Concurrent Strategies

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The notion of *deterministic/nondeterministic strategy*
is potentially as fundamental as the notion of *function/relation*.
The notion needs to be developed in sufficient generality.

Two-party concurrent games: Player (a team of players) against Opponent (a team of opponents) subject to constraints of the game.
For *Player/Opponent* read *process/environment, proof/refutation, ally/enemy.*

First: in a general model for concurrency. Later: a recent more geometrical view

Motivation from semantics and logic

In **Semantics** of computation it's become clear that we need an *intensional* theory (a generalized domain theory) to capture the *ways* of computing, to near operational and algorithmic concerns.

What are to replace functions?

A possible answer: **strategies**, *"functions (or relations) extended in time."* [AJ]
[*There are others, e.g. profunctors as maps between presheaf categories.*]

In **Logic** the well-known Curry-Howard correspondence:
Propositions as types, proofs as programs
is being recast: *Propositions as games, proofs as strategies*.

Traditional definitions of strategies in games are not general enough!
Too sequential, too alternating ...

From strategies to arrows

Two important operations on games:
parallel composition of games $G\|H$;
dual of a game $G^{\perp}$ (reversing the roles of Player and Opponent)

Joyal after Conway: A strategy σ **from** a game G **to** a game H, $\sigma : G \rightarrow\!\!\!\!\!+ H$, is a strategy in $G^{\perp}\|H$; strategies compose with identities given by 'copy-cat.'

A strategy in H corresponds to a strategy from the empty game $\emptyset$ to H. Note

$$\emptyset \rightarrow\!\!\!\!\!+ G \rightarrow\!\!\!\!\!+ H \text{ composes to give } \emptyset \rightarrow\!\!\!\!\!+ H ,$$

so a strategy in G gives rise to a strategy in H when $G \rightarrow\!\!\!\!\!+ H$.
Conway's surreal numbers are strengths of games.

Games in a model for concurrency (via Joyal-Conway)

Lead to

- Generalised domain theory

- Operations, including higher-order operations via "function spaces" $G^{\perp} \| H$, within the model for concurrency

- Techniques for Logic (via proofs as concurrent strategies) and possibly verification and algorithmics

1. EVENT STRUCTURES

Event structures are the analogue of trees in a concurrent setting, where the causal dependence and independence of events is made explicit.

E.g. just as a transition system unfolds to a tree, a more general model for concurrency such as a Petri net unfolds to an event structure.

Event structures

An **event structure** comprises $(E, \leq, \mathrm{Con})$, consisting of a set of *events* E

- partially ordered by $\leq$, the **causal dependency relation**, and

- a nonempty family Con of finite subsets of E, the **consistency relation**,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \;\&\; e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

In games the relation of **immediate dependency** $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role.

Configurations of an event structure

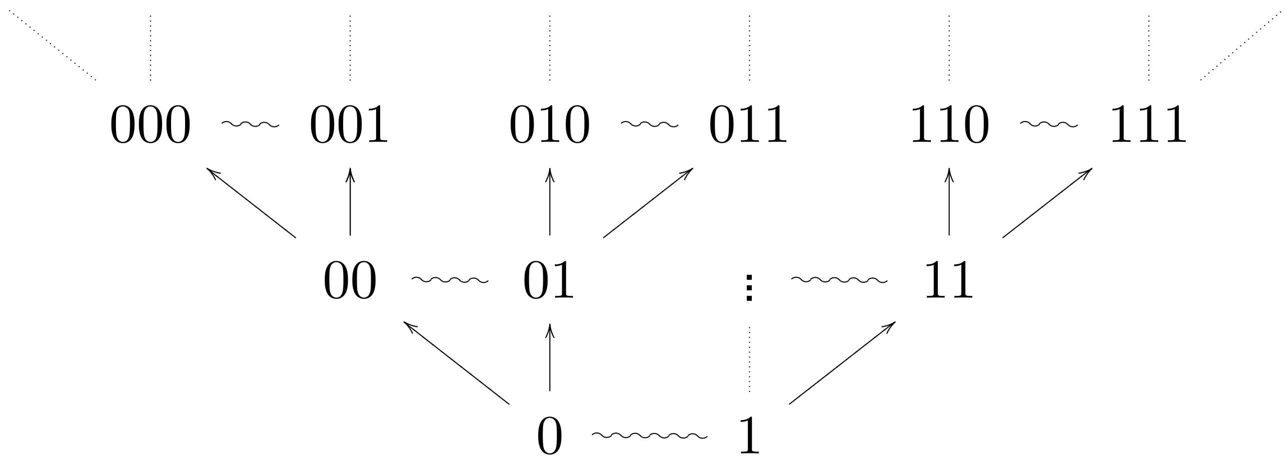
The **configurations**, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\rm fin} x.\ X \in {\rm Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

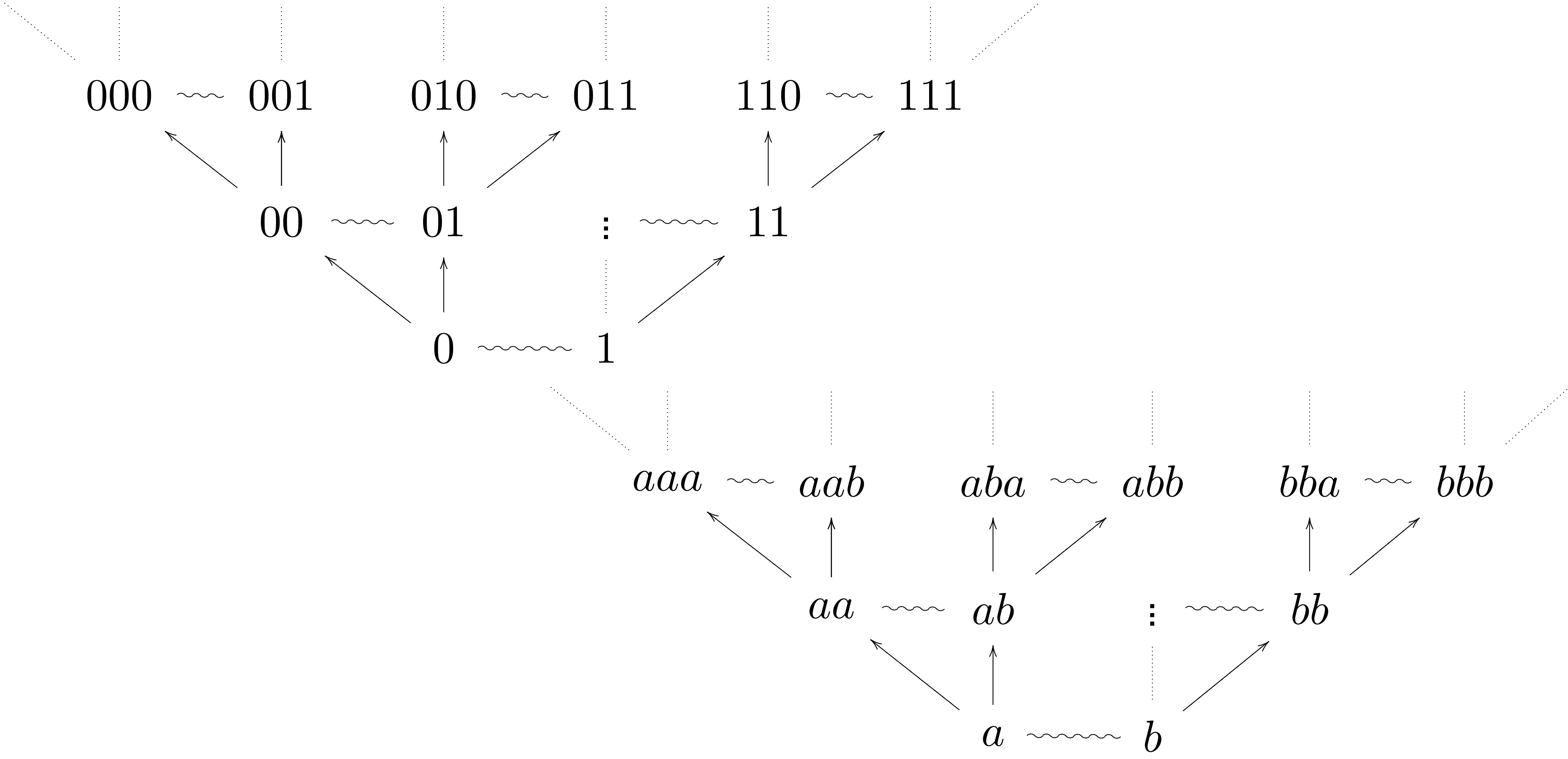
Often concentrate on the **finite configurations** $\mathcal{C}(E)$.

