Introduction to Coalgebra

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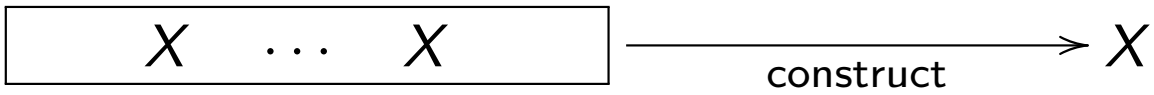
Outline

What are coalgebras, intuitively

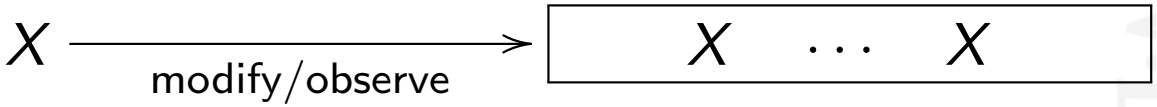
- Mathematical models for state-based computation
  - your computer has a complicated internal state
  - you as user can only modify it to some extend
  - also, you can observe only so much
- Basically only two kinds of operations:
  - move to a next state, somehow (deterministically, non-deterministically, probabilistically, . . .)
  - make an observation ("measurement") about the current state
  - These operations can be combined ("observation has side-effect")
- Describing such state-based systems and their dynamics requires a new kind of mathematics.
  - related to dynamical systems (and evolution via differential equations)

(Co)algebras, pictorially

Algebras with "carrier" X are maps into X, of the form:



Coalgebras with "state space" X are maps out of X, of the form:



Think of a fields & methods in an object-oriented class

- Formally, the boxes are functors, usually **Sets** $\to$ **Sets**, ...
- ... but at this stage you may think of them as types $\sigma(X)$ containing a single type variable X

Intermezzo: type constructors for "boxes"

- Identity X, and constants A
  - including empty set $\emptyset$, often written as 0
  - singleton set $1 = \{*\}$
- products $X \times Y = \{\langle x, y\rangle \mid x \in X, y \in Y\}$, with **projections**:

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y \qquad \frac{Z \xrightarrow{\langle f, g\rangle} X \times Y}{Z \underset{f}{\to} X \qquad Z \underset{g}{\to} Y}$$

  - infinite version usually written as $\prod_{i \in I} X_i$
- coproducts $X + Y = \{\langle x, 0\rangle \mid x \in X\} \cup \{\langle y, 1\rangle \mid y \in Y\}$, with **coprojections**

$$X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y \qquad \frac{X + Y \xrightarrow{[f, g]} Z}{X \underset{f}{\to} Z \qquad Y \underset{g}{\to} Z}$$

  - infinite version written as $\coprod_{i \in I} X_i$

Algebra examples $F(X) \to X$

- Natural numbers

$$1 + \mathbb{N} \xrightarrow{[\text{zero},\text{succ}]} \mathbb{N} \qquad \text{i.e.} \quad \begin{cases} \text{zero} \colon 1 \longrightarrow \mathbb{N} \\ \text{succ} \colon \mathbb{N} \longrightarrow \mathbb{N} \end{cases}$$

  The functor/type here is $F(X) = 1 + X$.

- Groups

$$1 + (G \times G) + G \xrightarrow{[0,+,-]} G \qquad \text{i.e.} \quad \begin{cases} 0 \colon 1 \longrightarrow G \\ + \colon G \times G \longrightarrow G \\ - \colon G \longrightarrow G \end{cases}$$

- Clearly this works for many algebraic structures: for each signature Σ there is a functor:

$$X \longmapsto \coprod_{f \in \Sigma} X^{\text{arity}(f)} \qquad \text{(disjoint union over all operations)}$$

  whose algebras are precisely the Σ-algebras.

Note: this captures algebraic operations, not equations

Coalgebra examples $X \to F(X)$

- Streams (infinite lists) $A^{\mathbb{N}}$ of elements of a set A:

$$A^{\mathbb{N}} \xrightarrow{\langle \mathsf{head}, \mathsf{tail} \rangle} A \times A^{\mathbb{N}}$$

for the functor $F(X) = A \times X$, where:

$$\begin{cases} \mathsf{head}(\alpha) = \alpha(0) \\ \mathsf{tail}(\alpha) = \alpha' = \lambda n \in \mathbb{N}.\, \alpha(n+1) \end{cases}$$

- Both finite and infinite lists $A^\infty = A^\star + A^{\mathbb{N}}$

$$A^\infty \xrightarrow{\text{possible head}} 1 + (A \times A^\infty)$$

for the functor $F(X) = 1 + A \times X$.
Note: observation with side-effect

Automata-theoretic coalgebra examples

- A deterministic automaton with input actions A is a coalgebra:

$$X \xrightarrow{\langle \text{step}, \text{final?} \rangle} X^A \times 2$$

  where $2 = \{0, 1\}$.

