ITT9131 Concrete Mathematics
Exercises from 13 December

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Exercise RET1

Find an explicit formula for the n-th *Lucas number*, defined by the recurrence $L_n = L_{n-1} + L_{n-2}$ for every $n \geq 2$ with the initial conditions $L_0 = 2$, $L_1 = 1$.
Solution. Let $L(z) = \sum_{n \geq 0} L_n z^n$ be the generating function of the Lucas numbers, with the convention that $L_n = 0$ if $n < 0$. The recurrence $L_n = L_{n-1} + L_{n-2}$ holds for every $n < 0$ and $n \geq 2$; for $n = 0$, we must have $2 = L_0 = L_{-1} + L_{-2} + 2$; for $n = 1$ we must have $1 = L_1 = L_0 + L_{-1} - 1$. Then,

$$\sum_n L_n z^n = \sum_n L_{n-1} z^n + \sum_n L_{n-2} z^n + 2 \sum_n [n=0]\, z^n - \sum_n [n=1]\, z^n \,,$$

that is,

$$L(z) = zL(z) + z^2 L(z) + 2 - z \ :$$

which yields

$$L(z) = \frac{2-z}{1-z-z^2} \,.$$

We know that $1 - z - z^2 = (1 - \phi z)(1 - \hat{\phi} z)$. In the notation of the Rational Expansion Theorem, we have $\rho_1 = \phi$, $\rho_2 = \hat{\phi}$, $d_1 = d_2 = 1$. The derivative of $Q(z) = 1 - z - z^2$ is $Q'(z) = -1 - 2z$. We can then use the formula for distinct roots and get

$$\begin{aligned}
a_1 &= \frac{-\phi(2-1/\phi)}{-1-2/\phi} \\
&= \frac{2\phi-1}{1+2/\phi} \\
&= \frac{\sqrt{5}}{1+\frac{4}{1+\sqrt{5}}} \\
&= \frac{\sqrt{5}(1+\sqrt{5})}{1+\sqrt{5}+4} \\
&= \frac{\sqrt{5}+5}{5+\sqrt{5}} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
a_2 &= \frac{-\hat{\phi}(2-1/\hat{\phi})}{-1-2/\hat{\phi}} \\
&= \frac{2\hat{\phi}-1}{1+2/\hat{\phi}} \\
&= \frac{-\sqrt{5}}{1+\frac{4}{1-\sqrt{5}}} \\
&= \frac{-\sqrt{5}(1-\sqrt{5})}{1-\sqrt{5}+4} \\
&= \frac{-\sqrt{5}+5}{5-\sqrt{5}} \\
&= 1\,.
\end{aligned}$$

Therefore, $L_n = \phi^n + \hat{\phi}^n$.

Exercise RET2

Solve the recurrence

$$g_n = 6g_{n-1} - 9g_{n-2} \;\; \forall n \geq 2 \tag{1}$$

with the initial conditions $g_0 = 1$, $g_1 = 9$.
Solution. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$, with the convention that $g_n = 0$ if $n < 0$. The recurrence $g_n z^n = 6g_{n-1}z^n - 9g_{n-2}z^n$ holds for every $n < 0$ and $n \geq 2$; for $n = 0$ we must have $1 = g_0 = 6g_{-1} - 9g_{n-2} + 1$; for $n = 1$ we must have $9 = g_1 = 6g_0 - 9g_{-1} + 3$. Then,

$$\sum_n g_n z^n = 6\sum_n g_{n-1}z^n - 9\sum_n g_{n-2}z^n + \sum_n [n=0]\, z^n + 3\sum_n [n=1]\, z^n\,,$$

that is,

$$G(z) = 6zG(z) - 9z^2G(z) + 1 + 3z\ :$$

which yields

$$G(z) = \frac{1+3z}{1-6z+9z^2} = \frac{1+3z}{(1-3z)^2}\,.$$

In the notation of the Rational Expansion Theorem, we have $P(z) = 1 + 3z$, $Q(z) = (1-3z)^2$, $\rho_1 = 3$, $d_1 = 2$. Therefore, $Q'(z) = -6 + 18z$, $Q''(z) = 18$, and $g_n = (a_1 n + c_1) \cdot 3^n$, where

