ITT9131 Concrete Mathematics
Solutions to final exam of 17 January 2017

Revision: 18 January 2017

Exercise 1

(12 points) Solve the recurrence:

$$\begin{aligned} g_0 &= 1\,; & g_1 &= 3\,; \\ g_n &= 4g_{n-1} - 4g_{n-2} & \forall n &\geq 2\,. \end{aligned} \tag{1}$$

Solution:

The recurrence (1) is easily solved with generating functions via the Rational Expansion Theorem. Let us follow the method step by step:

1. We must rewrite (1) so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$. We need to check the initial conditions:

   - For $n = 0$ it is $g_0 = 1$ but $4g_{-1} - 4g_{-2} = 0$: we thus need a correction summand 1.
   - For $n = 1$ it is $g_1 = 3$ but $4g_0 - 4g_{-1} = 4$: we thus need a correction summand -1.

   The recurrence (1) for arbitrary $n \in \mathbb{Z}$ is thus:

   $$g_n = 4g_{n-1} - 4g_{n-2} + [n = 0] - [n = 1]\ .$$

2. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. By multiplying the recurrence by z^n for every $n \in \mathbb{Z}$ and summing over n we

obtain:

$$\begin{aligned} G(z) &= \sum_n g_n z^n \\ &= 4\sum_n g_{n-1} z_n - 4\sum_n g_{n-2} z^n + \sum_n [n=0]\, z^n - \sum_n [n=1]\, z^n \\ &= 4\sum_n g_n z^{n+1} - 4\sum_n g_n z^{n+2} + 1 - z \\ &= 4zG(z) - 4z^2 G(z) + 1 - z\,. \end{aligned}$$

3. By solving the above with respect to $G(z)$ we get

$$G(z) \cdot \left(1 - 4z + 4z^2\right) = 1 - z\,,$$

which yields

$$G(z) = \frac{1-z}{1-4z+4z^2}\,.$$

4. The function $G(z)$ has the form $G(z) = P(z)/Q(z)$ where $P(z) = 1-z$ and $Q(z) = 1-4z+4z^2 = (1-2z)^2$. Then the solution of the recurrence is $(an+b)\cdot 2^n$ for suitable a and b. To find such numbers, we use the Rational Expansion Theorem: in our case, $\rho = 2$ and $d = 2$, so:

$$a = \frac{(-2)^2 \cdot P(1/2) \cdot 2}{Q''(1/2)} = \frac{4 \cdot (1/2) \cdot 2}{8} = \frac{1}{2}\,.$$

To find b, we compare the initial condition $g_0 = 1$ with the value $(a \cdot 0 + b) \cdot 2^0$: which yields $b = 1$. In conclusion,

$$g_n = \left(\frac{n}{2} + 1\right) \cdot 2^n\,.$$

Exercise 2

(10 points) For $n, r, s \geq 0$ all integers compute

$$S_n = \sum_{k=0}^{n} \binom{k}{r} \binom{n-k}{s}\,.$$

Solution: The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle \binom{n}{r} \right\rangle$ and

$\left\langle \binom{n}{s} \right\rangle$. We know that $\sum_{n\geq 0} \binom{n}{r} = \frac{z^r}{(1-z)^{r+1}}$ and $\sum_{n\geq 0} \binom{n}{s} = \frac{z^s}{(1-z)^{s+1}}$: then the generating function of $\langle S_n \rangle$ is

$$S(z) = \frac{z^{r+s}}{(1-z)^{r+s+2}} \,.$$

This writing is annoying, because the right-hand side does not have the convenient form $\frac{z^m}{(1-z)^{m+1}}$: which it would have if the exponent at the numerator was $r+s+1$ instead of $r+s$. But as $r+s \geq 0$, the constant coefficient of $\frac{z^{r+s+1}}{(1-z)^{r+s+2}} = \sum_{n\geq 0} \binom{n}{r+s+1} z^n$ is $\binom{0}{r+s+1} = 0$: by applying the formula $\frac{G(z)-g_0}{z} = \sum_{n\geq 0} g_{n+1} z^n$, we get

$$S(z) = \frac{1}{z} \cdot \left(\frac{z^{r+s+1}}{(1-z)^{r+s+2}} - 0 \right) = \sum_{n\geq 0} \binom{n+1}{r+s+1} z^n \,.$$

By comparison, we finally find:

$$\sum_{k=0}^{n} \binom{k}{r} \binom{n-k}{s} = \binom{n+1}{r+s+1} \,.$$

Exercise 3

(8 points) Determine the values of $n \geq 0$ such that $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 250.
Solution: As $250 = 2 \cdot 5^3$ as a product of powers of primes, we must show that $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by both 2 and 125. One part is easy: there are four summands, which are either all even or all odd, so the sum is even. For the other part, we factor the polynomial and obtain:

$$n^{14} - 3n^{10} + 3n^6 - n^2 = n^2 \cdot (n^{12} - 3n^8 + 3n^4 - 1) = n^2 \cdot (n^4 - 1)^3 \,.$$

If n is not a multiple of 5, then $n^4 - 1$ is by Fermat's little theorem, and as there are three such factors, $n^{14} - 3n^{10} + 3n^6 - n^2$ is indeed divisible by 125. If n is a multiple of 5, however, then $n^4 - 1$ is not, and the contributions to divisibility by 125 must come all from n: as there are two factors n in $n^{14} - 3n^{10} + 3n^6 - n^2$, if n is divisible by 5 but not by 25, then $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 525 but not by 125; while if n is divisible by 25, then $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 625, thus also by 125.

