Concrete Mathematics
Exercises from 25 October 2016

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Exercise 4.16

The *Euclid numbers* are defined for $n \geq 1$ by the relation:

$$\begin{array}{rcl} e_1 & = & 2 \\ e_n & = & e_1 \cdots e_{n-1} + 1 \quad \forall n \geq 2 \end{array}$$

What is the sum of the reciprocals of the first n Euclid numbers?
Solution.

Let us start counting: $1/e_1 = 1/2$; $1/e_1 + 1/e_2 = 1/2 + 1/3 = 5/6$; $1/e_1 + 1/e_2 + 1/e_3 = 5/6 + 1/7 = 41/42$; and so on.

Do we recognize any pattern? These seem to have the form $1 - 1/d_n$ where d_n is a *product* of Euclid numbers: this number is $d_1 = 2 = e_1 = e_2 - 1$ for $1/e_1$, $d_2 = 6 = e_1 e_2 = e_3 - 1$ for $1/e_1 + 1/e_2$, and so on.

Let us check it by induction. Suppose $1/e_1 + \ldots + 1/e_{n-1} = 1 - 1/(e_n - 1)$: then

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{n} \frac{1}{e_k} & = & \displaystyle\sum_{k=1}^{n-1} \frac{1}{e_k} + \frac{1}{e_k} \\ & = & 1 - \dfrac{1}{e_n - 1} + \dfrac{1}{e_n} \\ & = & 1 - \dfrac{e_n - (e_n - 1)}{(e_n - 1)e_n} \\ & = & 1 - \dfrac{1}{e_1 \cdots e_{n-1} \cdot e_n} \\ & = & 1 - \dfrac{1}{e_{n+1} - 1} \, . \end{array}$$

Esercise 4.17

Let f_n be the "Fermat number" $2^{2^n}+1$. Prove that $\gcd(f_m, f_n) = 1$ if $m < n$.
Solution. Let us construct the firs t Fermat numbers: $f_0 = 3$, $f_1 = 5$, $f_2 = 17$, $f_3 = 257$, $f_4 = 65537$. We observe that $f_0 = 3$ divides $f_1 - 2 = 3$, $f_2 - 2 = 15$, $f_3 - 2 = 255$, $f_4 - 2 = 65535$; and so on. We also observe that $f_1 = 5$ divides $f_2 - 2$, $f_3 - 2$, and $f_4 - 2$. We thus formulate the following conjecture: if $m < n$ then $f_m \mid f_n - 2$.

Is this conjecture of any utility for our objective? Yes, it is: if $f_m \mid f_n - 2$, then $\gcd(f_m, f_n) = \gcd(f_n \bmod f_m, f_m) = \gcd(2, f_m) = 1$ as f_m is odd.

Let us now prove the conjecture. If $m < n$ then 2^{n-m} is even: but $a^{2r} - 1 = (a+1)(a^{2r-1} - a^{2r-2} + \ldots + a - 1)$. Put then $a = 2^{2^m}$ and $2^{n-m} = 2r$: then $f_m = a + 1$ and $f_n - 2 = a^{2r} - 1$.

Exercise 4.18

Show that if $2^n + 1$ is prime then n is a power of 2.
Solution.

We reformulate the problem as follows: if n has an odd factor $m > 1$, then $2^n + 1$ has a nontrivial factor. So suppose $n = qm$ with $m > 1$ odd: then

$$2^n + 1 = 2^{qm} + 1 = (2^q + 1)(2^{(m-1)q} - 2^{(m-2)q} + \ldots + 2^{2q} - 2^q + 1) ,$$

and the factor $2^q + 1$ surely is nontrivial.