- A non-deterministic automaton, or transition system, with input actions A is a coalgebra:

$$X \xrightarrow{\text{succs}} \mathcal{P}(X)^A$$

- Other functors (actually "monads") than powerset $\mathcal{P}$ may be used: eg. partial automata via lift, or probabilistic automata via distribution.

Markov chains

For a set X, define the set of discrete distributions on X

$$\mathcal{D}(X) = \{\varphi \colon X \to [0,1] \mid \text{support}(\varphi) \text{ is finite, and } \textstyle\sum_x \varphi(x) = 1\}$$

Such $\varphi \in \mathcal{D}(X)$ is a formal convex combination:

$$r_1 x_1 + \cdots + r_n x_n \quad \text{where} \quad \begin{cases} \text{support}(\varphi) = \{x_1, \ldots, x_n\} \\ r_i = \varphi(x_i) > 0 \\ r_1 + \cdots + r_n = 1 \end{cases}$$

Coalgebras $X \to \mathcal{D}(X)$ are Markov chains, giving probabilistic transitions:

$$x \xrightarrow{r_i} x_i \quad \text{with} \quad \textstyle\sum_i r_i = 1.$$

They can also be used in labeled form, as

$$X \longrightarrow \mathcal{D}(X)^A \qquad \text{or as} \qquad X \longrightarrow \mathcal{D}(A \times X)$$

Coalgebra essentials & Plan

- Generic framework for describing state-based systems ✓
- Coinductive definition and reasoning principles Lect. II, III
- Bisimilarity, as indistinguishability of states Lect. III, IV
- Modal logic, also generically
- Coherence between operational & denotational semantics
- General trace semantics
- Current own interest: coagebra & quantum computing Lect. V
  - Quantum language has strong coalgebraic flavour: states, transitions, observations, indistinguishability, ...
  - Can this be made mathematically precise?

Functors, part I

- Before we can proceed we need to know a little bit more about functors, for the time being only on sets
- In functional programming "functoriality" is often described by "map" properties, such as `list_map`
- A functor F maps sets X to sets $F(X)$, and also functions to functions, in an orderly manner:
  - if $g\colon X \to Y$, then $F(g)\colon F(X) \to F(Y)$
  - F preserves identity maps: $F\left(X \xrightarrow{\mathrm{id}} X\right) = F(X) \xrightarrow{\mathrm{id}} F(X)$
  - F preserves composition: $F(h \circ g) = F(h) \circ F(g)$, in:

$$F\left(X \xrightarrow{g} Y \xrightarrow{h} Z\right) = \left(F(X) \xrightarrow{F(g)} F(Y) \xrightarrow{F(h)} F(Z)\right)$$

- Often, when the action on functions is "obvious", it is omitted from the description of a functor

Functor example I

- For a set X write $\mathcal{P}(X)$ for the powerset: the set of all subsets of X
- This $\mathcal{P}(-)$ forms a functor
- For a function $g\colon X \to Y$ there is a suitable function $\mathcal{P}(X) \to \mathcal{P}(Y)$
  - this is new function is written $\mathcal{P}(g)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$
  - it is given by image:

$$\begin{aligned}
\mathcal{P}(g)\Big(U \subseteq X\Big) &= \Big(\{g(x) \mid x \in U\} \subseteq Y\Big) \\
&= \{y \in Y \mid \exists x \in U.\, y = g(x)\} \\
&= g(U) \quad \text{as it is sometimes written} \\
&= g[U] \qquad \text{also possible} \\
&= \coprod_g(U) \quad \text{for categorical logicians}
\end{aligned}$$

Functor example I

We check explicitly:

$$\begin{aligned}\mathcal{P}(\mathrm{id})(U) &= \{\mathrm{id}(x) \mid x \in U\} \\ &= U\end{aligned}$$

