Exercises from December 2

Silvio Capobianco
Revised: December 2, 2014

Exercise on generating functions (complete)

(From the exercise sheet by Albert R. Meyer and Ronitt Rubinfeld) Let $a_n$ be the number of string on a ternary alphabet that contain a double character, i.e., a sequence $xx$ with $x$ a letter.

1. Find a recurrence for $a_n$.

2. Let $G(z)$ be the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Prove that

$$G(z) = \frac{-z}{1 - 2z} + \frac{z}{(1 - 2z)(1 - 3z)} \quad \text{(1)}$$

3. Find $r$ and $s$ such that

$$\frac{1}{(1 - 2z)(1 - 3z)} = \frac{r}{1 - 2z} + \frac{s}{1 - 3z} \quad \text{(2)}$$

4. Find a closed form for $a_n$.

Solution. **Point 1.** Call “good” a sequence with a double letter, “bad” a sequence without. Then $a_n$ is the number of good sequences of $n$ letters.

There are no good sequences of length 0 or 1, so $a_0 = a_1 = 0$. For $n \geq 2$, a good sequence of length $n$ can be obtained from a sequence of length $n - 1$ in two ways: Either we add any letter at the end of a good sequence, or we duplicate the last letter of a bad sequence. No string allows applying both
methods, so the number of good strings of length $n$ is three times the number of good strings of length $n - 1$, plus the number of bad strings of length $n - 1$. Thus,

$$a_n = 3a_{n-1} + (3^{n-1} - a_{n-1}) = 2a_{n-1} + 3^{n-1}$$  \tag{3}

for every $n \geq 2$.

**Point 2.** We want to rewrite (3) in terms of generating functions. Since the recurrence only holds for $n \geq 2$, we must consider formal power series whose constant and linear term are zero. Recall that $1/(1 - \alpha z)$ is the generating function of the sequence of the powers of $\alpha$.

Let $G(z)$ be the generating function of $\langle a_0, a_1, a_2, a_3, \ldots \rangle$. Then $zG(z)$ is the generating function of $\langle 0, a_0, a_1, a_2, \ldots \rangle$, while that of $\langle 0, 1, 3, 9, 27, \ldots \rangle$ is $z/(1 - 3z)$. Therefore, in terms of generating functions, (3) is rewritten as

$$G(z) - a_0 - a_1z = 2z(G(z) - a_0) + z \left( \frac{1}{1 - 3z} - 1 \right)$$

which, since $a_0 = a_1 = 0$, gives

$$(1 - 2z)G(z) = \frac{z}{1 - 3z} - z$$

which yields (1).

**Point 3.** Equation (2) is satisfied if and only if $r \cdot (1 - 3z) + s \cdot (1 - 2z) = 1$ whatever $z$ is. For $z = 1/2$ we find $r \cdot (1 - 3/2) = 1$, so $r = -2$. For $z = 1/3$ we find $s \cdot (1 - 2/3) = 1$, so $s = 3$. In conclusion,

$$\frac{1}{(1 - 2z)(1 - 3z)} = \frac{3}{1 - 3z} - \frac{2}{1 - 2z}.$$ 

**Point 4.** We can rewrite (1) as follows:

$$G(z) = \frac{-z(1 - 3z) + z}{(1 - 2z)(1 - 3z)} = \frac{-z + 3z^2 + z}{(1 - 2z)(1 - 3z)} = 3z^2 \cdot \left( \frac{3}{1 - 3z} - \frac{2}{1 - 2z} \right)$$

2
For \( n \geq 2 \) it must then be

\[
\begin{align*}
a_n &= [z^n]G(z) \\
&= 3 \cdot \left( [z^{n-2}] \left( \frac{3}{1-3z} \right) - [z^{n-2}] \left( \frac{2}{1-2z} \right) \right) \\
&= 9 \cdot 3^{n-2} - 6 \cdot 2^{n-2} \\
&= 3 \cdot (3^{n-1} - 2^{n-1})
\end{align*}
\]

