(n) = ((-1)^n - 1)/2 = -[n \text{ is odd}]$. This is a rather ugly function, and we would be very happy not to have to deal with it.

The third step will be to put $R_n = (-1)^n \cdot n$. This corresponds to $\alpha = 0$ and the recurrence equation

$$\begin{aligned}(-1)^n n &= (-1)^{n-1}(n-1) + (-1)^n \beta + (-1)^n \gamma n \\ &= (-1)^{n-1} n - (-1)^{n-1} + (-1)^n \beta + (-1)^n \gamma n + (-1)^n \delta n^2 ,\end{aligned}$$

which is satisfied for every $n \geq 1$ if and only if $\delta = 0$, $\beta = -1$, and $\gamma = 2$. We thus get the equation $-B(n) + 2C(n) = (-1)^n n$.

The fourth step will be to put $R_n = (-1)^n n^2$. This corresponds to $\alpha = 0$ and the recurrence equation

$$\begin{aligned}(-1)^n n^2 &= (-1)^{n-1}(n-1)^2 + (-1)^n(\beta + \gamma n + \delta n^2) \\ &= (-1)^{n-1}(n^2 - 2n + 1) + (-1)^n(\beta + \gamma n + \delta n^2) \\ &= ((-1)^{n-1} + (-1)^n \beta) \\ &\quad +((-1)^{n-1} \cdot (-2) + (-1)^n \gamma) n \\ &\quad +((-1)^{n-1} + (-1)^n \delta) n^2 ,\end{aligned}$$

which is satisfied for every $n \geq 1$ if and only if $\beta = 1$, $\gamma = -2$, and $\delta = 2$. We thus get $B(n) - 2C(n) + D(n) = (-1)^n n^2$.

At this point, we have a full system of equations:

$$\begin{array}{lllll} A(n) & & & & = 1 \\ A(n) & +2B(n) & & & = (-1)^n \\ & -B(n) & +2C(n) & & = (-1)^n n \\ & B(n) & -2C(n) & +2D(n) & = (-1)^n n^2 \end{array}$$

from which we want to find $D(n)$. But by adding together the third and fourth equation we immediately find $2D(n) = (-1)^n \cdot (n + n^2)$. Then $S_n = D(n) = (-1)^n(n^2 + n)/2 = (-1)^n T_n$, where T_n is the nth triangular number.

Exercise 2.20

Try to evaluate $\sum_{k=0}^{n} kH_k$ by the perturbation method, but deduce the value of $\sum_{k=0}^{n} H_k$ instead.

Solution. Call $\sum_{k=0}^{n} kH_k = S_n$. Let's try the perturbation method:

$$\begin{aligned}
S_n + (n+1)H_{n+1} &= \sum_{0\le k\le n+1} kH_k \\
&= \sum_{0\le k\le n} (k+1)H_{k+1} \\
&= \sum_{0\le k\le n} (k+1)\left(H_k + \frac{1}{k+1}\right) \\
&= \sum_{0\le k\le n} \left((k+1)\cdot H_k + 1\right) \\
&= S_n + \sum_{0\le k\le n} H_k + (n+1)\ .
\end{aligned}$$

We have S_n on both sides, so our attempt to evaluate $\sum_{k=0}^{n} kH_k$ has failed. However, $\sum_{k=0}^{n} H_k$ has popped out, and we can work on that one instead! A simple rearrangement of the summands yields

$$\begin{aligned}
\sum_{0\le k\le n} H_k &= (n+1)\cdot H_{n+1} - (n+1) \\
&= (n+1)\cdot\left(H_n + \frac{1}{n+1}\right) - (n+1) \\
&= (n+1)\cdot H_n + 1 - (n+1) \\
&= (n+1)\cdot H_n - n\ .
\end{aligned}$$