Hence $\mathcal{P}(\mathrm{id}) = \mathrm{id}$. Similarly, $\mathcal{P}$ preserves composition:

$$\begin{aligned}&\Big(\mathcal{P}(h) \circ \mathcal{P}(g)\Big)(U) \\ &= \mathcal{P}(h)\Big(\mathcal{P}(g)(U)\Big) \\ &= \mathcal{P}(h)\Big(\{y \in Y \mid \exists x \in U.\, y = g(x)\}\Big) \\ &= \{z \in Z \mid \exists y \in Y.\, \exists x \in U.\, y = g(x) \text{ and } z = h(y)\} \\ &= \{z \in Z \mid \exists x \in U.\, z = h(g(x))\} \\ &= \mathcal{P}(h \circ g)(U)\end{aligned}$$

Functor example II

- For a set X write $X^\star$ for the associated set of finite lists:

$$X^\star = \{\langle x_1, x_2, \ldots, x_n\rangle \mid x_i \in X\}$$

- The mapping $X \longmapsto X^\star$ is a functor
- For $g\colon X \to Y$ we have $g^\star\colon X^\star \to Y^\star$ given by:

$$g^\star\Big(\langle x_1, x_2, \ldots, x_n\rangle\Big) = \langle g(x_1), g(x_2), \ldots, g(x_n)\rangle$$

- It is not hard to check the functoriality properties:

$$\mathrm{id}^\star = \mathrm{id} \qquad \text{and} \qquad (h \circ g)^\star = h^\star \circ g^\star$$

Functor examples II

- Infinite lists: $X \mapsto X^{\mathbb{N}}$ with: $g^{\mathbb{N}} \colon X^{\mathbb{N}} \to Y^{\mathbb{N}}$, given by:

$$g^{\mathbb{N}}\Big(\langle x_0, x_1, x_2, \ldots, \rangle\Big) = \langle g(x_0), g(x_1), g(x_2), \ldots \rangle$$

- Product with a constant $X \mapsto X \times A$, with:

$$g \times A \colon X \times A \longrightarrow Y \times A \quad \text{given by} \quad (g \times A)(x, a) = (g(x), a)$$

- Coproduct with a constant $X \mapsto X + A$, with

$$(g+A) \colon X+A \longrightarrow Y+A \quad \text{given by} \quad \begin{cases} (g + A)(\kappa_1 x) = \kappa_1 g(x) \\ (g + A)(\kappa_2 a) = \kappa_2 a \end{cases}$$

Definition

Let F be a functor

- An algebra for F is a map $F(X) \to X$
- A coalgebra for F is a map $X \to F(X)$

Definition

- An algebra homomorphism from $F(X) \xrightarrow{a} X$ to $F(Y) \xrightarrow{b} Y$ is a map $f : X \to Y$ for which the rectangle on the left commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(Y) \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle d} \\ X & \xrightarrow{g} & Y \end{array}$$

- A coalgebra homomorphism from $X \xrightarrow{c} F(X)$ to $Y \xrightarrow{d} F(Y)$ is a map $g : X \to Y$ for which the diagram on the right commutes.

Algebra homomorphism example

- Consider the signature functor $F(X) = 1 + (X \times X)$ for monoids
- The natural numbers $\mathbb{N}$ form an algebra, via:

$$1 + (\mathbb{N} \times \mathbb{N}) \xrightarrow{[0,+]} \mathbb{N}$$

- Also lists $A^\star$ for an algebra via empty list $\langle\rangle$ and list concatenation ; in:

$$1 + (A^\star \times A^\star) \xrightarrow{[\langle\rangle,;]} A^\star$$

- There is an algebra homomorphism:

$$\begin{array}{ccc} 1 + (\mathbb{N} \times \mathbb{N}) & \xrightarrow{\mathsf{id}+(f\times f)} & 1 + (A^\star \times A^\star) \\ {\scriptstyle [0,+]}\downarrow & & \downarrow{\scriptstyle [\langle\rangle,;]} \\ \mathbb{N} & \xrightarrow{\quad f \quad} & A^\star \end{array}$$

where $f(n) = \underbrace{\langle a, a, \ldots, a \rangle}_{n \text{ times}}$, for some $a \in A$.