The Rational Expansion Theorem

Let the sequence \( \langle g_0, g_1, g_2, \ldots \rangle \) have as its generating function

\[
G(z) = \frac{P(z)}{Q(z)}
\]

where \( P(z), Q(z) \) are polynomials with complex coefficients such that

1. \( \deg P < \deg Q \), and
2. \( Q(z) = (1 - \rho_1 z)^{d_1} (1 - \rho_2 z)^{d_2} \cdots (1 - \rho_\ell z)^{d_\ell} \) for suitable \( \rho_1, \ldots, \rho_\ell \in \mathbb{C} \) and positive integers \( d_1, \ldots, d_\ell \).

Then there exist polynomials \( f_1(z), \ldots, f_\ell(z) \) with complex coefficients such that:

1. For every \( k = 1, \ldots, \ell \), \( \deg f_k = d_k - 1 \).
2. For every \( k = 1, \ldots, \ell \), the leading coefficient of \( f_k \) is

\[
a_k = \frac{(-\rho_k)^{d_k} P(1/\rho_k) d_k}{Q^{(d_k)}(1/\rho_k)} = \frac{P(1/k)}{(d_k - 1)! \prod_{j \neq k} (1 - \rho_j/\rho_k)^{d_j}}.
\]

3. For every \( n \geq 0 \),

\[
g_n = \sum_{k=1}^\ell f_k(n) \rho_k^n.
\]

In particular, if \( G \) has only simple roots, then

\[
g_n = \sum_{k=1}^\ell a_k \rho_k^n
\]

for every \( n \geq 0 \), where \( a_k = -\rho_k P(1/\rho_k)/Q'(1/\rho_k) \).
Exercise on the Rational Expansion Theorem

Find an explicit formula for the $n$-th Lucas number, defined by the recurrence $L_{n+1} = L_n + L_{n-1}$ with the initial conditions $L_0 = 2$, $L_1 = 1$.

**Solution.** Let $L(z) = \sum_{n \geq 0} L_n z^n$ be the generating function of the Lucas numbers. As the recurrence $L_n = L_{n-1} + L_{n-2}$ holds for every $n > 1$, we must have

$$\sum_{n \geq 2} L_n z^n = \sum_{n \geq 2} L_{n-1} z^n + \sum_{n \geq 2} L_{n-2} z^n,$$

that is,

$$L(z) - L_0 - L_1 z = z(L(z) - L_0) + z^2 L(z),$$

that is, as $L_0 = 2$ and $L_1 = 1$,

$$L(z) - 2 - z = zL(z) - 2z + z^2 L(z),$$

which yields

$$L(z) = \frac{2 - z}{1 - z - z^2}.$$

We know that $1 - z - z^2 = (1 - \phi z)(1 - \hat{\phi} z)$. In the notation of the Rational Expansion Theorem, we have $\rho_1 = \phi$, $\rho_2 = \hat{\phi}$, $d_1 = d_2 = 1$. The derivative of $Q(z) = 1 - z - z^2$ is $Q'(z) = -1 - 2z$. We can then use the formula for distinct roots and get

$$a_1 = \frac{-\phi(2 - 1/\phi)}{-1 - 2/\phi} = \frac{2\phi - 1}{1 + 2/\phi} = \frac{\sqrt{5}}{1 + \frac{4}{1+\sqrt{5}}} = \frac{\sqrt{5}(1 + \sqrt{5})}{1 + \sqrt{5} + 4} = \frac{\sqrt{5} + 5}{5 + \sqrt{5}} = 1.$$
and

\[ a_2 = \frac{-\phi(2 - 1/\hat{\phi})}{-1 - 2/\hat{\phi}} = \frac{2\hat{\phi} - 1}{1 + 2/\hat{\phi}} = \frac{-\sqrt{5}}{1 + \frac{4}{1 - \sqrt{5}}} = \frac{-\sqrt{5}(1 - \sqrt{5})}{1 - \sqrt{5} + 4} = \frac{-\sqrt{5} + 5}{5 - \sqrt{5}} = 1. \]

