Exercise 6.11

Compute $\sum_k (-1)^k \binom{n}{k}$.

Solution. We know from the textbook that

$$\sum_k \binom{n}{k} x^k = x^n$$

For $x = -1$ we get

$$\sum_k (-1)^k \binom{n}{k} = (-1)^n$$

This is 1 for $n = 0$, $-1$ for $n = 1$, and 0 otherwise. A one-liner is:

$$\sum_k (-1)^k \binom{n}{k} = [n = 0] - [n = 1]$$

Exercise 6.22

Let $z$ be a complex number. Consider the sum

$$\sum_{k \geq 1} \left( \frac{1}{k} - \frac{1}{k + z} \right) \quad (1)$$

1. Prove that (1) converges for every complex number $z$ except the negative integers.
2. Observe that (1) equals $H_n$ when $z = n$ is a positive integer.

Solution. If $z$ is a negative integer, then some of the summands are undefined. Otherwise, the general term is

$$a_k = \frac{1}{k} - \frac{1}{k + z} = \frac{k + z - k}{k(k + z)} = \frac{z}{k^2 + kz}$$

By the second triangle inequality, $|a - b| \geq ||a| - |b||$ : for $a = k$, $b = -z$, and $k > |z|$ we have $|k^2 + kz| \geq |k| \cdot |k - |z|| > (k - |z|)^2$. Then $\sum_{k \geq 1} |a_k|$ converges by comparison, and (1) converges. If $z = n$ is a positive integer, then the $m$-th partial sum is

$$\sum_{k=1}^{m} \left( \frac{1}{k} - \frac{1}{k + n} \right) = \sum_{k=1}^{m} \frac{1}{k} - \sum_{j=n+1}^{n+m} \frac{1}{j}$$

$$= H_m - (H_{m+n} - H_n)$$

$$= H_n - (H_{m+n} - H_m)$$

$$= H_n - \frac{1}{m+1} - \frac{1}{m+2} - \ldots - \frac{1}{m+n}$$

that is, $H_n$ minus $n$ summands that vanish for $m \to \infty$.

Exercise on generating functions (first part)

(From the exercise sheet by Albert R. Meyer and Ronitt Rubinfeld) Let $a_n$ be the number of string on a ternary alphabet that contain a double character, i.e., a sequence $xx$ with $x$ a letter.

1. Find a recurrence for $a_n$.

2. Let $G(z)$ be the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. 
   Prove that

$$G(z) = \frac{-z}{1 - 2z} + \frac{z}{(1 - 2z)(1 - 3z)}$$

2
in two ways: Either we add any letter at the end of a good sequence, or we duplicate the last letter of a bad sequence. No string allows applying both methods, so the number of good strings of length \( n \) is three times the number of good strings of length \( n - 1 \), plus the number of bad strings of length \( n - 1 \). Thus,

\[
a_n = 3a_{n-1} + (3^{n-1} - a_{n-1}) = 2a_{n-1} + 3^{n-1}
\]

(3) for every \( n \geq 2 \).

**Point 2.** We want to rewrite (3) in terms of generating functions. Since the recurrence only holds for \( n \geq 2 \), we must consider formal power series whose constant and linear term are zero. Recall that \( 1/(1 - \alpha z) \) is the generating function of the sequence of the powers of \( \alpha \).

Let \( G(z) \) be the generating function of \( \langle a_0, a_1, a_2, a_3, \ldots \rangle \). Then \( zG(z) \) is the generating function of \( \langle 0, a_0, a_1, a_2, \ldots \rangle \), while that of \( \langle 0, 1, 3, 9, 27, \ldots \rangle \) is \( z/(1 - 3z) \). Therefore, in terms of generating functions, (3) is rewritten as

\[
G(z) - a_0 - a_1 z = 2z(G(z) - a_0) + z \left( \frac{1}{1 - 3z} - 1 \right)
\]

which, since \( a_0 = a_1 = 0 \), gives

\[
(1 - 2z)G(z) = \frac{z}{1 - 3z} - z
\]

which yields (2).