For $A = 1 = \{*\}$ we get $\mathbb{N} \xrightarrow{\cong} 1^\star$

Initiality and finality

Definition

- An algebra $F(A) \xrightarrow{\alpha} A$ is initial if for each algebra $F(X) \xrightarrow{b} X$ there is a unique algebra map:

$$\begin{array}{ccc} F(A) & \overset{F(!_b)}{\dashrightarrow} & F(X) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle b} \\ A & \overset{!_b}{\dashrightarrow} & X \end{array}$$

- A coalgebra $Z \xrightarrow{\zeta} F(Z)$ is final if for each coalgebra $X \xrightarrow{c} F(X)$ there is a unique coalgebra map:

$$\begin{array}{ccc} F(X) & \overset{F(!_c)}{\dashrightarrow} & F(Z) \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle\zeta} \\ X & \overset{!_c}{\dashrightarrow} & Z \end{array}$$

Fixed point results

Fact

- An initial algebra is an isomorphism $F(A) \xrightarrow{\cong} A$
- a final coalgebra is an isomorphism $Z \xrightarrow{\cong} F(Z)$

Consequence: By Cantor's theorem, the powerset functor $\mathcal{P}$ has neither an initial algebra nor a final coalgebra.

Unique existence

- existence is used for definition by (co)induction
- uniqueness is used for reasoning by (co)induction

Induction example, I

Lemma

Fix a set A and consider the functor $F(X) = 1 + (A \times X)$.

The initial algebra of F is given by finite lists $A^\star$ with algebra structure given by empty list (nil) $\langle\rangle$ and prefix $\cdot$ in:

$$1 + (A \times A^\star) \xrightarrow[\cong]{[\langle\rangle, \cdot]} A^\star$$

Proof: The dashed map f in:

$$\begin{array}{ccc} 1 + (A \times A^\star) & \overset{\mathsf{id}+(\mathsf{id}\times f)}{\dashrightarrow} & 1 + (A \times X) \\ {\scriptstyle [\langle\rangle,\cdot]}\downarrow & & \downarrow{\scriptstyle [p,q]} \\ A^\star & \overset{f}{\dashrightarrow} & X \end{array}$$

is given by: $f(\langle\rangle) = p$ and $f(a \cdot \sigma) = q(a, f(\sigma))$.
It is a homomorphism by construction.

Induction example, II

We wish to define the length function $\mathsf{len}\colon A^\star \to \mathbb{N}$ formally, by initiality.

This requires an algebra map ?? on $\mathbb{N}$ in:

$$\begin{array}{ccc} 1 + (A \times A^\star) & \xrightarrow{\mathsf{id}+(\mathsf{id}\times\mathsf{len})} & 1 + (A \times \mathbb{N}) \\ {\scriptstyle [\langle\rangle,\cdot]}\downarrow & & \downarrow{\scriptstyle ??} \\ A^\star & \xrightarrow{\mathsf{len}} & \mathbb{N} \end{array}$$

This ?? must be of the form $[p, q]$, with $p\colon 1 \to \mathbb{N}$ and $q\colon A \times \mathbb{N} \to \mathbb{N}$ satisfy:

$$\mathsf{len}(\langle\rangle) = p \qquad \text{and} \qquad \mathsf{len}(a \cdot \sigma) = q(a, n)$$

Hence it is clear how to choose p, q, namely as:

$$p = 0 \qquad \text{and} \qquad q(a, n) = n + 1 \quad \text{ie. as:} \quad q = \lambda(a, n).\, n + 1$$

Induction example, III

- Consider the functor $F(X) = 1 + (X \times A \times X)$, for a fixed set A
- its operations give binary, A-labeled trees
  - what if we had taken $F(X) = A + (X \times A \times X)$
  - or: $F(X) = B + (X \times A \times X)$
  - or: $F(X) = B + (X \times X \times X)$
- We write its initial algebra as:

$$1 + \big(\mathsf{BinTree}(A) \times A \times \mathsf{BinTree}(A)\big) \xrightarrow[\cong]{[\mathsf{nil},\mathsf{node}]} \mathsf{BinTree}(A)$$

- We do not really need to know what is inside the set $\mathsf{BinTree}(A)$; its universal properties suffice to be able to use it
  - a form of mathematical behaviourism

Induction example, III (cntd)

We sketch how to do tree traversal $\mathrm{BinTree}(A) \to A^\star$, in two ways ("inorder" and "preorder")

$$\begin{array}{ccc} 1 + \big(\mathrm{BinTree}(A) \times A \times \mathrm{BinTree}(A)\big) & \dashrightarrow & 1 + \big(A^\star \times A \times A^\star\big) \\ \cong\downarrow & & \downarrow [\langle\rangle, ??] \\ \mathrm{BinTree}(A) & \dashrightarrow & A^\star \end{array}$$

This map $??: A^\star \times A \times A^\star \longrightarrow A^\star$ is:

$$\begin{cases} \text{for inorder traversal} & (\sigma, a, \tau) \longmapsto \sigma; a \cdot \tau \\ \text{for preorder traversal} & (\sigma, a, \tau) \longmapsto a \cdot \sigma; \tau \end{cases}$$

