ITT9131 Concrete Mathematics
Exercises from November 15

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Exercise 5.2

Find the values of k for which $\binom{n}{k}$ is a maximum. Prove the answer.
Solution. Write $\binom{n}{k} = \frac{n!}{k!(n-k)!}$: as k varies between 0 and n, the numerator is constant, which means that $\binom{n}{k}$ is maximum when the denominator is *minimum*. The symmetry rule $\binom{n}{k} = \binom{n}{n-k}$ and the equality $\binom{n}{} = \binom{n}{n} = 1$ prompts us to conjecture that $\binom{n}{k}$ is maximum for $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$. We can verify our conjecture for $n \geq 2$ as it is clearly true for $n = 0$ and $n = 1$. By symmetry, we only need to prove it for $k \leq n/2$.

First, suppose $n = 2m$ is even: then $m = n/2 = \lfloor n/2 \rfloor = \lceil n/2 \rceil$. For $k \leq m$ the denominator of $\binom{n}{m}$ is $(m!)^2 = \left(m^{\underline{m-k}}\, k!\right)^2$, while that of $\binom{n}{k}$ is:

$$k!(2m-k)! = k!\,(2m-k)^{\underline{2m-2k}}\,k! = (k!)^2\,(2m-k)^{\underline{m-k}}\,m^{\underline{m-k}}\,.$$

Our thesis is then equivalent to

$$\left(m^{\underline{m-k}}\right)^2 \leq (2m-k)^{\underline{m-k}}\,m^{\underline{m-k}}\ :$$

which is clearly true as $2m - k \geq m$.

Next, suppose $n = 2m+1$ is even: then $m = (n-1)/2 = \lfloor n/2 \rfloor$ and $m+1 = (n+1)/2 = \lceil n/2 \rceil$, while $\binom{n}{m} = \binom{n}{m+1}$. For $k \leq m$ the denominator of $\binom{n}{m}$ is $m!\,(m+1)! = \left(m^{\underline{m-k}}\,k!\right)^2 (m+1)$, while that of $\binom{n}{k}$ is:

$$\begin{aligned} k!(2m+1-k)! &= k!\,(2m+1-k)^{\underline{2m+1-2k}}\,k! \\ &= (k!)^2\,(2m+1-k)^{\underline{m-k}}\,(m+1)\,m^{\underline{m-k}}\,. \end{aligned}$$

Our thesis is then equivalent to

$$\left(m^{\underline{m-k}}\right)^2 (m+1) \leq (2m+1-k)^{\underline{m-k}} (m+1)\, m^{\underline{m-k}} \ :$$

which is true because $2m+1-k \geq m$.

Other methods involve putting $f(k) = \binom{n}{k}$ for $0 \leq k \leq n$, and verify either that Δf is nonnegative for $k \leq \lfloor n \rfloor$ and negative for $k > \lceil n \rceil$, or that $f(k+1)/f(k)$ is greater or equal to 1 for $k \leq \lfloor n \rfloor$ and smaller than 1 for $k \leq \lfloor n \rfloor$.

Exercise 5.5

Let p be a prime. Prove that $\binom{p}{k} \equiv 0 \mod p$ for $0 < k < p$. Find a consequence about $\binom{p-1}{k}$.

Solution. Recall that $a \equiv b \mod m$ means that $a - b$ is a multiple of m. By definition,

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!}$$

If p is prime and k is neither 0 nor p, there is no way to make the p at numerator disappear by dividing by $k!$.

Now, $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$: since the left-hand side is 0 modulo p, going from $\binom{p-1}{k-1}$ to $\binom{p-1}{k}$ only involves a change of sign modulo p. Since $\binom{p-1}{0} = 1$, we get:

$$\binom{p-1}{k} \equiv (-1)^k \mod p\,.$$

Exercise 5.15

Equation (5.29) in the book states that, for every a, b, c nonnegative integers,

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}\,.$$

Use (5.29) to compute $\sum_k \binom{n}{k}^3 (-1)^k$.

Solution. Set $\sum_k \binom{n}{k}^3 (-1)^k = S(n)$. We preliminarly observe that $S(n) = 0$

if $n = 2m+1$ is odd, because in this case, as the sum is on every k integer,

$$\begin{aligned} \sum_k \binom{2m+1}{k}^3 (-1)^k &= \sum_k \binom{2m+1}{2m+1-k}^3 (-1)^{2m+1-k} \\ &= -\sum_k \binom{2m+1}{k}^3 (-1)^k , \end{aligned}$$

as k and $2m+1-k$ have opposite parity. So let $n = 2m$ be even: in this case,

$$S(n) = \sum_k \binom{2m}{k}\binom{2m}{k}\binom{2m}{k}(-1)^k$$

looks dangerously similar to (5.29) with $a = b = c = m$, except that the lower summation index is k instead of $m+k$. Is this a real problem? No, because the sum is on *every* integer, and does not change by translation of the index, so we can safely do:

