ITT9131 – Concrete Mathematics
Midterm exam – Recovery
Solutions

23 November 2016

Exercise 1

(10 points) Solve the recurrence:

$$\begin{aligned} T_0 &= 1\,; \\ nT_n &= 2T_{n-1} + \frac{2^n}{n!}\left(1 + \frac{n}{3^n}\right) \quad \forall n \geq 1\,. \end{aligned} \tag{1}$$

Solution: The form of the recurrence equation (1) suggests to make a sostitution such that the factors n on the left-hand side and 2 on the right-hand side disappear. Intuition would suggest to put

$$T_n = \frac{2^n U_n}{n!} \;: \tag{2}$$

with this substitution and some manipulations, (1) becomes

$$\begin{aligned} U_0 &= 1\,; \\ \frac{2^n}{(n-1)!}U_n &= \frac{2^n}{(n-1)!}U_{n-1} + \frac{2^n}{(n-1)!}\left(\frac{1}{n} + \frac{1}{3^n}\right) \quad \forall n \geq 1\,, \end{aligned}$$

which clearly has the solution

$$\begin{aligned} U_n &= 1 + H_n + \sum_{k=1}^{n} \frac{1}{3^n} \\ &= H_n + \sum_{k=0}^{n} \frac{1}{3^n} \\ &= H_n + \frac{1 - \frac{1}{3^{n+1}}}{1 - \frac{1}{3}} \\ &= H_n + \frac{1}{2} \left(3 - \frac{1}{3^n} \right) . \end{aligned}$$

If we want to try a summation factor, we have to be careful to the fact that a_n is n for $n > 0$, but 1 for $n = 0$. Then

$$s_n = \prod_{j=1}^{n} \frac{a_{j-1}}{b_j} = \frac{(n-1)!}{2^n}$$

for $n \geq 1$, and $s_0 = 1$ as usual for the method: for $n \geq 1$ we then have

$$U_n = s_n a_n T_n = \frac{(n-1)!}{2^n} \cdot n = \frac{n!}{2^n} T_n \, ,$$

which matches our previous intuition. In the end,

$$T_n = \frac{2^n}{n!} H_n + \frac{2^{n-1}}{n!} \left(3 - \frac{1}{3^n} \right) . \tag{3}$$

Exercise 2

(8 points) Express $\sum_{1 \leq k \leq n} k \cdot 2^{-k}$ as a function of n, and evaluate $\sum_{k \geq 1} k \cdot 2^{-k}$.

Solution: We can compute $\sum_{1 \leq k \leq n} k \cdot 2^{-k}$ in two different ways:

- Perturbation method:

Let $S_n = \sum_{1 \le k \le n} k \cdot 2^{-k}$: then

$$\begin{aligned} S_n + (n+1) \cdot 2^{-n-1} &= \frac{1}{2} + \sum_{k=2}^{n+1} k \cdot 2^{-k} \\ &= \frac{1}{2} + \sum_{k=1}^{n} (k+1) \cdot 2^{-k-1} \\ &= \frac{1}{2} + \frac{1}{2} \left(\sum_{k=1}^{n} k \cdot 2^{-k} + \sum_{k=1}^{n} 2^{-k} \right) , \end{aligned}$$

so that by multiplying both sides by 2 we get

$$2S_n + (n+1) \cdot 2^{-n} = 1 + S_n + \sum_{k=1}^{n} 2^{-k} . \qquad (4)$$

As the last summand on the right-hand side of (4) is $1 - 2^{-n}$, we get

$$S_n = 2 - (n+2) \cdot 2^{-n} .$$

- Discrete calculus:

  We look at $k \cdot 2^{-k}$ as an object of the form $u \Delta v$, where $u(x) = x$ (so that $\Delta u(x) = 1$) and $\Delta v(x) = 2^{-x}$. Recall that $\Delta c^x = (c-1)c^x$ for $c > 0$: which means that

  $$\Delta 2^{-x} = \Delta \left(\frac{1}{2} \right)^x = \left(\frac{1}{2} - 1 \right) \left(\frac{1}{2} \right)^x = -\frac{1}{2} \cdot 2^{-x} .$$