Summary, so far

- Definitions by induction illustrated via initiality
- Initiality formulation is "generic": it works for every functor / signature
- This genericity is convenient when formalising induction in (functional) programming languages and theorem provers
- Examples of reasoning by induction (via uniqueness) are missing so far, but will be given next, for coinduction

Coinduction example I

Lemma

The final coalgebra of the functor $F(X) = A \times X$ is the set $A^{\mathbb{N}}$ of infinite sequences $\langle a_0, a_1, \ldots \rangle$ of elements $a_i \in A$, with coalgebra structure:

$$A^{\mathbb{N}} \xrightarrow[\cong]{\langle hd, tl \rangle} A \times A^{\mathbb{N}} \qquad \begin{cases} hd(\sigma) = \sigma(0) \\ tl(\sigma) = \langle \sigma(1), \sigma(2), \ldots \rangle \end{cases}$$

Proof: The required unique dashed map f in:

$$\begin{array}{ccc} A \times X & \overset{\mathrm{id} \times f}{\dashrightarrow} & A \times A^{\mathbb{N}} \\ {\scriptstyle \langle p, q \rangle}\uparrow & & \uparrow{\scriptstyle \langle \mathrm{hd}, \mathrm{tl} \rangle} \\ X & \overset{f}{\dashrightarrow} & A^{\mathbb{N}} \end{array}$$

is given by $f(x) = \langle p(x), p(q(x)), p(q^2(x)), p(q^3(x)), \ldots \rangle$.

Coinduction example II

- We wish to define even : $A^{\mathbb{N}} \to A^{\mathbb{N}}$
- This involves a coalgebra $??$ in:

$$\begin{array}{ccc} A \times A^{\mathbb{N}} & \xrightarrow{\mathsf{id} \times \mathsf{even}} & A \times A^{\mathbb{N}} \\ {\scriptstyle ??}\uparrow & & \cong\uparrow{\scriptstyle \langle \mathsf{hd}, \mathsf{tl} \rangle} \\ A^{\mathbb{N}} & \xrightarrow{\mathsf{even}} & A^{\mathbb{N}} \end{array}$$

- If we write $?? = \langle p, q \rangle$, then commutation says:

$$\begin{cases} p(\sigma) = \mathsf{hd}(\mathsf{even}(\sigma)) = \sigma(0) = \mathsf{hd}(\sigma) \\ \mathsf{even}\big(q(\sigma)\big) = \mathsf{tl}(\mathsf{even}(\sigma)) = \langle \sigma(2), \sigma(4), \ldots \rangle \end{cases}$$

- Thus we use:

$$?? = \langle \mathsf{hd}, \mathsf{tl} \circ \mathsf{tl} \rangle$$

Definition by coinduction: from single-step to "and-so-forth"

Coinduction example III

- We also wish $\mathsf{odd}\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$
- Which coalgebra do we need now on the left?

$$\begin{array}{ccc} A \times A^{\mathbb{N}} & \xrightarrow{\mathsf{id}\times\mathsf{odd}} & A \times A^{\mathbb{N}} \\ {\scriptstyle \langle \mathsf{hd}\circ\mathsf{tl},\mathsf{tl}\circ\mathsf{tl}\rangle}\uparrow & & \cong\uparrow{\scriptstyle \langle \mathsf{hd},\mathsf{tl}\rangle} \\ A^{\mathbb{N}} & \xrightarrow{\mathsf{odd}} & A^{\mathbb{N}} \end{array}$$

- How to prove $\mathsf{odd}(\sigma) = \mathsf{even}(\mathsf{tl}(\sigma))$?
- Use uniqueness, and show that $f(\sigma) = \mathsf{even}(\mathsf{tl}(\sigma))$ is also a coalgebra homomorphism in the above diagram!

$$\begin{array}{lll} \mathsf{hd}(f(\sigma)) & = \mathsf{hd}(\mathsf{even}(\mathsf{tl}(\sigma))) & \mathsf{tl}(f(\sigma)) = \mathsf{tl}(\mathsf{even}(\mathsf{tl}(\sigma))) \\ & = \mathsf{hd}(\mathsf{tl}(\sigma)) & \qquad\quad = \mathsf{even}(\mathsf{tl}(\mathsf{tl}(\mathsf{tl}(\sigma)))) \\ & & \qquad\quad = f(\mathsf{tl}(\mathsf{tl}(\sigma))) \end{array}$$