Example: Streams as event structures

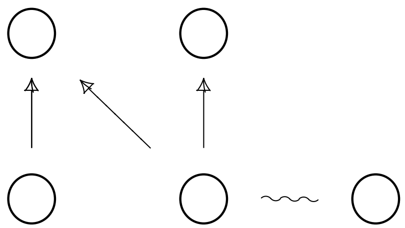


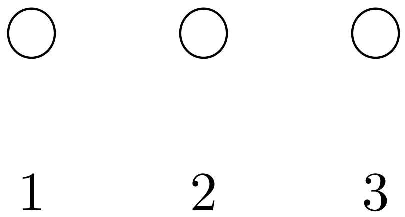
~~ conflict (inconsistency) → immediate causal dependency

Simple parallel composition



Other examples





$$\mathrm{Con} = \{\ \emptyset,\ \{1\},\{2\},\{3\},\ \{1,2\},\{1,3\},\{2,3\}\ \}$$

Maps of event structures

A (simulation) **map** of event structures $f : E \to E'$
is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$fx \in \mathcal{C}(E')$ and

if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$, then $e_1 = e_2$. *('event linearity')*

Idea: *the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined.*

$\rightsquigarrow$

- Semantics of synchronising processes [Hoare, Milner] can be expressed in terms of universal constructions on event structures.
- Relations between models via adjunctions.

Process constructions on event structures

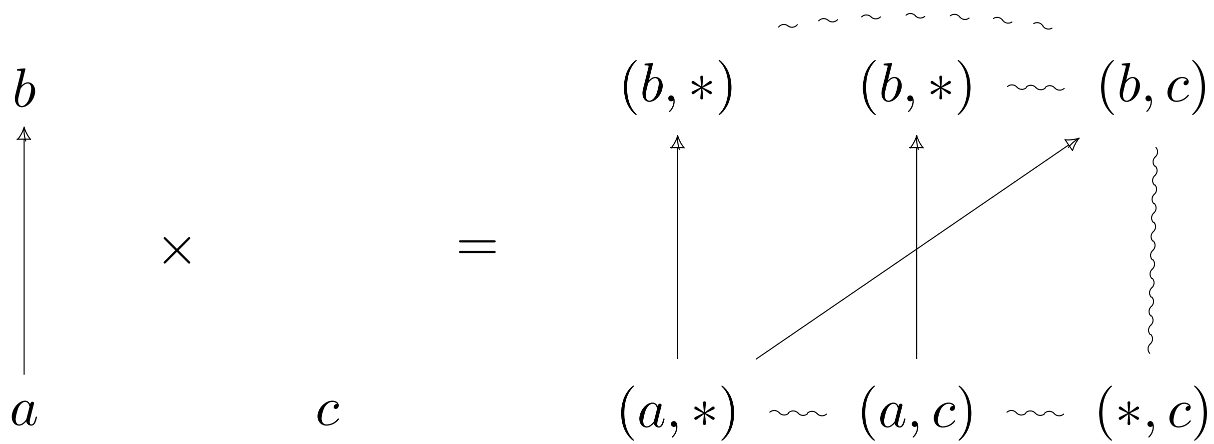
"Partial synchronous" product: $A \times B$ with projections Π_1 and Π_2, *cf.* synchronized composition where all events of A *can* synchronize with all events of B. (*Hard to construct directly, use e.g. coreflection with stable families.*)

Restriction: $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, has events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency restricted from E.

Synchronized compositions: restrictions of products $A \times B \upharpoonright R$, where R specifies the allowed synchronized and unsynchronized events.

Projection: Let E be an event structure. Let V be a subset of 'visible' events. The *projection* of E on V, $E{\downarrow}V$, has events V with causal dependency and consistency restricted from E.

Product—an example



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2. CONCURRENT GAMES

The paradigm of Joyal-Conway carried out in event structures.

We define and characterize concurrent strategies, those pre-strategies, i.e. nondeterministic plays, for which copy-cat strategies act as identities.

Concurrent games—basics

Games and strategies are represented by **event structures with polarity**, where events carry a polarity $+/-$ (Player/Opponent), respected by maps.

(Simple) Parallel composition: $A \| B$, by juxtaposition.

Dual, $B^{\perp}$, of an event structure with polarity B is a copy of the event structure B with a reversal of polarities; $\overline{b} \in B^{\perp}$ is complement of $b \in B$, and *vice versa*.

A (nondeterministic) concurrent **pre-strategy** in game A is a total map

$$\sigma : S \to A$$

of event structures with polarity (*a nondeterministic play in game A*).

Pre-strategies as arrows

A pre-strategy $\sigma : A \mathrel{+\!\!\!\!\rightarrow} B$ is a total map of event structures with polarity

$$\sigma : S \to A^\perp \parallel B\,.$$

It corresponds to a *span* of event structures with polarity

$$A^\perp \xleftarrow{\ \sigma_1\ } S \xrightarrow{\ \sigma_2\ } B$$

where σ_1, σ_2 are *partial* maps of event structures with polarity; one and only one of σ_1, σ_2 is defined on each event of S.

Pre-strategies are isomorphic if they are isomorphic as spans.

Concurrent copy-cat

Identities on games A are given by copy-cat strategies

$$\gamma_A : \mathbb{C}\!\mathbb{C}_A \to A^\perp \parallel A$$

—strategies for player based on copying the latest moves made by opponent.

$\mathbb{C}\!\mathbb{C}_A$ has the same events, consistency and polarity as $A^\perp \parallel A$ but with causal dependency $\leq_{\mathbb{C}\!\mathbb{C}_A}$ given as the transitive closure of the relation

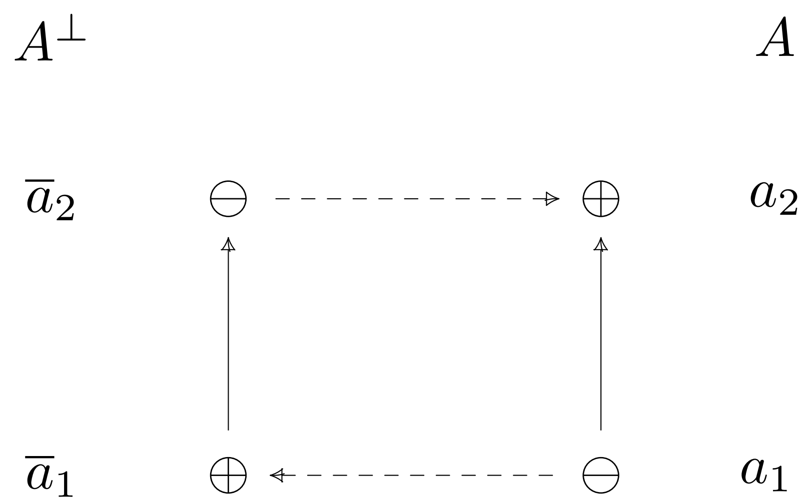
$$\leq_{A^\perp \parallel A} \cup \{(\overline{c}, c) \mid c \in A^\perp \parallel A \;\;\&\;\; pol_{A^\perp \parallel A}(c) = +\}$$

where $\overline{c} \leftrightarrow c$ is the natural correspondence between $A^\perp$ and A.

