ITT9131 Concrete Mathematics
Exercises from 20 December

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Exercise RET4

Solve the recurrence

$$g_n = 3g_{n-2} - 2g_{n-3} \ \ \forall n \geq 3 \tag{1}$$

with the initial conditions $g_0 = 0$, $g_1 = 1$, $g_2 = 3$.
Solution. As (1) is a third-order relation, we need three initial conditions. We apply our four-step technique:

1. We want the relation (1) to hold for every integer n, up to some correction summand, with the usual convention that $g_n = 0$ if $n < 0$. For $n < 0$ and $n \geq 3$ we have no problem: but we must check the cases $n = 0$, $n = 1$, $n = 2$.

   $n = 0$. The recurrence gives $g_0 = 3g_{-2} - 2g_{-3} = 0$: as $g_0 = 0$, no correction is needed.

   $n = 1$. The recurrence gives $g_1 = 3g_{-1} - 2g_{-2} = 0$: as $g_1 = 1$, we need a correction summand $[n = 1]$.

   $n = 2$. The recurrence gives $g_2 = 3g_0 - 2g_{-1} = 0$: as $g_1 = 1$, we need a correction summand $3\,[n = 2]$.

2. Multiplying both sides of the nth equation by z^n and summing over all integers n, we find:

$$\sum_n g_n z^n = 3 \sum_n g_{n-2} z^n - 2 \sum_n g_{n-3} z^n + \sum_n [n = 1]\, z^n + 3 \sum_n [n = 2]\, z^n \,.$$

Let then $G(z) = \sum_{n\geq 0} g_n z^n$: the above becomes

$$G(z) = 3z^2 G(z) - 2z^3 G(z) + z + 3z^2 .$$

3. We easily solve the above with respect to $G(z)$ and obtain:

$$G(z) = \frac{z + 3z^2}{1 - 3z^2 + 2z^3} . \tag{2}$$

4. Let $P(z) = z + 3z^2$ and $Q(z) = 1 - 3z^2 + 2z^3$ then $G(z) = P(z)/Q(z)$ with $\deg P < \deg Q$, and we can use the Rational Expansion Theorem.

   To find the roots of $Q(z)$, we observe that $Q(1) = 0$, therefore $Q(z) = (1-z)(a + bz + cz^2)$ for suitable a, b, and c: comparing the coefficients yields $a = 1$, $b = a$, and $c = -2$. In turn, $1 + z - 2z^2$ also vanishes for $z = 1$, so it has the form $(1-z)(r + sz)$: again, comparing the coefficients yields $r = 1$ and $s = 2$. We then have:

$$Q(z) = (1-z)^2(1+2z) . \tag{3}$$

   To apply the Rational Expansion Theorem we put $\rho_1 = 1$, $d_1 = 2$, $\rho_2 = -2$, and $d_2 = 1$: then

$$g_n = (a_1 n + b_1)1^n + a_2(-2)^n ,$$

   where:

   - $a_1 = 1/\rho_1 = \dfrac{(-1)^2 \cdot P(1) \cdot 2}{Q''(1)}$ because $\alpha_1 = 1$ is a double root;
   - $a_1 = 1/\rho_2 = \dfrac{2 \cdot P(-1/2)}{Q'(-1/2)}$ because $\alpha_2 = 1/\rho_2 = -1/2$ is a simple root.

   As $Q'(z) = -6z + 6z^2$ and $Q''(z) = -6 + 12z$ we find:

$$a_1 = \frac{1 \cdot (1 + 3 \cdot 1^2) \cdot 2}{-6 + 12} = \frac{4}{3}$$

   and

$$a_2 = \frac{2 \cdot (-1/2 + 3 \cdot 1/4)}{6 \cdot 1/2 + 6 \cdot 1/4} = \frac{1}{9}$$

To find b_1, we put $n = 0$ and apply the initial condition: we get

$$\left(\frac{4}{3} \cdot 0 + b_1\right) \cdot 1^0 + \frac{1}{9} \cdot (-2)^0 = 0 \, ,$$

which yields $b_1 = -1/9$.