Coinduction example IV

- The next aim is merge: $A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$
- We thus have to provide a coalgebra c in a diagram:

$$\begin{array}{ccc} A \times (A^{\mathbb{N}} \times A^{\mathbb{N}}) & \xrightarrow{\mathsf{id} \times \mathsf{merge}} & A \times A^{\mathbb{N}} \\ {\scriptstyle c}\uparrow & & \cong\uparrow{\scriptstyle \langle \mathsf{hd}, \mathsf{tl} \rangle} \\ A^{\mathbb{N}} \times A^{\mathbb{N}} & \xrightarrow{\mathsf{merge}} & A^{\mathbb{N}} \end{array}$$

- This coalgebra $c \colon A^{\mathbb{N}} \times A^{\mathbb{N}} \longrightarrow A \times (A^{\mathbb{N}} \times A^{\mathbb{N}})$ is:

$$c(\sigma, \tau) = \langle \mathsf{hd}(\sigma), (\tau, \mathsf{tl}(\sigma)) \rangle$$

- Then, for instance:
  - $\mathsf{hd}(\mathsf{merge}(\sigma, \tau)) = \mathsf{hd}(\sigma)$
  - $\mathsf{hd}(\mathsf{tl}(\mathsf{merge}(\sigma, \tau))) = \mathsf{hd}(\mathsf{merge}(\pi_2 c(\sigma, \tau))) = \mathsf{hd}(\mathsf{merge}(\tau, \mathsf{tl}(\sigma))) = \mathsf{hd}(\tau)$
  - $\mathsf{hd}(\mathsf{tl}^2(\mathsf{merge}(\sigma, \tau))) = \cdots = \mathsf{hd}(\mathsf{tl}(\sigma))$ etc.

Coinduction example V

- Now we wish to prove $\mathsf{merge}(\mathsf{even}(\sigma), \mathsf{odd}(\sigma)) = \sigma$
- Obviously, the identity $\mathrm{id}\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ is the unique coalgebra homomorphism g in:

$$\begin{array}{ccc} A \times A^{\mathbb{N}} & \xrightarrow{\mathrm{id} \times g} & A \times A^{\mathbb{N}} \\ {\scriptstyle \langle \mathsf{hd}, \mathsf{tl} \rangle}\uparrow & & \cong \uparrow {\scriptstyle \langle \mathsf{hd}, \mathsf{tl} \rangle} \\ A^{\mathbb{N}} & \xrightarrow{g} & A^{\mathbb{N}} \end{array}$$

- Hence it suffices to prove that the map

$$f(\sigma) = \mathsf{merge}(\mathsf{even}(\sigma), \mathsf{odd}(\sigma))$$

is also such a coalgebra homomorphism.

Coinduction example VI

Thus we calculate:

$$\begin{aligned}
\mathsf{hd}(f(\sigma)) &= \mathsf{hd}(\mathsf{merge}(\mathsf{even}(\sigma), \mathsf{odd}(\sigma))) \\
&= \mathsf{hd}(\mathsf{even}(\sigma)) \\
&= \mathsf{hd}(\sigma)
\end{aligned}$$

$$\begin{aligned}
\mathsf{tl}(f(\sigma)) &= \mathsf{tl}(\mathsf{merge}(\mathsf{even}(\sigma), \mathsf{odd}(\sigma))) \\
&= \mathsf{merge}(\mathsf{odd}(\sigma), \mathsf{tl}(\mathsf{even}(\sigma))) \\
&= \mathsf{merge}(\mathsf{even}(\mathsf{tl}(\sigma)), \mathsf{even}(\mathsf{tl}(\mathsf{tl}(\sigma)))) \\
&= \mathsf{merge}(\mathsf{even}(\mathsf{tl}(\sigma)), \mathsf{odd}(\mathsf{tl}(\sigma))) \\
&= f(\mathsf{tl}(\sigma))
\end{aligned}$$

Conclusion: $\mathsf{id} = f = \lambda\sigma.\, \mathsf{merge}(\mathsf{even}(\sigma), \mathsf{odd}(\sigma))$

Summary on induction and coinduction

- Induction and coinduction, formulated via initiality and finality are each other's duals
- Initiality & finality mechanism is:
  - generic / uniform
  - powerful
  - abstract (can be formulated in other categories than sets)
- Initiality & finality can also be used in binary form; this is the next topic

Bisimilarity

- Formalisation of the intuitive idea: indistinguishability of states, via the available operations
- Several definitions possible, but they all coincide for "reasonable" functors
- Important relation with final coalgebras, and coinductive proof techniques
- Also relevant for Hennessey-Milner result in coalgebraic modal logic: states are bisimilar if and only if they satisfy the same formulas.