In conclusion, $n^{14} - 3n^{10} + 3n^6 - n^2$ is divisible by 250 if and only if n is either divisible by 25, or not divisible by 5.

Questions

1. If 100 people are put in circle and every second person is eliminated, which one will be left in the end?
   $100 = 64 + 36$, *and* $2 \cdot 36 + 1 = 73$*: hence, the seventy-third person will be left in the end.*

2. Explain the main idea of the method of the summation factor.
   If the recurrence equation has the form $a_n T_n = b_n T_{n-1} + c_n$*, and we find nonzero* $\langle s_n \rangle$ *such that* $s_n b_n = s_{n-1} a_{n-1}$ *for every* $n \geq 1$*, then by putting* $U_n = s_n a_n T_n$ *for every* $n \geq 0$ *we can rewrite* $U_n = U_{n-1} + s_n c_n$*, which is easy to solve.*

3. Write the formula of integration by parts of discrete calculus.
   $u \, \Delta v = \Delta(uv) - Ev \, \Delta u$*, where* E *is the shift operator:* $Ev(x) = v(x+1)$.

4. Write the definition of the sum of a sequence of real numbers, of which at most finitely many are negative.
   Any of the following answers is acceptable:

   - $\sum_{k \geq 0} a_k = \sum_{a_k \geq 0} a_k + \sum_{a_k < 0} a_k$.
   - $\sum_{k \geq 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k$.
   - $\sum_{k \geq 0} a_k = \sum_{k \geq 0} a_k^+ - \sum_{k \geq 0} a_k^-$*, where* $a_k^+ = \max(a_k, 0)$ *and* $a_k^- = \max(-a_k, 0)$.

5. How many integers $1 \leq k \leq n$ are in the union of the spectra of $\sqrt{3}$ and $(3 + \sqrt{3})/2$?
   n*. As* $1/\sqrt{3} + 2/(3 + \sqrt{3}) = 1$ *and the two numbers are irrational, the spectra of* $\sqrt{3}$ *and* $(3 + \sqrt{3})/2$ *form a partition of the positive integers.*

6. State Bézout's theorem.
   The greatest common divisor of two positive integers m *and* n *is the smallest positive integer which can be written as a linear combination of* m *and* n *with integer coefficients.*

7. Is $105^{72} - 1$ divisible by 37?
   Yes, because $105^{72} - 1 = (105^{36} - 1) \cdot (105^{36} + 1)$*, and the first factor is divisible by 37 by Fermat's little theorem.*

8. State Euler's theorem.
If a and m are positive integer and $\gcd(a,m) = 1$*, then* $a^{\phi(m)} \equiv 1 \pmod{m}$*, where ϕ is Euler's totient function.*

9. Write the Vandermonde convolution.

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}.$$

10. Write the recurrence relation for the Stirling numbers of the second kind.

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}.$$

11. How many ways are there of arranging 6 objects into 2 nonempty cycles?

$$\left[{6 \atop 2} \right] = 5!H_5 = 120 + 60 + 40 + 30 + 24 = 274\,.$$

12. Let $m \geq 0$. What is $\sum_n \left[{m \atop n} \right]$?
$m!$. This can be seen by either using the formulas $\sum_n \left[{m \atop n} \right] z^n = z^{\overline{m}}$ and $1^{\overline{m}} = m!$, or by observing that there is a bijection between the arrangements of m objects into nonempty cycles and the permutations of m objects.

13. Write the generalized Cassini's identity.
For every k and n integer, $f_{n+k} = f_k f_{n+1} + f_{k-1} f_n$.

14. Can an analytic function be the generating function of two different sequences?
No, because of the identity principle for analytic functions.

15. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\langle g_{2n} \rangle$?
$H(z)$, where $H(z^2) = \frac{G(z)+G(-z)}{2}$.

16. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\left\langle \sum_{i+j+k=n} g_i g_j g_k \right\rangle$?
$(G(z))^3$. *This is the convolution of three copies of the sequence* $\langle g_n \rangle$.

17. What is the generating function of the sequence $\langle 2^n + n \rangle$?
$\dfrac{1}{1-2z} + \dfrac{z}{(1-z)^2}$, *because the generating function of the sum is the sum of the generating functions.*

18. What is the generating function of the sequence of harmonic numbers?
$\dfrac{1}{1-z} \ln \dfrac{1}{1-z}$.

19. How many complete binary trees with 8 leaves exist?
The number of such trees is the Catalan number of index 7:

$$C_7 = \frac{1}{8}\binom{14}{7} = \frac{1}{8} \cdot \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 429\,.$$

20. What is the binomial convolution of two sequences?
The binomial convolution of $\langle f_n \rangle$ *and* $\langle g_n \rangle$ *is the sequence* $\left\langle \sum_k \binom{n}{k} f_k g_{n-k} \right\rangle$.