Therefore, \( L_n = \phi^n + \hat{\phi}^n. \)

**Exercise on the Rational Expansion Theorem**

Solve the recurrence

\[ g_n = 6g_{n-1} - 9g_{n-2} \quad \forall n \geq 2 \]  

with the initial conditions \( g_0 = 1, \) \( g_1 = 9. \)

**Solution.** From Equation (6) follows that \( g_n z^n = 6g_{n-1}z^n - 9g_{n-2}z^n \) for every \( n \geq 2; \) thus,

\[ \sum_{n \geq 2} g_n z^n = 6 \sum_{n \geq 2} g_{n-1} z^n - 9 \sum_{n \geq 2} g_{n-2} z^n, \]

that is,

\[ G(z) - g_0 - g_1 z = 6z(G(z) - g_0) - 9z^2G(z) : \]

as \( g_0 = 1 \) and \( g_1 = 9, \) this yields

\[ G(z) - 1 - 9z = 6z(G(z) - 1) - 9z^2G(z), \]

that is,

\[ G(z) = \frac{1 + 3z}{1 - 6z + 9z^2} = \frac{1 + 3z}{(1 - 3z)^2}. \]
In the notation of the Rational Expansion Theorem, we have \( P(z) = 1 + 3z, \) 
\( Q(z) = (1 - 3z)^2, \) \( \rho_1 = 3, \) \( d_1 = 2. \) Therefore, 
\[ g_n = (a_1 n + c_1) \cdot 3^n \text{ where } a_1 = \frac{(1 + 3/3)}{(2 - 1)!} \cdot 1 = 2. \]

For \( n = 0 \) we find \( 1 = g_0 = (0 + c_1) \cdot 1, \) yielding \( c_1 = 1. \) Therefore,
\[ g_n = (2n + 1) \cdot 3^n. \]

Exercise 7.3
What is \( \sum_{n \geq 0} H_n/10^n? \)
Solution. The fact that the sum depends on the powers of \( 1/10, \) should lead us to the search for a solution in terms of a generating function. But the generating function of harmonic numbers is
\[ H(z) = \sum_{n \geq 0} H_n z^n = \frac{1}{1 - z} \log \frac{1}{1 - z}; \]
our idea will work because the series \( H(z) \) has convergence radius 1, which is greater than \( 1/10. \) In fact, for every \( n \geq 1 \) we have \( 1 \leq H_n \leq n, \) so that \( \limsup_{n \geq 0} \sqrt[n]{|H_n|} = 1 \) by comparison. Therefore, the required sum is
\[ H \left( \frac{1}{10} \right) = \frac{1}{1 - \frac{1}{10}} \log \frac{1}{1 - \frac{1}{10}} = \frac{10}{9} \log \frac{10}{9}. \]

Exercise 7.11 (first part)
Let \( a_n = b_n = c_n = 0 \) for \( n < 0, \) and
\[ A(z) = \sum_n a_n z^n ; \quad B(z) = \sum_n b_n z^n ; \quad C(z) = \sum_n c_n z^n. \]

1. Express \( C \) in terms of \( A \) and \( B \) when \( c_n = \sum_{j+2k \leq n} a_j b_k. \)

Solution.
Point 1. We know that, if \( a_n = [z^n]G(z), \) then \( \sum_{k \leq n} a_k = [z^n]G(z)/(1 - z). \) Then we can solve point 1 as soon as we find \( G(z) \) such that \( [z^n]G(z) = \sum_{j+2k=n} a_j b_k. \) But the latter is the coefficient of index \( n \) of the convolution
of $A$ with a power series whose odd-indexed coefficients are 0, and whose coefficient of index $2k$ is $b_k$: such function is precisely $B(z^2)$. Therefore,

$$C(z) = \frac{A(z)B(z^2)}{1 - z}$$