ITT9131 Concrete Mathematics, Fall 2016
Program of the course

Revision: 29 December 2016

1 Recurrent Problems

Hanoi tower. Lines in the plane. Josephus' problem for every second person. Structural induction. The repertoire method. The Josephus problem in arbitrary base.

2 Sums

Sequences. Notation for sums. Sums and recurrences. Methods for solving recurrences: repertoire, perturbation, reduction to known solutions. Summation factors. Efficiency of quicksort. Harmonic numbers. Manipulation of sums.

Multiple sums. Sums with independent indices. Sums with dependent indices. Mutual upper bounds. General methods: looking up, guessing the answer, perturbation, repertoire, integrals, expansion and contraction.

Finite and infinite calculus. The difference operator. Elementary functions in discrete calculus and their differences: falling factorials, exponentials, harmonic numbers. Indefinite and definite summation. Summation by parts.

Infinite sums. Paradoxes of infinite summation. Definition for sequences of nonnegative terms. General definition.

3 Integer Functions

Floor and ceiling. Properties. The generalized Dirichlet box principle. Fractional part. Applications: base-b expansion, continuous functions with floor and ceiling, number of integer elements in an interval of the real line. Spectra.

Recurrences with floors and ceilings. The Knuth numbers. Solution of the generalized Josephus problem. The modulo as a binary operation. Sums with floors and ceilings.

4 Number Theory

Division with remainder. Positive and negative divisors. Divisibility. Properties of divisors.

The greatest common divisors (GCD). The Euclidean algorithm: definition, termination, correctness and complexity. GCD as a linear combination: Bézout's theorem. Linear Diophantine equations: solvability and solutions.

Prime numbers. The fundamental theorem of arithmetics. Prime-exponent representation of positive integers. Further properties of the GCD. Relative primality. Distribution of prime numbers.

Modular arithmetics. Congruences. Arithmetics modulo m. Division modulo a prime p. Linear equations modulo a prime.

Primality tests. Fermat's little theorem. Pseudoprimes. Carmichael numbers. Fermat's test. The Miller-Rabin algorithm.

Euler's totient function. Euler's theorem. Möbius function. Möbius inversion formula.

5 Binomial Coefficients

Definition of the binomial coefficient $\binom{r}{k}$ for arbitrary r real and k integer. Basic identities. Pascal's triangle. The hexagon property. The polynomial argument. Tricks of the trade: unfolding, summation on the upper index, summations with $1/2$ in the upper index, nth difference operator. Binomial inversion formula.

Power series. Convergence radius. Abel-Hadamard theorem, Laurent's theorem, uniqueness of analytic continuation.

Generating functions: definition, basic principles and operations. Convolution of two sequences. Counting with generating functions: number of integer solutions of equations, distributing objects into bins, change a 100 euro banknote into smaller size bills.

Identities in Pascal's triangle. The Vandermonde convolution. Generating functions for sequences of binomial coefficients. Solving recurrences with generating functions.

6 Special Numbers

Stirling numbers of the second kind: definition, special values, recurrence equation. Stirling numbers of the first kind: definition, special values, recurrence equation. Generating functions for Stirling numbers up to the sign. Falling and rising factorials as generating functions. Stirling's inversion formula.

Fibonacci numbers: definition. Golden ratio. Generating function of Fibonacci numbers. Cassini's identity; matrix form. Applications: non-overlapping dominos, partitions of integers. Fibonacci numbers with negative index. Generalized Cassini's identity.

Harmonic numbers. Harmonic summation. Euler's γ constant. Euler numbers. Bernoulli numbers. Bernoulli numbers and the Riemann zeta function.

7 Generating functions

Basic maneuvers: linear combination, shift, differentiation, integration, convolution. Basic sequences and their generating functions. Extracting the even- or odd-numbered terms of a sequence. Generating functions and infinite sums.

Solving recurrences. Fibonacci numbers revisited. Partial fraction expansion. Rational Expansion Theorem: special case for simple roots, general case.

Convolutions. Convolution of the Fibonacci sequence with itself. m-fold convolution: number of spanning trees of the fan of order n. Catalan numbers.

Exponential generating functions.