The map γ_A is the identity on the common underlying set of events.

Copy-cat—an example

$\mathbb{C}\mathbb{C}_A$



Composing pre-strategies

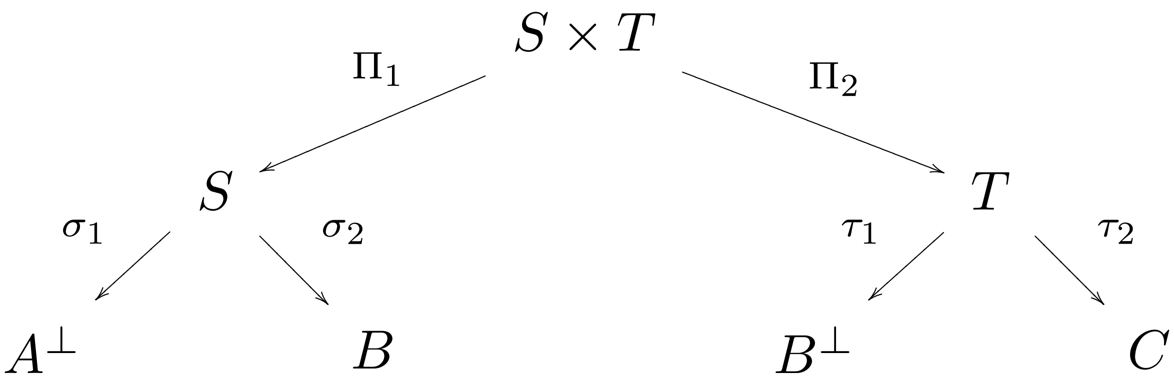
Two pre-strategies $\sigma : A \mathrel{+\mkern-14mu\rightarrow} B$ and $\tau : B \mathrel{+\mkern-14mu\rightarrow} C$ as spans:

$$A^{\perp} \xleftarrow{\ \sigma_1\ } S \xrightarrow{\ \sigma_2\ } B \qquad B^{\perp} \xleftarrow{\ \tau_1\ } T \xrightarrow{\ \tau_2\ } C\,.$$

Their composition

$$A^{\perp} \xleftarrow{\ (\tau\odot\sigma)_1\ } T\odot S \xrightarrow{\ (\tau\odot\sigma)_2\ } C$$

where $T\odot S =_{\rm def} (S \times T \restriction \textbf{Syn}) \downarrow \textbf{Vis}$ where ...



Their composition: $T \odot S =_{\mathrm{def}} (S \times T \upharpoonright \textbf{Syn}) \downarrow \textbf{Vis}$ where

$$
\begin{aligned}
\textbf{Syn} \;=\;& \{p \in S \times T \mid \sigma_1\Pi_1(p) \text{ is defined } \& \ \Pi_2(p) \text{ is undefined}\} \cup \\
& \{p \in S \times T \mid \sigma_2\Pi_1(p) = \overline{\tau_1\Pi_2(p)} \text{ with both defined}\} \cup \\
& \{p \in S \times T \mid \tau_2\Pi_2(p) \text{ is defined } \& \ \Pi_1(p) \text{ is undefined}\}\,, \\
\textbf{Vis} \;=\;& \{p \in S \times T \upharpoonright \textbf{Syn} \mid \sigma_1\Pi_1(p) \text{ is defined}\} \cup \\
& \{p \in S \times T \upharpoonright \textbf{Syn} \mid \tau_2\Pi_2(p) \text{ is defined}\}\,.
\end{aligned}
$$

Theorem characterizing concurrent strategies

Receptivity $\sigma : S \to A^\perp \parallel B$ is *receptive* when $\sigma(x) {-\!\!\!\subset}^- y$ implies there is a *unique* $x' \in \mathcal{C}(S)$ such that $x {-\!\!\!\subset} x'$ & $\sigma(x') = y$.

$$\begin{array}{ccc} x & {-\!\!\!\subset} & x' \\ \downarrow & & \downarrow \\ \sigma x & {-\!\!\!\subset}^- & y \end{array}$$

A strategy should be receptive to all possible moves of opponent.

Innocence $\sigma : S \to A^\perp \parallel B$ is *innocent* when it is
$+$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$ and
$-$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.
A strategy should only adjoin immediate causal dependencies $\ominus \rightarrowtriangle \oplus$.

Theorem Receptivity and innocence are necessary and sufficient for copy-cat to act as identity w.r.t. composition: $\gamma_B \odot \sigma \odot \gamma_A \cong \sigma$. *[Silvain Rideau, GW]*

Definition A **strategy** is a receptive, innocent pre-strategy.

$\rightsquigarrow$ A bicategory, $\mathbf{Games}$, whose

objects are event structures with polarity—the games,

maps are strategies $\sigma : A \nrightarrow B$

2-cells are maps of spans.

The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition).

$\rightsquigarrow$ A sub-$category$ where maps are **deterministic strategies** and objects are 'race-free' games. [Melliès & Mimram's *receptive ingenuous* strategies]

Winning strategies

The paradigm of Joyal-Conway carries through in concurrent games A with winning conditions $W \subseteq \mathcal{C}^\infty(A)$:

A strategy $\sigma : S \to A$ in (A, W) is **winning** (for Player)

iff

any maximal play against a counter-strategy results in a win for Player

iff

$\sigma x \in W$, for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$.

A winning strategy **from** (A, W_A) **to** (B, W_B) is a winning strategy in $(A, W_A)^\perp \| (B, W_B)$ where

$(A, W_A)^\perp = (A^\perp, W_{A^\perp})$ where $W_{A^\perp}$ is the complement of W_A.

$(A, W_A) \| (B, W_B) = (A \| B, W_{A\|B})$ where

$$x \in W_{A\|B} \iff x_A \in W_A \text{ or } x_B \in W_B\,.$$

To win in $G\|H$ is to win in either game G or H.

Winning strategies compose $\rightsquigarrow$ *a bicategory of winning strategies.*

Extensions

Determinacy for well-founded race-free concurrent games with winning conditions; concurrent game semantics for PC and a version of Hintikka's IF Logic [LICS 2012 with Julian Gutierrez and Pierre Clairambault]

Games with **neutral configurations** and **imperfect information** via access levels [Dexter Kozen festschrift]

Back-tracking? To do so developing **games with symmetry** to support copying monads/comonads. $\rightsquigarrow$ fully-fledged generalized domain theory for concurrency

Concurrent games with **pay-off**, early stages [with Pierre Clairambault]

Linear strategies and full completeness for MALL [FOSSACS paper is wrong in claiming the linear strategies as defined there yield a monoidal closed category!]

• **Games on categories** with a factorization system $\rightsquigarrow$ a geometrical view •

An alternative description of strategies

A strategy in a game A comprises $\sigma : S \to A$, a total map of event structures with polarity, such that
(i) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that
$x \subseteq x' \;\&\; \sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x & \subseteq & x' \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ \sigma x & \subseteq^- & y\,, \end{array}$$

and
(ii) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that
$x' \subseteq x \;\&\; \sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x' & \subseteq & x \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ y & \subseteq^+ & \sigma x\,. \end{array}$$

Corollary

Defining a partial order — *the Scott order* — on configurations of A

$$x \sqsubseteq y \iff_{\text{def}} x \supseteq^{-} x \cap y \subseteq^{+} y\,,$$

we obtain a factorization system $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^{-}, \subseteq^{+})$, *i.e.* $\exists! z.\quad x \supseteq^{-} z \subseteq^{+} y$, with $x \sqsubseteq y$.