In conclusion,

$$g_n = \frac{4}{3} n - \frac{1}{9} + \frac{(-1)^n}{9} \cdot 2^n \, .$$

Exercise 7.11

Let $a_n = b_n = c_n = 0$ for $n < 0$, and

$$A(z) = \sum_n a_n z^n \ \ ; \ \ B(z) = \sum_n b_n z^n \ \ ; \ \ C(z) = \sum_n c_n z^n$$

3. Express A in terms of B when $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$. Then, construct $\{f_n(r)\}_{n \geq 0}$ such that $b_n = \sum_{k=0}^{n} f_k(r) a_{n-k}$.

Solution. A must be the convolution of B with a power series whose coefficient of index n is $\binom{r+n}{n}$. The tables in Section 7.2 provide the formula $\sum_{n \geq 0} \binom{c+n-1}{n} z^n = 1/(1-z)^c$: therefore, such function is $1/(1-z)^{r+1}$. This means

$$A(z) = \frac{B(z)}{(1-z)^{r+1}}$$

But then, $B(z) = (1-z)^{r+1} A(z)$: by the generalized binomial theorem (also displayed in the tables) $(1-z)^{r+1} = \sum_{n \geq 0} \binom{r+1}{n} (-z)^n$. Therefore,

$$f_n(r) = [z^n](1-z)^{r+1} = (-1)^n [z^n](1+z)^{r+1} = (-1)^n \binom{r+1}{n}$$

Exercise 7.35

Evaluate the sum $\sum_{0<k<n} 1/k(n-k)$ in two ways:

1. Expand the summand in partial fractions.

2. Treat the sum as a convolution and use generating functions.

Solution. Expanding $1/k(n-k)$ in partial fractions means finding constants A and B such that

$$\frac{1}{k(n-k)} = \frac{A}{k} + \frac{B}{n-k} \ :$$

from $\frac{1}{k} + \frac{1}{n-k} = \frac{n}{k(n-k)}$ we easily get $A = B = \frac{1}{n}$. Then

$$\sum_{0<k<n} \frac{1}{k(n-k)} = \frac{1}{n} \sum_{0<k<n} \left(\frac{1}{k} + \frac{1}{n-k} \right) = \frac{2}{n} H_{n-1} \, .$$

We can also observe that $g_n = \sum_{0<k<n} \frac{1}{k(n-k)}$ is the term of index n of the convolution of the sequence of generic term $h_n = \frac{1}{n}[n > 0]$ with itself. Let $G(z)$ and $H(z)$ be the generating functions of the sequences $\langle g_n \rangle$ and $\langle h_n \rangle$, respectively: we know that $H(z) = \ln \frac{1}{1-z}$, so

$$G(z) = H(z)^2 = \left(\ln \frac{1}{1-z} \right)^2 \, . \tag{4}$$

This looks hard to manage until we remember that, if $G(z) = \sum_n g_n z^n$, then $zG'(z) = \sum_n n g_n z^n$. Said, done:

$$\begin{aligned} zG'(z) &= z \frac{d}{dz} \left(\ln \frac{1}{1-z} \right)^2 \\ &= z \cdot \left(2 \ln \frac{1}{1-z} \right) \cdot \frac{1}{\frac{1}{1-z}} \cdot \frac{1}{(1-z)^2} \\ &= 2z \cdot \left(\frac{1}{1-z} \ln \frac{1}{1-z} \right) . \end{aligned}$$

The function in parentheses on the last line is the generating function of the harmonic numbers[1]: by pre-multiplying by z, H_n becomes the coefficient of z^{n+1} instead of z^n. Equating the power series,

$$\sum_n n g_n z^n = 2 \sum_n H_n z^{n+1} = 2 \sum_n H_{n-1} z^n \ :$$

then $n g_n = 2H_{n-1}$ for every n, which is equivalent to what we had found before.

Lesson learned: if you need to kill a mosquito, don't use a cannon!

[1]More in general, if $G(z)$ is the generating function of $\langle g_n \rangle$, then $\frac{G(z)}{1-z}$ is the generating function of $\left\langle \sum_{0 \leq k \leq n} g_k \right\rangle$. Recall the convention that undefined by zero is zero.