# Concrete Mathematics
# Exercises from 11 October 2016

Silvio Capobianco

## Exercise 2.14

Use multiple sums to evaluate

$$\sum_{k=1}^{n} k \cdot 2^k$$

*Solution.* Write $k = \sum_{j=1}^{k} 1$. Then

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot 2^k &= \sum_{k=1}^{n} \left( \sum_{j=1}^{k} 1 \right) \cdot 2^k \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k} 1 \cdot 2^k \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} 2^k
\end{aligned}$$

Clearly,

$$\begin{aligned}
\sum_{k=j}^{n} 2^k &= 2^j \cdot \sum_{k=0}^{n-j} 2^k \\
&= 2^j \cdot \left( 2^{n-j+1} - 1 \right) \\
&= 2^{n+1} - 2^j
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot 2^k &= \sum_{j=1}^{n} \left(2^{n+1} - 2^j\right) \\
&= \sum_{j=1}^{n} 2^{n+1} - \sum_{j=1}^{n} 2^j \\
&= n \cdot 2^{n+1} - 2 \cdot \sum_{j=0}^{n-1} 2^j \\
&= n \cdot 2^{n+1} - 2 \cdot (2^n - 1) \\
&= n \cdot 2^{n+1} - 2^{n+1} + 2 \\
&= (n-1) \cdot 2^{n+1} + 2
\end{aligned}$$

## Exercise 2.15

Evaluate $\boxcube_n = \sum_{k=1}^{n} k^3$ by the text's Method 5 as follows: First write $\boxcube_n + \square_n = 2 \sum_{1 \le j \le k \le n} jk$ ; then apply (2.33).

*Solution.* Recall that $\square_n = \sum_{k=1}^{n} k^2$. Then:

$$\begin{aligned}
\boxcube_n + \square_n &= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= \sum_{k=1}^{n} k^2 (k+1) \\
&= 2 \sum_{k=1}^{n} k \cdot \frac{k(k+1)}{2} \\
&= 2 \sum_{k=1}^{n} k \cdot \sum_{j=1}^{k} j \\
&= 2 \sum_{1 \le j \le k \le n} jk \ .
\end{aligned}$$

By (2.33), whatever the summands $a_k$ are,

$$\sum_{1 \le j \le k \le n} a_j a_k = \frac{1}{2} \left( \sum_{k=1}^{n} a_k^2 + \left( \sum_{k=1}^{n} a_k \right)^2 \right) \quad :$$

in our case, $a_k = k$, and

$$\cube_n + \square_n = \sum_{k=1}^{n} k^2 + \left( \sum_{k=1}^{n} k \right)^2 = \square_n + \left( \sum_{k=1}^{n} k \right)^2 ,$$

which yields $\cube_n = \left(\sum_{k=1}^{n} k\right)^2 = (n(n+1)/2)^2$.

## Exercise 2.23

Evaluate $\sum_{k=1}^{n} (2k+1)/k(k+1)$ in two ways:

1. Replace $1/k(k+1)$ by the "partial fractions" $1/k - 1/(k+1)$.
2. Sum by parts.

*Solution.* By method 1 we get:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} &= 2 \sum_{k=1}^{n} \frac{1}{k+1} + \sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&= 2 \cdot (H_{n+1} - 1) + \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) \\
&= 2H_{n+1} - 2 + 1 - \frac{1}{n+1} \\
&= 2H_n - \frac{n}{n+1} ,
\end{aligned}$$

as $H_{n+1} = H_n + 1/(n+1)$.

As a variant, we can observe that $\frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$ and compute:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k+1} \\
&= H_n + (H_{n+1} - 1) \\
&= H_n + \left( H_n + \frac{1}{n+1} - 1 \right) \\
&= 2H_n - \frac{n}{n+1} .
\end{aligned}$$

To use method 2, we need to express $(2k+1)/k(k+1)$ as $u\Delta v$ for suitable $u$ and $v$. If we choose $u(x) = 2x + 1$ and $\Delta v(x) = 1/x(x+1) = (x-1)^{\underline{-2}}$, then $\Delta u(x) = 2$ and $v(x) = -(x-1)^{\underline{-1}} = -1/x$, thus:

$$\begin{aligned} \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} &= \sum_{1}^{n+1} u(x)\Delta v(x)\delta x \\ &= u(x)v(x)\big|_{x=1}^{x=n+1} - \sum_{1}^{n+1} Ev(x)\Delta u(x)\delta x \\ &= -\frac{2x+1}{x}\bigg|_{x=1}^{x=n+1} + \sum_{1}^{n+1} \frac{2}{x+1}\delta x \,. \end{aligned}$$

The first summand is $3 - (2n+3)/(n+1) = n/(n+1)$. For the other one, we know that $\sum_{1}^{n+1} \Delta g(x)\delta x = g(n+1) - g(1)$: for $\Delta g(x) = 1/(x+1)$ it is clearly $g(x) = H_x$, thus

$$\sum_{1}^{n+1} \frac{2}{x+1} = 2(H_{n+1} - H_1) = 2H_n + \frac{2}{n+1} - 2 \,.$$

Putting everything together, $\sum_{k=1}^{n} (2k+1)/k(k+1) = 2H_n - n/(n+1)$, as we had previously found.