$$a_1 = \frac{(-3)^2 \cdot (1 + 3/3) \cdot 2}{18} = 2\,.$$

For $n = 0$ we find $1 = g_0 = (0 + c_1) \cdot 1$, yielding $c_1 = 1$. Therefore,

$$g_n = (2n+1) \cdot 3^n\,.$$

Exercise RET3

Solve the recurrence

$$g_n = 3g_{n-1} - 4g_{n-3}\ \ \forall n \geq 3 \tag{2}$$

with the initial conditions $g_0 = 0$, $g_1 = 1$, $g_2 = 3$.
Solution. Observe that (2) is a recurrence of the *third* order, since g_n depends on g_{n-1} and g_{n-3}: therefore, we need *three* initial conditions.

Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$, with the convention that $g_n = 0$ if $n < 0$. The recurrence $g_n = 3g_{n-1} - 4g_{n-3}$ holds for every $n < 0$ and $n \geq 3$; for $n = 0$ we have $0 = g_0 = 3g_{-1} - 4g_{-3}$; for $n = 1$ we have $1 = g_1 = 3g_0 - 4g_{-2} + 1$; for $n = 2$ we have $3 = g_2 = 3g_1 - 4g_{-1}$. Then,

$$\sum_n g_n z^n = 3\sum_n g_{n-1}z^n - 4\sum_n g_{n-3}z^n + \sum_n [n=1]\, z^n\,,$$

that is,

$$G(z) = 3zG(z) - 4z^3G(z) + z \ :$$

which yields

$$G(z) = \frac{z}{1 - 3z + 4z^3} \,.$$

We observe that $Q(1/2) = Q(-1) = 0$: and in fact, if we divide $Q(z)$ by $1 + z$, we get $1 - 4z + 4z^2 = (1 - 2z)^2$. Therefore,

$$G(z) = \frac{z}{(1 + z)(1 - 2z)^2} \,.$$

In the notation of the Rational Expansion Theorem, we have $\rho_1 = -1$, $d_1 = 1$, $\rho_2 = 2$, $d_2 = 2$; also, $P(z) = z$ and

$$Q(z) = 1 - 3z + 4z^3$$

, from which $Q'(z) = -3 + 12z^2$ and $Q''(z) = 24z$. Then $g_n = a_1 \cdot (-1)^n + (a_2 n + c_2) \cdot 2^n$ for suitable a_1, a_2, c_2, where

$$a_1 = \frac{1^1 \cdot (-1)}{-3 + 12} = -\frac{1}{9}$$

and

$$a_2 = \frac{(-2)^2 \cdot (1/2) \cdot 2}{24 \cdot 1/2} = \frac{1}{3} \,.$$

For $n = 0$ we find $0 = -\frac{(-1)^0}{9} + (0 + c_2) \cdot 1$, yielding $c_2 = 1/9$. Therefore,

$$g_n = \frac{(-1)^{n+1}}{9} + \left(\frac{n}{3} + \frac{1}{9}\right) 2^n = \frac{(-1)^{n+1}}{9} + \frac{3n + 1}{9} \cdot 2^n \,.$$

Exercise 7.11

Let $a_n = b_n = c_n = 0$ for $n < 0$, and

$$A(z) = \sum_n a_n z^n \ ; \ B(z) = \sum_n b_n z^n \ ; \ C(z) = \sum_n c_n z^n$$

2. Express A in terms of B when $nb_n = \sum_{k=0}^{n} 2^k a_k/(n-k)!$

Solution. We know that $nb_n = [z^{n-1}]B'(z) = [z^n]zB'(z)$, that is, $\sum_n nb_n z^n = zB'(z)$. Moreover, nb_n must be the coefficient of index n of the convolution of $A(2z)$ (because of the 2^k factor) with a power series whose coefficient of index n is $1/n!$: such function is e^z. This means

$$zB'(z) = e^z A(2z)$$

and consequently

$$A(z) = \frac{z}{2} e^{-z/2} B'\left(\frac{z}{2}\right) .$$