Theorem *Strategies $\sigma : S \to A$ correspond to a discrete fibrations*

$$\sigma“ : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)\,, \quad i.e. \quad \exists! x'.\ x' \sqsubseteq_S x \text{ with } \sigma“ x' = y,\quad y \sqsubseteq_A \sigma“(x)\,,$$

preserving $\supseteq^{-}$, $\subseteq^{+}$ *and* $\emptyset$.

From strategies to profunctors

A strategy $\sigma : A \nrightarrow B$ determines a discrete fibration so a presheaf over

$$(\mathcal{C}(A^{\perp} \| B), \sqsubseteq_{A^{\perp}\|B}) \;\cong\; (\mathcal{C}(A), \sqsubseteq_{A})^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_{B})$$

$i.e.$ a profunctor $\sigma``: (\mathcal{C}(A), \sqsubseteq_A) \nrightarrow (\mathcal{C}(B), \sqsubseteq_B)$.

$\rightsquigarrow$ a lax pseudo functor $(\_)`` : \mathbf{Games} \to \mathbf{Prof}$; have $(\tau \odot \sigma)`` \Rightarrow \tau\,`` \circ \sigma\,``$.
The profunctor composition introduces extra 'unreachable' elements.

Not lax for 'rigid' strategies including : simple games; sub-bicategory of "stable spans" with objects $A, B, \cdots$ purely $+$ve.

3. GAMES AS FACTORIZATION SYSTEMS

A **rooted factorization system** $(\mathbb{C}, L, R, 0)$ comprises a small category $\mathbb{C}$ on which there is a factorization system $(\mathbb{C}, L, R)$,

so all maps $c \to c'$ factor uniquely up to iso as

$$\begin{array}{ccc} & & c' \\ & \nearrow & \uparrow R \\ c & \xrightarrow[L]{} & c'' \end{array} ,$$

with an object 0 s.t.

$$0 \leftarrow_L \cdot \rightarrow_R \cdots \leftarrow_L \cdot \rightarrow_R c$$

for all objects c in $\mathbb{C}$.

Example $(\,(\mathcal{C}(A), \sqsubseteq_A)\,,\, \supseteq^-,\, \subseteq^+,\, \emptyset)$ for a concurrent game A.

Strategies

A strategy on a rooted factorization system $(\mathbb{A}, L_A, R_A, 0_A)$ is a discrete fibration

$$F : (\mathbb{S}, L_S, R_S, 0_S) \to (\mathbb{A}, L_A, R_A, 0_A)\,,$$

from another rooted factorization system $(\mathbb{S}, L_S, R_S, 0_S)$, which preserves L, R maps and 0.

Example: The map σ" $: ((\mathcal{C}(S), \sqsubseteq_S), \supseteq^-, \subseteq^+, \emptyset) \to ((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+, \emptyset)$ induced by a strategy $\sigma : S \to A$.

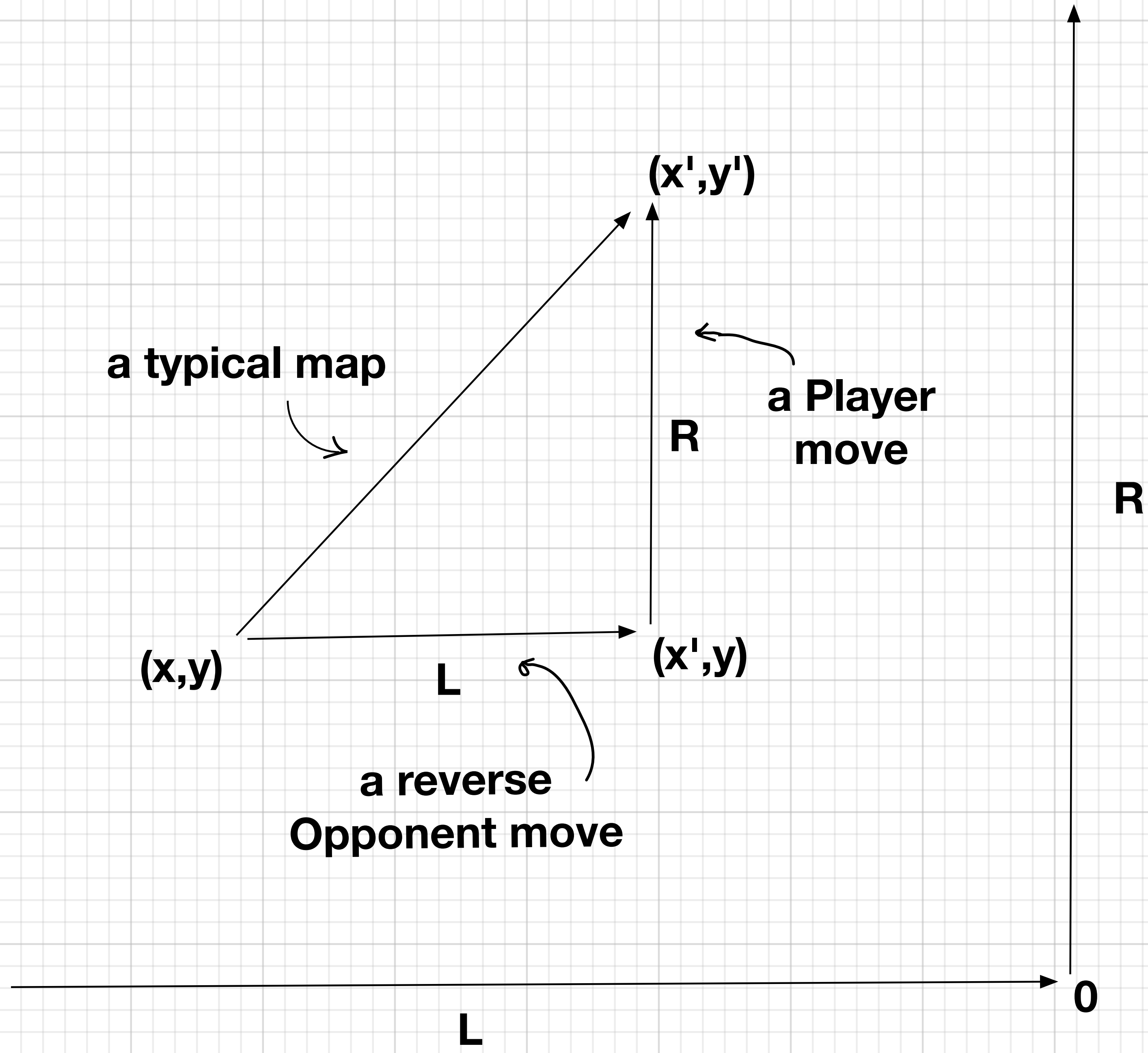
Operations $\quad (\mathbb{C}, L, R, 0)^\perp =_{\text{def}} (\mathbb{C}^{\text{op}}, R^{\text{op}}, L^{\text{op}}, 0)$

$(\mathbb{B}, L_B, R_B, 0_B) \| (\mathbb{C}, L_C, R_C, 0_C) =_{\text{def}} (\mathbb{B} \times \mathbb{C}, L_B \times L_C, R_B \times R_C, (0_B, 0_C))$