Exercise 4.20

For every positive integer n there's a prime p such that $n < p \leq 2n$. (This is essentially "Bertrand's postulate", which Joseph Bertrand verified for $n < 3000000$ in 1845 and Chebyshev proved for all n in 1850.) Use Bertrand's postulate to prove that there's a constant $b \approx 1.25$ such that the numbers

$$\lfloor 2^b \rfloor \; . \; \left\lfloor 2^{2^b} \right\rfloor \; , \; \left\lfloor 2^{2^{2^b}} \right\rfloor \; , \; \ldots \tag{1}$$

are all prime.
Solution. Call lg the binary (base-2) logarithm. Let us define a "simple" sequence of primes by putting $p_1 = 2$, and p_n as the smallest prime larger

than $2^{p_{n-1}}$. By Bertrand's postulate, $2^{p_{n-1}} < p_n < 2^{p_{n-1}+1}$ for every $n \geq 2$: we can switch to strict inequality because such p_n are odd. Hence,

$$p_{n-1} < \lg p_n < p_{n-1} + 1 \tag{2}$$

for every $n \geq 2$. The left-hand inequality of (2) tells us that the sequence

$$b_n = \lg^{(n)} p_n \,, \tag{3}$$

where $\lg^{(n)}$ is the nth iteration of lg, is nondecreasing. To prove that it is bounded from above, we set $a_1 = 2$ and $a_n = 2^{a_{n-1}}$ for every $n \geq 2$, so that $a_2 = 4$, $a_3 = 16$, and so on: we prove by induction that $p_n < a_{n+1}$ for every $n \geq 1$, from which follows $b_n < 2$ for every $n \geq 1$ as $\lg^{(n)} a_{n+1} = 2$. This is true for $n = 1$ and $n = 2$ as $p_2 = 5$; for $n \geq 3$, if $p_{n-1} < a_n$, then, as p_{n-1} and a_n are both integers, $p_{n-1} + 1 \leq a_n$, and the right-hand inequality of (2) tells us that $p_n < 2^{p_{n-1}+1} \leq 2^{a_n} = a_{n+1}$. We then set:

$$b = \lim_{n\to\infty} b_n = \sup_{n\geq 1} \lg^{(n)} p_n \,. \tag{4}$$

To prove that this is the b we were looking for, we set $u_1 = 2^b$ and $u_n = 2^{u_{n-1}}$ for every $n \geq 2$: we will show that $\lfloor u_n \rfloor = p_n$ for every $n \geq 1$, which will solve the exercise. Clearly $\lfloor u_n \rfloor \geq p_n$ as $b_n < b$; also, as $b = 1.25164\ldots$ and $2^{1.26} < 2.4$, $\lfloor u_1 \rfloor = p_1$. If for some $n > 1$ it is $\lfloor u_n \rfloor > p_n$, let n be the minimum value for which this happens: then $u_n \geq p_n + 1$, and

$$u_{n-1} = \lg u_n \geq \lg(p_n + 1) > \lg p_n > p_{n-1} \,,$$

against minimality of n.

Exercise 4.21

Let P_n be the n-th prime number. Find a constant K such that

$$\left\lfloor (10^{n^2} K) \bmod 10^n \right\rfloor = P_n \,. \tag{5}$$

Solution. We use again Bertrand's postulate to make the basic observation that $P_n < 10^n$. Then the series $\sum_{k\geq 1} 10^{-k^2} P_k$ converges: let K be its sum. Then

$$10^{n^2} K = \left(\sum_{1\leq k<n} 10^{n^2-k^2} P_k \right) + P_n + \sum_{k>n} 10^{n^2-k^2} P_k \ :$$

we want to show that the first sum is divisible by 10^n, and the second one is smaller than 1.

First, for $1 \leq k < n$ it is $n^2 - k^2 \geq n^2 - (n-1)^2 = 2n - 1 \geq n$ as $n \geq 1$: therefore, each summand in the first sum is divisible by 10^n. Next, for $k > n$ it is $k = n + t$ with $t \geq 1$, and

$$k^2 - n^2 - k = 2nt + t^2 - n - t = (2t-1)n + t(t-1) \geq t \ :$$

then, as $P_k < 10^k$, $\sum_{k>n} 10^{n^2-k^2} P_k \leq \sum_{t \geq 1} 10^{-t} = 1/9$.