ITT9131 Concrete Mathematics
Exercises from 22 November

Revision: 22 November 2016

Exercise 6.2

There are m^n functions from a set of n elements to a set of m elements. How many of them range over exactly k different function values?

Solution.

Suppose A has n elements, B has m, and $f : A \to B$ takes exactly k values $b_1, \ldots, b_k$. Then $P = \{f^{-1}(b_i) \mid 1 \leq i \leq k\}$ is a partition of A in k *nonempty* subsets; moreover, f is completely determined by P and the b_i's.

We have $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ ways of choosing P. We have $m^{\underline{k}}$ ways of choosing the b_i's. Therefore, we have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \cdot m^{\underline{k}}$$

ways of constructing f.

Exercise 6.11

Compute $\sum_k (-1)^k \left[\begin{matrix} n \\ k \end{matrix} \right]$.

Solution. We know from the textbook that

$$\sum_k \left[\begin{matrix} n \\ k \end{matrix} \right] x^k = x^{\overline{n}}$$

For $x = -1$ we get

$$\sum_k (-1)^k \left[\begin{matrix} n \\ k \end{matrix} \right] = (-1)^{\overline{n}}$$

This is 1 for $n = 0$, -1 for $n = 1$, and 0 otherwise. A one-liner is:

$$\sum_k (-1)^k \left[\begin{matrix} n \\ k \end{matrix} \right] = [n = 0] - [n = 1]$$

Exercise 6.13

The differential operators $D = \frac{d}{dz}$ and $\vartheta = zD$ are mentioned in Chapters 2 and 5. We have

$$\vartheta^2 = z^2 D^2 + zD\,,$$

because $\vartheta^2 f(z) = \vartheta z f'(z) = z(f'(z) + z f''(z)) = z^2 f''(z) + z f'(z)$, which is $(z^2D^2 + zD)f(z)$. Similarly it can be shown that $\vartheta^3 = z^3D^3 + 3z^2D^2 + zD$. Prove the general formulas

$$\vartheta^n = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} z^k D^k \tag{1}$$

and

$$z^n D^n = \sum_k \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} \vartheta^k\,. \tag{2}$$

Solution. For (1) we can proceed by induction. Suppose $\vartheta^n = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} z^k D^k$ for a given value of $n \geq 0$: then for every function f which is differentiable at least $n+1$ times,

$$\begin{aligned}
\vartheta^{n+1} f(z) &= \vartheta(\vartheta^n f(z)) \\
&= z\frac{d}{dz}\sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} z^k f^{(k)}(z) \\
&= z\sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} (kz^{k-1} f^{(k)}(z) + z^k f^{(k+1)}(z)) \\
&= \sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} (kz^k f^{(k)}(z) + z^{k+1} f^{(k+1)}(z)) \\
&= \sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} kz^k f^{(k)}(z) + \sum_k \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\} z^k f^{(k)}(z) \\
&= \sum_k \left(k\left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\} \right) z^k f^{(k)}(z) \\
&= \sum_k \left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} z^k f^{(k)}(z) \ :
\end{aligned}$$

as f is arbitrary, the equality $\vartheta^{n+1} = \sum_k \left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} z^k D^k$ follows.

For (2) we use Stirling's inversion formula in a clever way. Let h be a function differentiable at least n times: for $n \in \mathbb{N}$ define $f(n;z) = (-1)^n z^n D^n h(z)$ and $g(n;z) = \vartheta^n h(z)$. Then (1) can be rewritten

$$g(n;z) = \sum_k \left\{ \begin{array}{c} n \\ k \end{array} \right\} (-1)^k f(k;z)\,,$$

with the usual convention that an undefined quantity multiplied by zero equals zero. By Stirling's inversion formula,

$$f(n;z) = \sum_k \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^k g(k;z)\,,$$

which in our case means

$$(-1)^n z^n D^n h(z) = \sum_k \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^k \vartheta^k h(z) \ :$$

by multiplying both sides by $(-1)^n$ and recalling that $n+k$ and $n-k$ are either both odd or both even,

$$z^n D^n h(z) = \sum_k \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^{n-k} \vartheta^k h(z) \, .$$

From the arbitrariness of $h(z)$ we deduce (2).

Exercise 6.16

What is the general solution of the double recurrence

$$\begin{array}{ll} A_{n,0} = a_n[n \geq 0] \, ; & A_{0,k} = 0 \, , \quad \text{if } k > 0 \, ; \\ A_{n,k} = kA_{n-1,k} + A_{n-1,k-1} \, , & k, n \in \mathbb{Z} \, , \end{array} \qquad (3)$$

when k and n range over the set of *all* integers?

Solution.

The double recurrence (3) is linear: if $A_{n,k} = U_{n,k}$ is the solution for $a_n = u_n$ and $A_{n,k} = W_{n,k}$ is the solution for $a_n = w_n$, then $A_{n,k} = \lambda U_{n,k} + \mu W_{n,k}$ is the solution for $a_n = \lambda u_n + \mu w_n$. We also know that $A_{n,k} = \left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the solution for $a_n = [n = 0]$.

Let us search for the solution of (3) in the more general case $a_n = [n = m]$, where m is an arbitrary integer. This seems difficult, but we observe that (3) is invariant by translations on n: as a consequence, if $A_{n,k}$ is the solution associated to the initial conditions a_n, then $A_{n-m,k}$ is the solution associated to the initial condition a_{n-m}. It follows that $A_{n,k} = \left\{ \begin{array}{c} n-m \\ k \end{array} \right\}$ is the solution for $a_n = [n = m]$.

By linearity, if $a_n \neq 0$ only for finitely many values of n, then

$$A_{n,k} = \sum_{m \geq 0} a_m \left\{ \begin{array}{c} n-m \\ k \end{array} \right\} \qquad (4)$$

is the solution to (3). Can we conclude that (4) is the solution to (3) also when infinitely many of the values a_n are nonzero? Yes, because $\left\{ \begin{array}{c} n-m \\ k \end{array} \right\}$ is zero if $m > n$ or $k > n-m$, thus only the values a_m with $0 \leq m \leq n-k$ contribute to the sum.