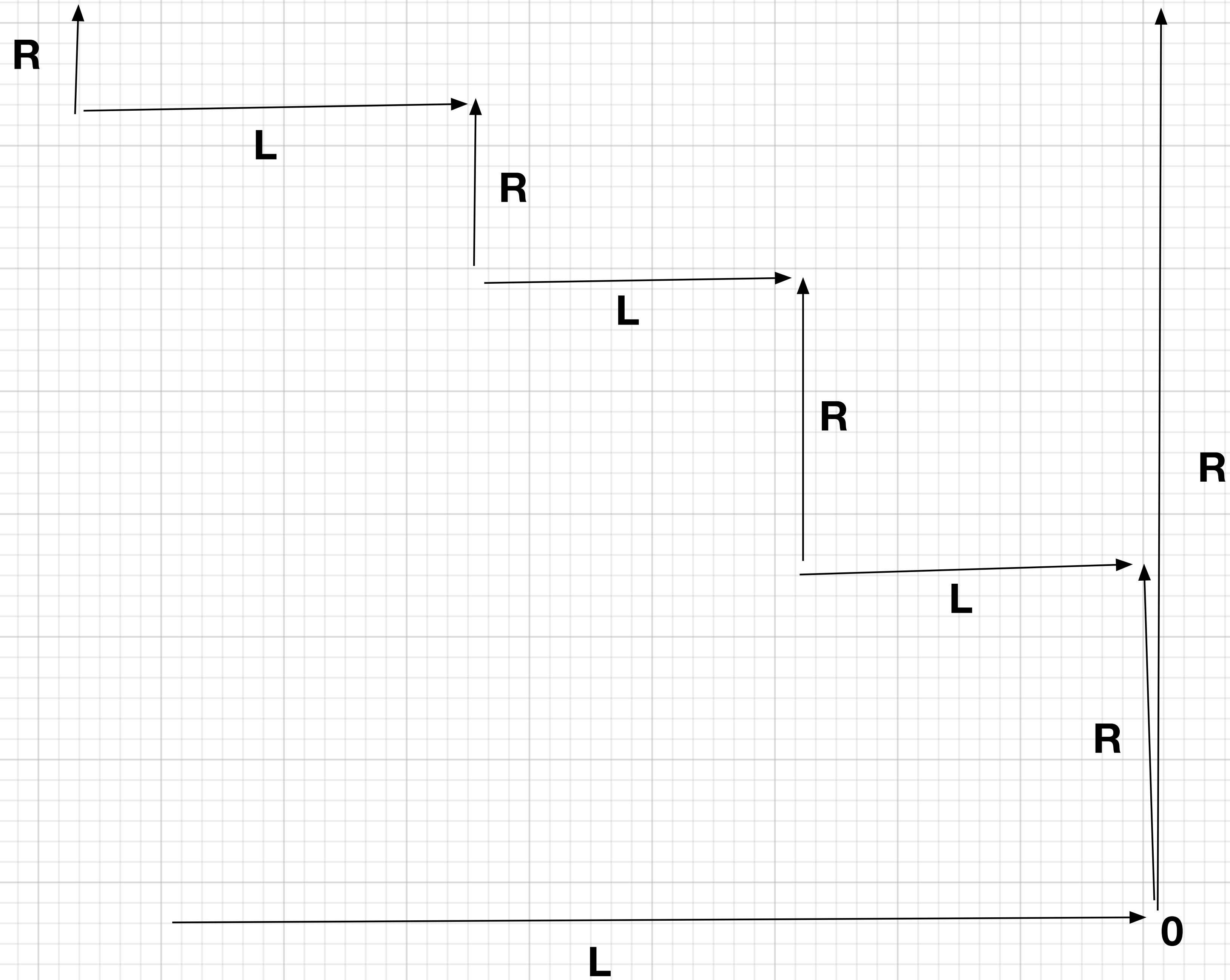
Composition: *reachable part of* profunctor composition.