  To have $\Delta v(x) = 2^{-x}$ we must then set $v(x) = -2 \cdot 2^{-x}$. If we make the additional observation that $\sum_{1 \le k \le n} k \cdot 2^{-k} = \sum_{0 \le k \le n} k \cdot 2^{-k}$, we can compute:

$$\begin{aligned}
\sum_{1\le k\le n} k\cdot 2^{-k} &= \sum_0^{n+1} x\cdot\left(\frac{1}{2}\right)^x \delta x \\
&= -2x\cdot 2^{-x}\Big|_0^{n+1} - \sum_0^{n+1}(-2)\left(\frac{1}{2}\right)^{x+1}\delta x \\
&= -(n+1)\cdot 2^{-n} + \sum_0^{n+1}\left(\frac{1}{2}\right)^{x+1}\delta x \\
&= -(n+1)\cdot 2^{-n} + \sum_{k=0}^{n} 2^{-k} \\
&= -(n+1)\cdot 2^{-n} + \left(1+\sum_{k=1}^{n} 2^{-k}\right) \\
&= -(n+1)\cdot 2^{-n} + 1 + 1 - 2^{-n} \\
&= 2-(n+2)\cdot 2^{-n}\,,
\end{aligned}$$

which is the same result we had found by the perturbation method. Then $\sum_{k\ge 1} k\cdot 2^k = \lim_{n\to\infty}\sum_{1\le k\le n} k\cdot 2^k = 2$.

Exercise 3

(4 points) Prove that $\left\lceil x-\frac{1}{2}\right\rceil \le \left\lfloor x+\frac{1}{2}\right\rfloor$ for every $x\in\mathbb{R}$, and give a closed formula for the difference.

Solution: The closed interval $\left[x-\frac{1}{2}, x+\frac{1}{2}\right]$ contains two integers if $\{x\} = x-\lfloor x\rfloor = \frac{1}{2}$, otherwise it contains a single integer. In this second case, such single integer must be the common value of $\left\lceil x-\frac{1}{2}\right\rceil$ and $\left\lfloor x+\frac{1}{2}\right\rfloor$; otherwise, $x-\frac{1}{2}$ and $x+\frac{1}{2}$ are both integer, so they coincide with both their floors and their ceilings, and the former is smaller than the latter. Then

$$\left\lfloor x+\frac{1}{2}\right\rfloor - \left\lceil x-\frac{1}{2}\right\rceil = \left[x-\lfloor x\rfloor = \frac{1}{2}\right]\,.$$

Exercise 4

(8 points) Prove that $n^{13}-n$ is divisible by 105 for every positive integer n.

Solution: As $105 = 3 \cdot 5 \cdot 7$ as a product of (powers of) primes, $n^{13} - n$ is divisible by 105 if and only if it is divisible by 3, 5, and 7. Write $n^{13} - n = n \cdot (n^{12} - 1)$: to apply Fermat's little theorem with prime p, we must collect a factor $n^p - n$ from $n^{13} - n$, or equivalently, a factor $n^{p-1} - 1$ from $n^{12} - 1$. For $p = 3$ we must show that $n^{12} - 1$ is divisible by $n^2 - 1$: but this is true, because

$$n^{12} - 1 = (n^2)^6 - 1 = (n^2 - 1)(n^{10} + n^8 + n^6 + n^4 + n^2 + 1)\,.$$

Similarly, for $p = 5$ we must show that $n^{12} - 1$ is divisible by $n^4 - 1$: which is the case, because

$$n^{12} - 1 = (n^4)^3 - 1 = (n^4 - 1)(n^8 + n^4 + 1)\,.$$

Finally, for $p = 7$ we must show that $n^{12} - 1$ is divisible by $n^6 - 1$: which is true, because $n^{12} - 1 = (n^6 - 1)(n^6 + 1)$.