$$\begin{aligned} \sum_k \binom{2m}{k}\binom{2m}{k}\binom{2m}{k}(-1)^k &= \sum_k \binom{2m}{m+k}\binom{2m}{m+k}\binom{2m}{m+k}(-1)^{m+k} \\ &= (-1)^m \sum_k \binom{2m}{m+k}\binom{2m}{m+k}\binom{2m}{m+k}(-1)^k \\ &= (-1)^m \cdot \frac{(3m)!}{(m!)^3} . \end{aligned}$$

In conclusion,

$$\sum_k \binom{n}{k}^3 (-1)^k = (-1)^{\lfloor n/2 \rfloor} \frac{(3\lfloor n/2 \rfloor)!}{(\lfloor n/2 \rfloor !)^3} [n \,\text{is even}] .$$

Exercise 5.37

Show that an analog of the binomial theorem holds for factorial powers. That is, prove the identities

$$\begin{aligned} (x+y)^{\underline{n}} &= \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} , \\ (x+y)^{\overline{n}} &= \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} , \end{aligned}$$

for all nonnegative integers n.

Solution. Let's prove the statement for falling powers. The case with rising powers would then follow easily, because it is true in general that $x^{\overline{n}} = (-1)^n -x^{\underline{n}}$, so that if the relation for falling powers is true, then

$$\begin{aligned}
(x+y)^{\overline{n}} &= (-1)^n(-x-y)^{\underline{n}} \\
&= (-1)^n \sum_k \binom{n}{k} (-x)^{\underline{k}}(-y)^{\underline{n-k}} \\
&= \sum_k \binom{n}{k} (-1)^k x^{\overline{k}} (-1)^{n-k} y^{\overline{n-k}},
\end{aligned}$$

and the corresponding relation for rising powers is also true.

The thesis can be proved by induction on n, as it is true for $n = 0$, and if it is true for n, then

$$\begin{aligned}
(x+y)^{\underline{n+1}} &= (x+y)^{\underline{n}}(x+y-n) \\
&= \left(\sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} \right) (x-k+y-(n-k)) \\
&= \sum_k \binom{n}{k} x^{\underline{k}}(x-k)y^{\underline{n-k}} + \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}(y-(n-k)) \\
&= \sum_k \binom{n}{k} x^{\underline{k+1}} y^{\underline{n-k}} + \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n+1-k}} \\
&= \sum_k \binom{n}{k-1} x^{\underline{k}} y^{\underline{n+1-k}} + \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n+1-k}} \\
&= \sum_k \binom{n+1}{k} x^{\underline{k}} y^{\underline{n+1-k}},
\end{aligned}$$

as it was claimed. Observe that we have used the trick of summing for every k integer.

But there is another, more interesting way to get to that result! By writing $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the first equation becomes

$$(x+y)^{\underline{n}} = n! \sum_k \frac{x^{\underline{k}}}{k!} \frac{y^{\underline{n-k}}}{(n-k)!} = n! \sum_k \binom{x}{k} \binom{y}{n-k},$$

which is equivalent to

$$\binom{x+y}{n} = \sum_k \binom{x}{k}\binom{y}{n-k} :$$

that is, the binomial theorem for falling powers is simply a rewriting of the Vandermonde convolution (5.22)!

Exercise 5.35

The writing $\sum_{k\le n}\binom{n}{k}2^{k-n}$ is ambiguous without context. Evaluate it

1. as a sum on k,

2. as a sum on n.

Solution. First, let us interpret the formula as a sum on k. Such sum vanishes if $n < 0$, otherwise

$$\sum_{k\le n}\binom{n}{k}2^{k-n} = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}2^k 1^{n-k} = 2^{-n}\cdot(2+1)^n = \left(\frac{3}{2}\right)^n .$$

In conclusion, $\sum_{k\le n} 2^{k-n} = (3/2)^n\,[n\ge 0]$ as a sum on k.

Next, let us interpret the formula as a sum over n. If $k < 0$ the sum clearly vanishes; otherwise,

$$\begin{aligned}\sum_{k\le n}\binom{n}{k}2^{k-n} &= \sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{1}{2}\right)^{n-k} \\ &= \sum_{n=0}^{\infty}\binom{k+n}{k}\left(\frac{1}{2}\right)^{n} \\ &= \frac{1}{(1-1/2)^{k+1}} \\ &= 2^{k+1}\end{aligned}$$

In conclusion, $\sum_{k\le n}\binom{n}{k}2^{k-n} = 2^{k+1}\,[k\ge 0]$ as a sum on n.