Concrete Mathematics
Exercises from Chapter 3

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Exercise 3.10

Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor \tag{1}$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?
Solution. We observe that

$$\begin{aligned}\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor &= \left\lceil \frac{2x+1}{2} \right\rceil - \left(\left\lceil \frac{2x+1}{4} \right\rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor \right) \\ &= \left\lceil x + \frac{1}{2} \right\rceil - \left[\frac{2x+1}{4} \text{ is not an integer} \right] .\end{aligned}$$

(Do not forget that x is a real number.) But $(2x+1)/4 = k$ is an integer if and only if $x = (4k-1)/2 = 2k - 1/2$: in this case, $\lceil x + 1/2 \rceil = 2k = \lceil x \rceil$. Otherwise, we have to distinguish the two cases $0 \leq \{x\} < 1/2$, $1/2 \leq \{x\} - 1$. In the second case, $\lceil (2x+1)/2 \rceil = \lceil x + 1/2 \rceil = \lceil x \rceil + 1$, and the expression (1) equals $\lceil x \rceil$. In the first case, if $\{x\} = 0$ (*i.e.*, x is an integer) then $\lceil x + 1/2 \rceil = x + 1 = \lceil x \rceil + 1$ and the expression (1) equals $\lceil x \rceil$; while if $0 < x < 1/2$, then $\lceil x + 1/2 \rceil = \lceil \lfloor x \rfloor + \{x\} + 1 \rceil = \lfloor x \rfloor + 1$ and the expression (1) equals $\lfloor x \rfloor$. In conclusion,

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lfloor x \rfloor \text{ if } 0 < \{x\} < \frac{1}{2} \text{ else } \lceil x \rceil .$$

Exercise 3.12

Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \tag{2}$$

for all integers n and all positive integers m. (This identity gives us another way to convert ceilings to floors and vice versa, instead of using the reflective law (3.4).)

Solution.

The closed interval $[n/m, (n+m-1)/m]$ has size $1-1/m$, and can thus contain at most one integer: in this case, such integer must coincide with both $\lceil n/m \rceil$ and $\lfloor (n+m-1)/m \rfloor$. However, of the m consecutive integers $n, n+1, \ldots, n+m-1$, exactly one is divisible by m: if x is this number, then $x/m \in [n/m, (n+m-1)/m]$ is the common value of $\lceil n/m \rceil$ and $\lfloor (n+m-1)/m \rfloor$.

Exercise 3.13

Let α and β be positive reals. Prove that the following are equivalent:

1. Spec(α) and Spec(β) partition the positive integers, *i.e.*, every positive integer n belongs to exactly one between Spec(α) and Spec(β).

2. α and β are irrational and $1/\alpha + 1/\beta = 1$.

Solution. We recall that, for a positive real x, the number $N(x, n)$ of elements in Spec(x) not greater than n satisfies

$$N(x, n) = \left\lceil \frac{n+1}{x} \right\rceil - 1$$

Suppose that point 2 is satisfied. Then α and β, being irrational, must be different (otherwise $\alpha = \beta = 2$). Also, $(n+1)/\alpha$ is not an integer (because α is irrational) and

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1 = \left\lfloor \frac{n+1}{\alpha} \right\rfloor = \frac{n+1}{\alpha} - \left\{ \frac{n+1}{\alpha} \right\},$$

and similarly for $(n+1)/\beta$. Hence,

$$N(\alpha, n) + N(\beta, n) = \left(\frac{1}{\alpha} + \frac{1}{\beta} \right)(n+1) - \left(\left\{ \frac{n+1}{\alpha} \right\} + \left\{ \frac{n+1}{\beta} \right\} \right)$$

By hypothesis, $1/\alpha + 1/\beta = 1$. Then the rightmost term in open parentheses is the sum of the fractional parts of two non-integer numbers whose sum is an integer, and is therefore equal to 1. Therefore $N(\alpha, n) + N(\beta, n) = n + 1 - 1 = n$ for every positive integer n: then also, for every n, either $N(\alpha, n+1) = N(\alpha, n)+1$ and $N(\beta, n+1) = N(\beta, n)$, or $N(\alpha, n+1) = N(\alpha, n)$ and $N(\beta, n+1) = N(\beta, n) + 1$: that is, each integer larger than 1 goes into exactly one of the two spectra. As $1/\alpha + 1/\beta = 1$ and $\alpha \neq \beta$, one of them is smaller than 2 and the other is greater, and $n = 1$ goes into the spectrum of the former: this allows us to conclude that $\mathrm{Spec}\,(\alpha)$ and $\mathrm{Spec}\,(\beta)$ partition the positive integers.

On the other hand, suppose that point 1 holds. Then the difference between $N(\alpha, n)$ and $(n+1)/\alpha - \{(n+1)/\alpha\}$ is at most 1, and similar for $N(\beta, n)$: therefore, for every positive integer n, we have:

$$n = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(n+1) - 1 + \text{a bounded quantity}$$

that is,

$$\left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) n = \frac{1}{\alpha} + \frac{1}{\beta} - 1 + \text{a bounded quantity}$$

By hypothesis, this equality must hold whatever n is: since α and β are constant, this is only possible if $1/\alpha + 1/\beta = 1$. In turn, this implies that α and β are either both rational or both irrational.

Suppose, for the sake of contradiction, that they are both rational: then there exist integers a, b, m such that $\alpha = m/a$ and $\beta = m/b$. (We are doing a thing slightly different than usual, reducing α and β to *common numerator* instead of common denominator. This is not a problem, since it is equivalent to reducing $1/\alpha$ and $1/\beta$ to common denominator.) Then $m = \lfloor a\alpha \rfloor = \lfloor b\beta \rfloor$ occurs in both $\mathrm{Spec}\,(\alpha)$ and $\mathrm{Spec}\,(\beta)$, so that $\mathrm{Spec}\,(\alpha)$ and $\mathrm{Spec}\,(\beta)$ do not form a partition of the positive integers: which goes against the hypothesis of point 1.

But the actual situation is even worse than that! Since $1/\alpha + 1/\beta = 1$, α and β are both greater than 1: therefore m must be greater than 1, and $m - 1$ is a positive integer. But

$$\lfloor (a-1)\alpha \rfloor = \lfloor m - \alpha \rfloor = m + \lfloor -\alpha \rfloor = m - \lceil \alpha \rceil \leq m - 2$$

which together with $\lfloor a\alpha \rfloor = m$ implies $m - 1 \notin \mathrm{Spec}\,(\alpha)$. Similarly, $m - 1 \notin \mathrm{Spec}\,(\beta)$. Therefore, if α and β are rational, then $\mathrm{Spec}\,(\alpha)$ and $\mathrm{Spec}\,(\beta)$ don't even *cover* the positive integers.