Bisimilarity (as behavioural equivalence)

Definition (cospan style)

Assume two coalgebras $X \xrightarrow{c} F(X)$ and $Y \xrightarrow{d} F(Y)$ of the same functor F.

Two states $x \in X$ and $y \in Y$ will be called bisimilar, written as $x \leftrightarrows y$, if there is a third coalgebra $W \xrightarrow{e} F(W)$ with two homomorphisms:

$$\Big(X \xrightarrow{c} F(X)\Big) \xrightarrow{\ f\ } \Big(W \xrightarrow{e} F(W)\Big) \xleftarrow{\ g\ } \Big(Y \xrightarrow{d} F(Y)\Big)$$

satisfying $f(x) = g(y)$ in W.

The definition is often used where coalgebras c and d are the same. This gives bisimilarity for two states of the same coalgebra.

Bisimilarity alternative I

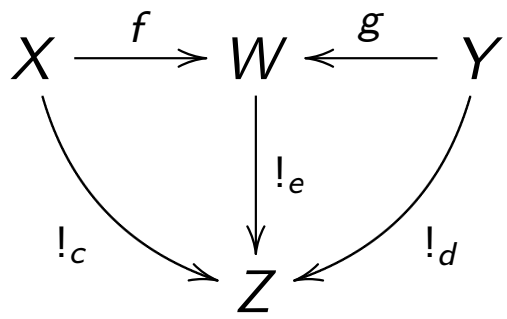
Lemma

Assume F as a final coalgebra $Z \underset{\cong}{\xrightarrow{\zeta}} F(Z)$.

Assume $X \xrightarrow{c} F(X)$, $Y \xrightarrow{d} F(Y)$, and $x \in X$ and $y \in Y$ as before.

$$\begin{aligned} x \leftrightarrows y &\iff !_c(x) = !_d(y) \text{ in } Z \\ &\iff x, y \text{ are mapped to the same element} \\ &\qquad \text{of the final coalgebra} \end{aligned}$$

Proof: ($\Leftarrow$) obvious; ($\Rightarrow$) Assume $f(x) = g(y)$ in $W \xrightarrow{e} F(W)$ in:



By finality:

$$!_e \circ f = !_c \quad !_e \circ g = !_d$$

Hence:

$$!_c(x) = !_e(f(x)) = !_e(g(y)) = !_d(y)$$

Bisimilarity alternative II

Lemma (Bisimilarity as greatest bisimulation)

Assume the functor F is "reasonable" (preserves weak pullbacks)

States $x \in X$ and $y \in Y$ are bisimilar iff the are contained in a bisimulation $R \subseteq X \times Y$

- *such a relation R is closed under the coalgebras*
- *formally, $\langle r_1, r_2 \rangle \colon R \hookrightarrow X \times Y$ carries a coalgebra itself in:*

$$\begin{array}{ccccc} F(X) & \xleftarrow{F(r_1)} & F(R) & \xrightarrow{F(r_2)} & F(Y) \\ {\scriptstyle c}\uparrow & & \uparrow & & \uparrow{\scriptstyle d} \\ X & \xleftarrow{r_1} & R & \xrightarrow{r_2} & Y \end{array}$$

Final coalgebras

- Not all functors have final coalgebras (recall powerset $\mathcal{P}$)
- but under suitable size conditions they do exist (boundedness, accessibility)
- Given such restrictions, there are several constructions:
  - as limits of chains
  - via modal formulas
  - as quotients of known final coalgebras
- Here we do not care very much about how they are built, but focus on its finality and how to use it

Final automaton example I

- Fix two sets: A for inputs, B for outputs
- We consider coalgebras, or deterministic automata:

$$X \xrightarrow{\langle \delta, \epsilon \rangle} X^A \times B$$

- For each state $x \in X$, it gives:

$$\begin{cases} \text{a successor state} & \delta(x)(a) \in X, \text{ for each input } a \in A \\ \text{an observation} & \epsilon(x) \in B \end{cases}$$

- For a finite sequence $\sigma = \langle a_1, \ldots, a_n \rangle \in A^\star$ we put:

$$\delta^\star(x)(\sigma) = \delta\big(\cdots \delta(x)(a_1) \cdots\big)(a_n)$$

This gives multiple-step transitions.