$\rightsquigarrow$ *'Venn diagrams' for games and strategies.*

Example: the Euclidean quarter plane



A typical play



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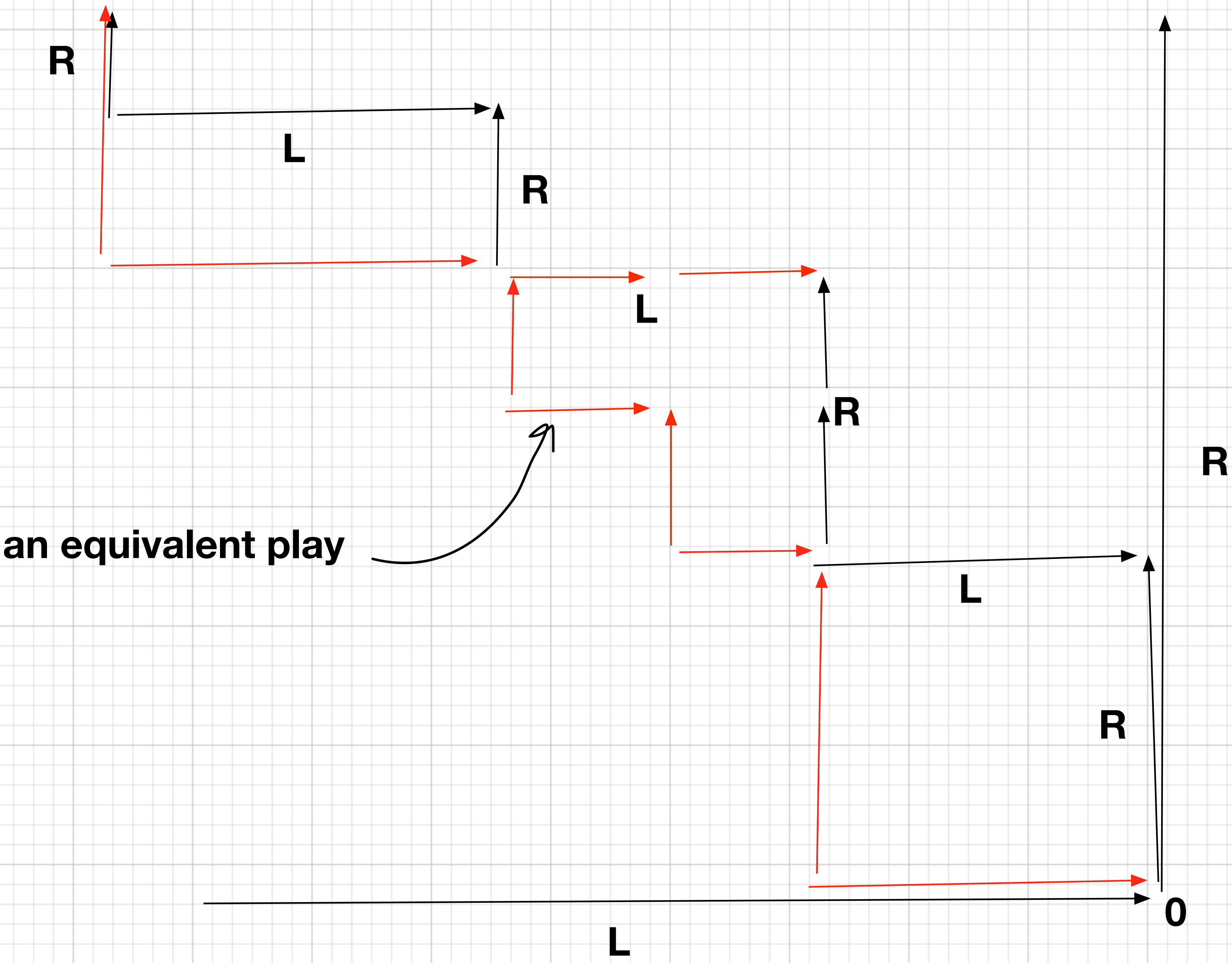
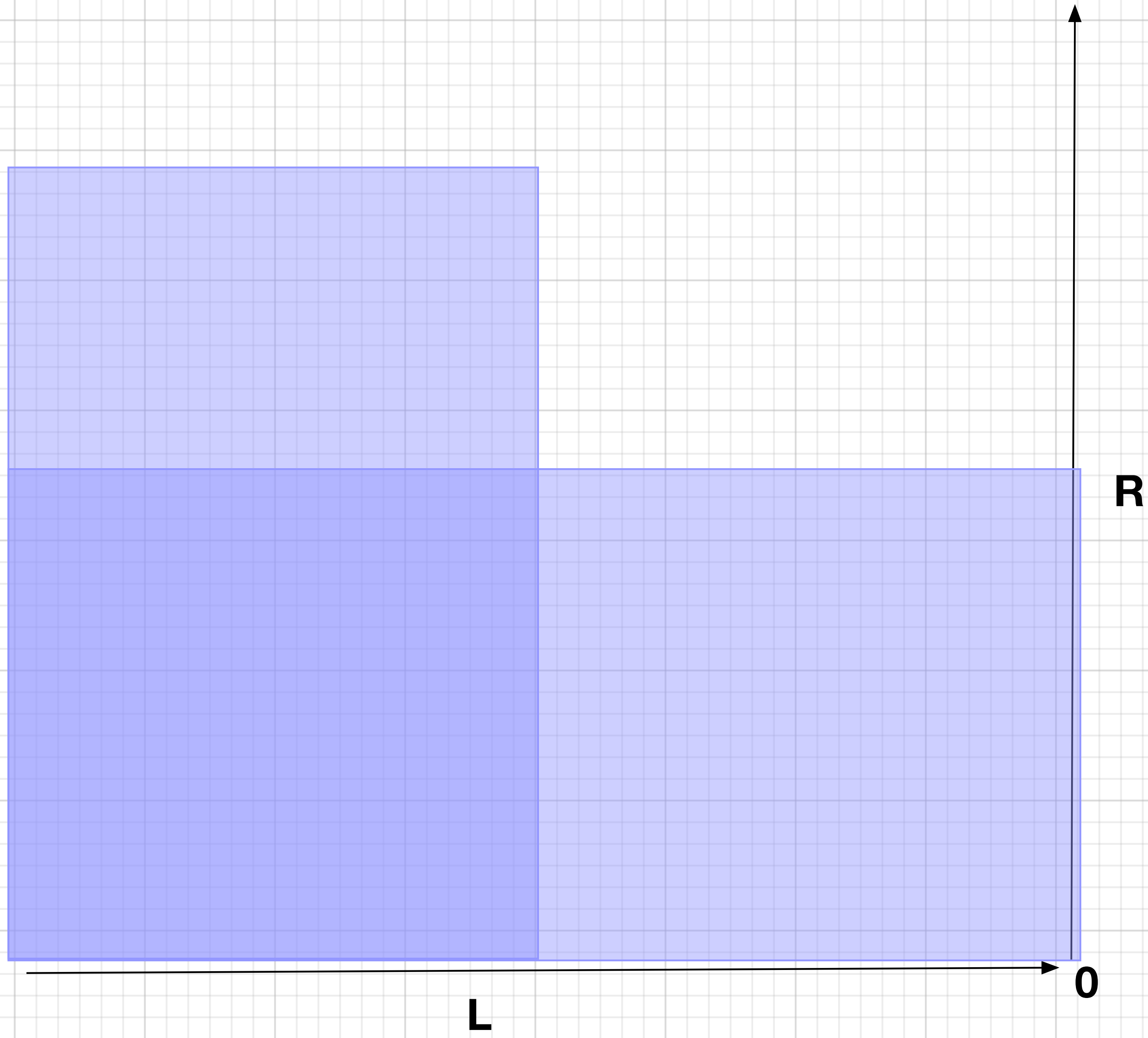
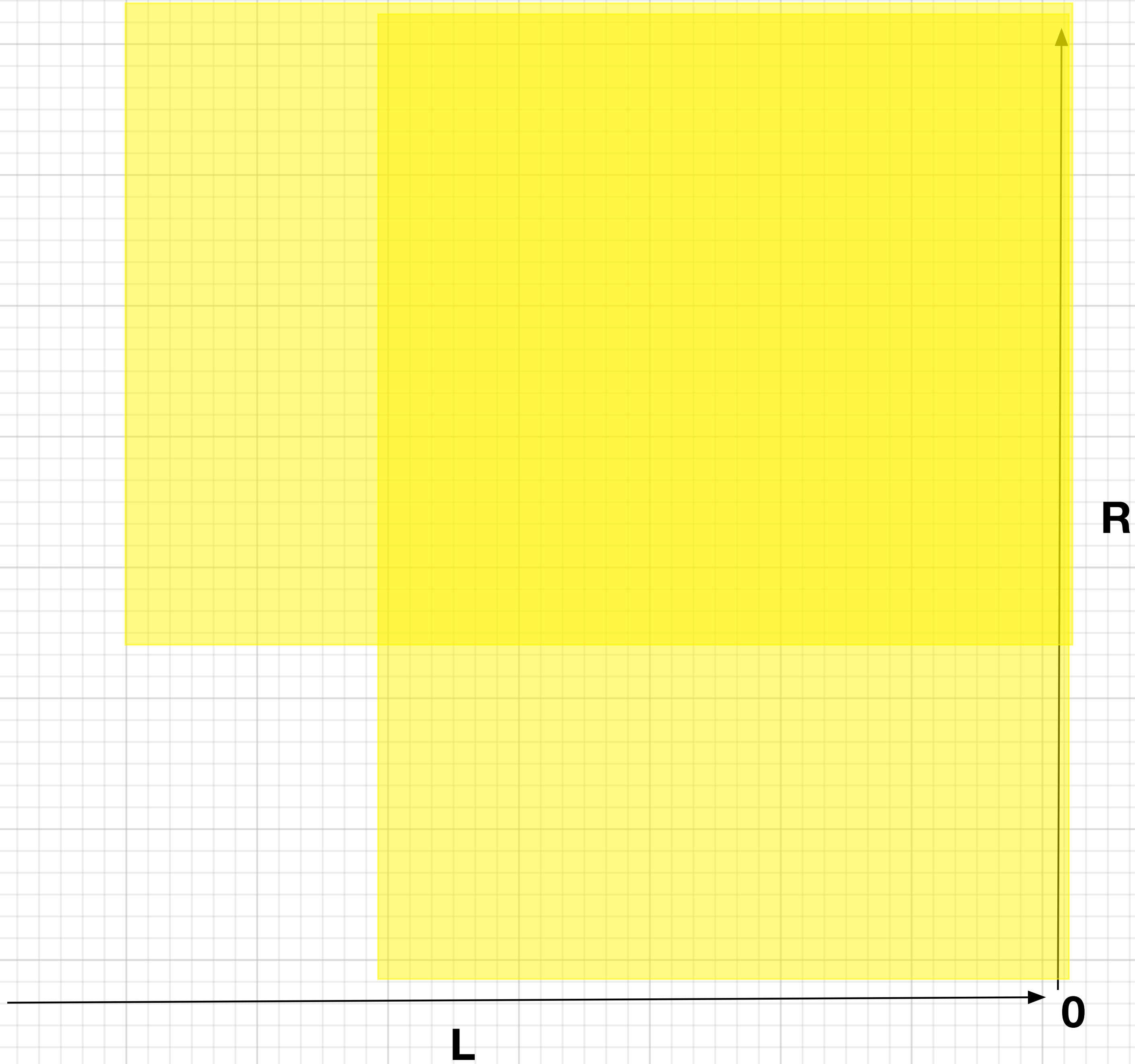


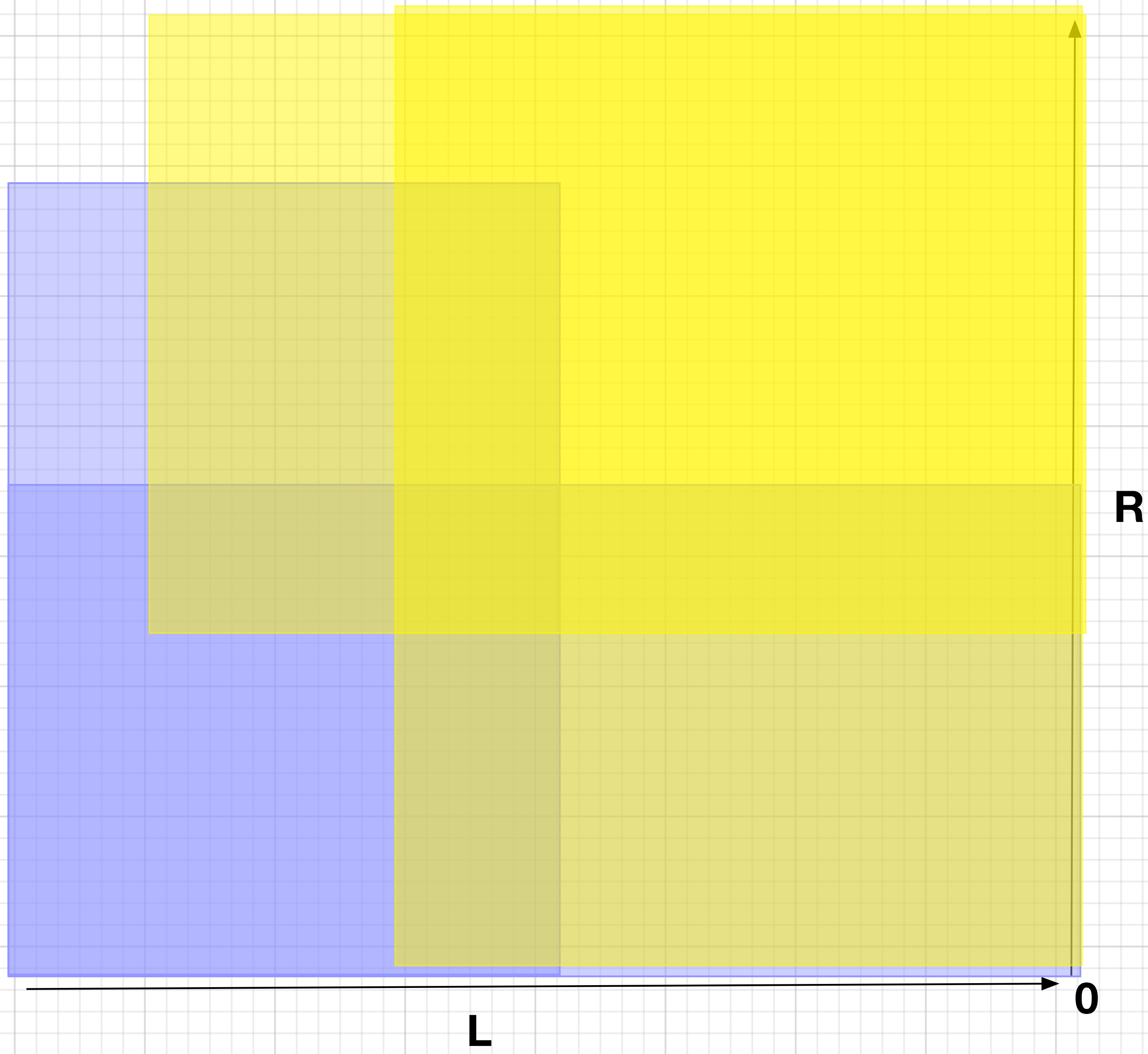
A typical deterministic strategy



A typical deterministic counter-strategy



Their play against each other



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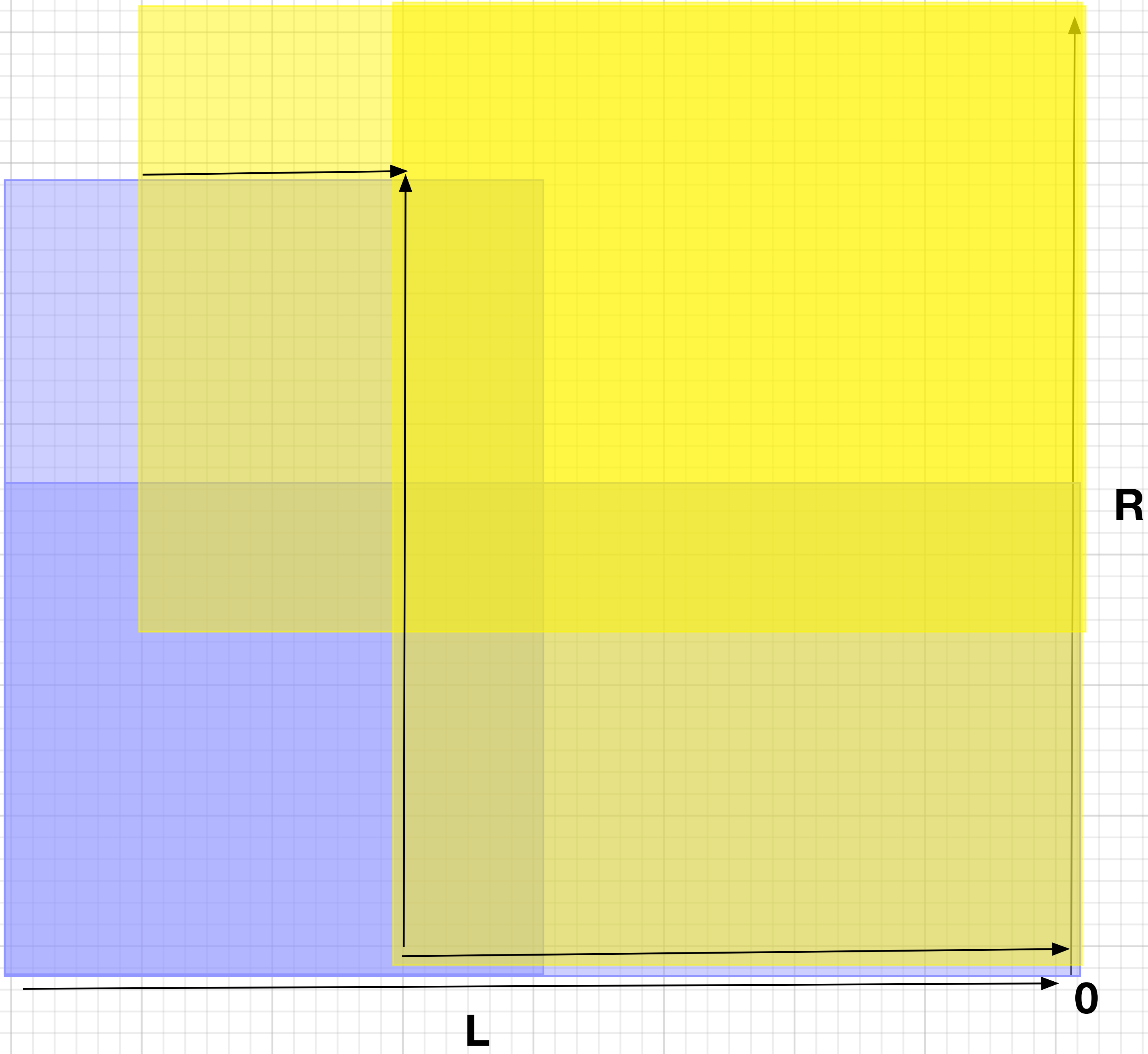
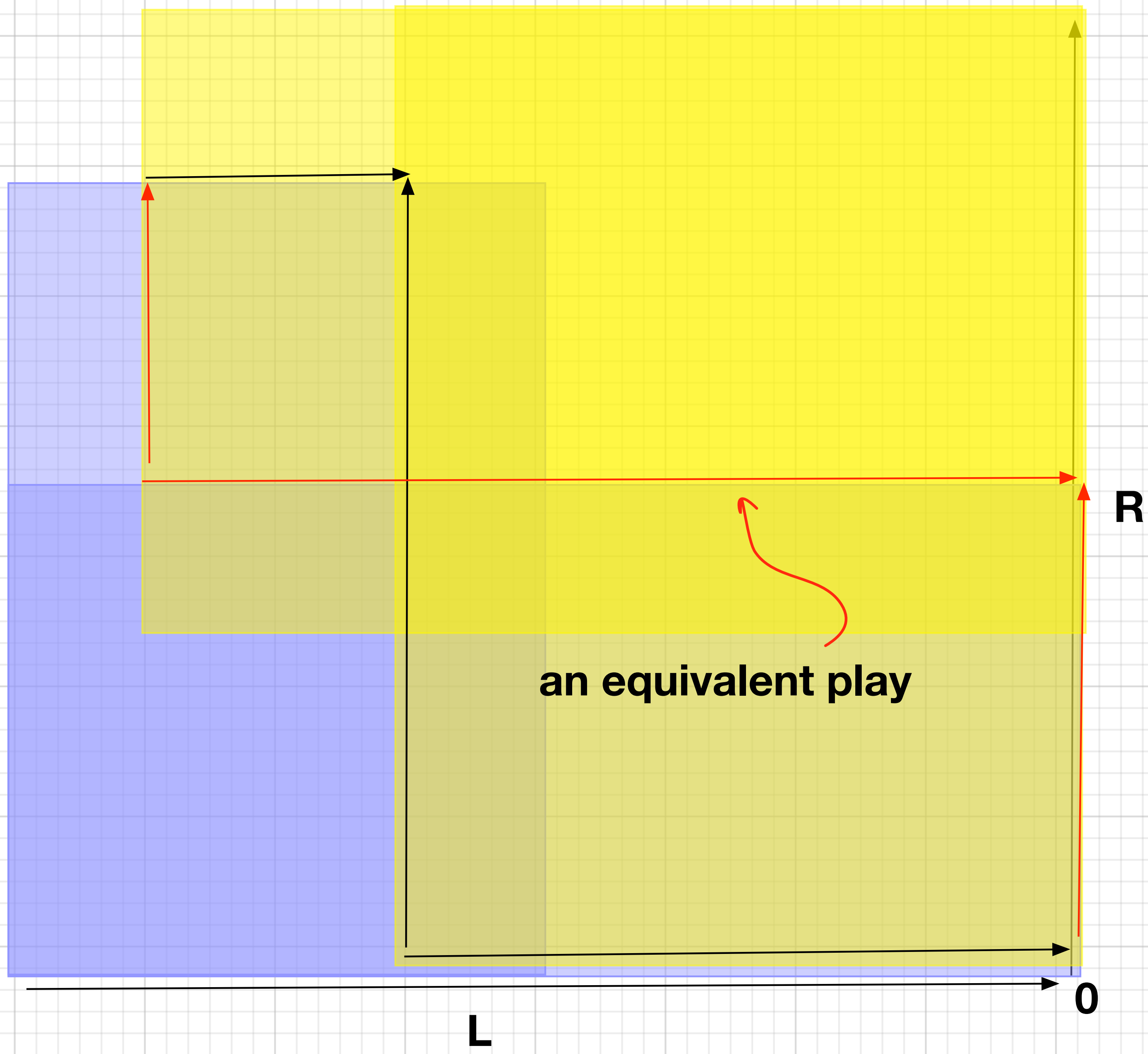
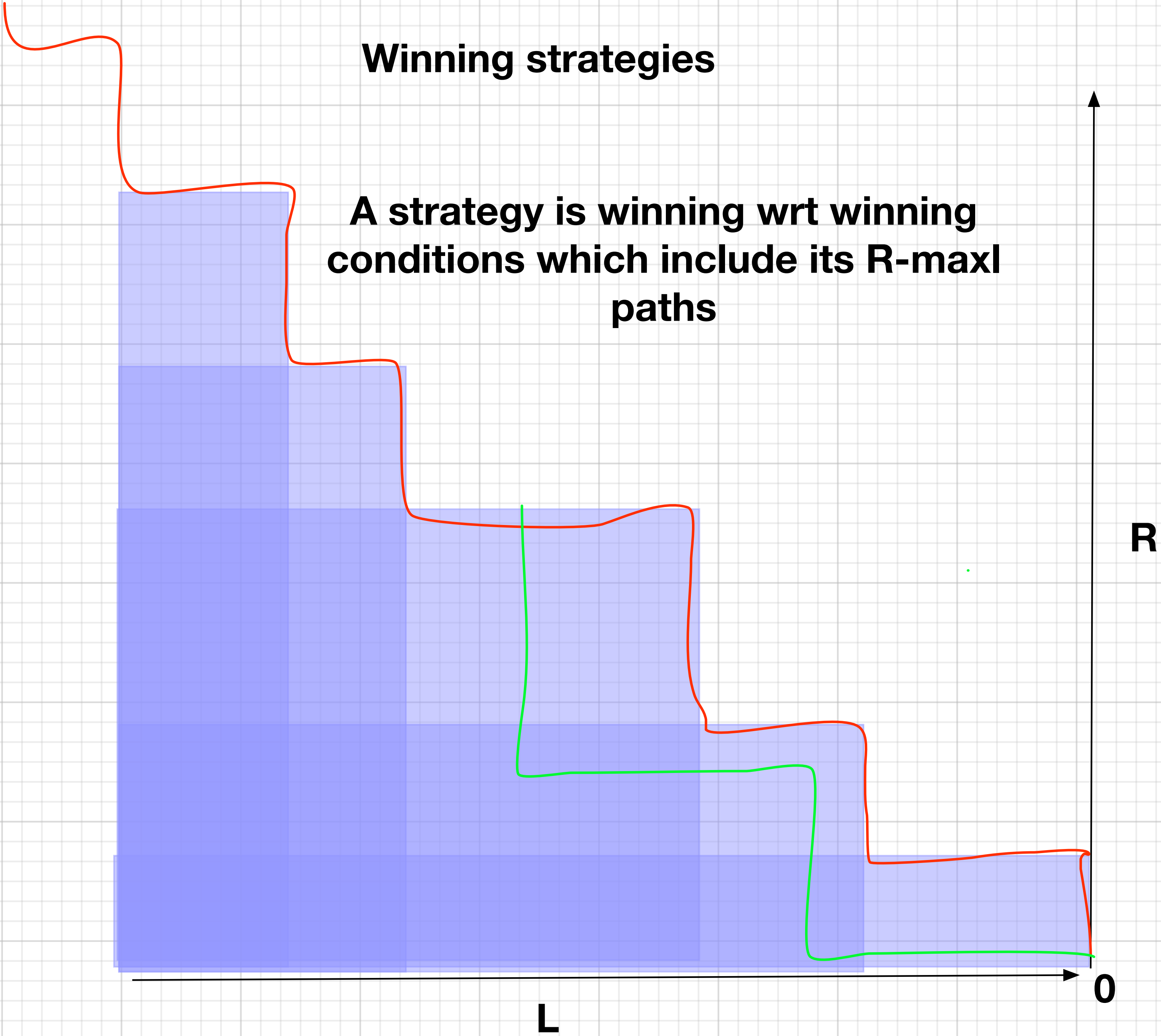


Their play against each other



Winning strategies

A strategy is winning wrt winning conditions which include its R-maxl paths



A similar example: the game of chase

Player: the **Hunter**, velocity vector $\mathbf{h}$; its moves are changes in velocity $\mathbf{\Delta h}$

Opponent: the **Prey**, velocity vector $\mathbf{p}$; its moves are changes in velocity $\mathbf{\Delta p}$

A strategy for Hunter (observed in people): run (towards Prey) so Prey appears to be moving in a fixed straight line (direction vector $\mathbf{d}$) from Hunter's viewpoint, $i.e.$ adjust velocity to maintain the winning condition

$$\mathbf{p} - \mathbf{h} = c.\mathbf{d} \quad \text{for some positive real } c$$

within a game with positions $(\mathbf{p}, \mathbf{h})$.

[*BBC Horizon programme "The Unconscious Mind"*]

THANK YOU!