Final automaton example II

Theorem (Arbib & Manes, Reichel)

The final coalgebra of the automaton functor $F(X) = X^A \times B$ *is the set of functions* $B^{A^\star}$ *(from input sequences to observations) with:*

$$B^{A^\star} \xrightarrow[\cong]{\zeta = \langle \zeta_1, \zeta_2 \rangle} \left(B^{A^\star}\right)^A \times B$$

where for a function/state $\varphi \in B^{A^\star}$

$$\begin{cases} \zeta_1(\varphi)(a) = \lambda\sigma \in A^\star.\, \varphi(a \cdot \sigma) \\ \zeta_2(\varphi) = \varphi(\langle\rangle) \end{cases}$$

Final automaton example III

For an arbitrary coalgebra $X \xrightarrow{\langle \delta, \epsilon \rangle} X^A \times B$ we have:

$$\begin{array}{ccc} X^A \times B & \overset{\mathsf{beh}^A \times \mathsf{id}}{\dashrightarrow} & \left(B^{A^\star}\right)^A \times B \\ {\scriptstyle \langle \delta, \epsilon \rangle}\uparrow & & \cong \uparrow {\scriptstyle \zeta = \langle \zeta_1, \zeta_2 \rangle} \\ X & \overset{\mathsf{beh}}{\dashrightarrow} & B^{A^\star} \end{array}$$

where: $\mathsf{beh}(x)(\sigma) = \epsilon\Big(\delta^*(x)(\sigma)\Big)$.

This make the diagram commute:

$$\begin{aligned} \zeta_2(\mathsf{beh}(x)) &= \mathsf{beh}(x)(\langle\rangle) \\ &= \epsilon\Big(\delta^\star(x)(\langle\rangle)\Big) \;=\; \epsilon(x) \\ \zeta_2(\mathsf{beh}(x))(a)(\sigma) &= \mathsf{beh}(x)(a \cdot \sigma) \\ &= \epsilon\Big(\delta^\star(x)(a \cdot \sigma)\Big) \\ &= \epsilon\Big(\delta^\star\big(\delta(x)(a)\big)(\sigma)\Big) \\ &= \mathsf{beh}\big(\delta(x)(a)\big)(\sigma) \;=\; \mathsf{beh}^A\big(\delta(x)\big)(a)(\sigma). \end{aligned}$$

Final automaton example IV

Automaton bisimulations

For a (single) automaton $X \xrightarrow{\langle \delta, \epsilon \rangle} X^A \times B$ a bisimulation is a relation $R \subseteq X \times X$ which is closed under the operations:

- $R(x, x') \Longrightarrow \epsilon(x) = \epsilon(x')$
- $R(x, x') \Longrightarrow R\big(\delta(x)(a), \delta(x')(a)\big)$, for all inputs $a \in A$.

Thus, states within R cannot be distinguished.

Final automaton example V

1. **Special case I**: $A = 1$, so functor is $F(X) = X^1 \times B \cong X \times B$
   - $A^\star = 1^\star \cong \mathbb{N}$
   - final coalgebra consists of infinite sequences / streams $B^{A^\star} \cong B^{\mathbb{N}}$
   - structure map is tail and head $B^{\mathbb{N}} \xrightarrow{\cong} B^{\mathbb{N}} \times B$

2. **Special case I**: $B = 2 = \{0, 1\}$, so functor is $F(X) = X^A \times 2$
   - final coalgebra is $2^{A^\star} = \mathcal{P}(A^\star)$
   - this is the set of languages over alphabet A
   - behaviour map beh : $X \to \mathcal{P}(A^\star)$ maps a state to the accepted language in that state
   - bisimilar states thus accept the same language

Final language coalgebra (special case II)

Write $\mathcal{L}(A) = \mathcal{P}(A^\star) = 2^{A^\star}$ for the final coalgebra of languages over alphabet A

$$\mathcal{L}(A) \xrightarrow[\cong]{\langle \delta, \epsilon \rangle} \mathcal{L}(A)^A \times 2$$

where, for $L \in \mathcal{L}(A)$,

$$\begin{array}{rcll} \delta(L)(a) & = & L_a, & \text{the Brzozowski derivative of } L \\ & = & \{\sigma \in A^\star \mid a \cdot \sigma \in L\} & \\ \epsilon(L) = 1 & \iff & \langle\rangle \in L & \end{array}$$

Two languages $L, K \in \mathcal{L}(A)$ are bisimilar if there is a bisimulation $R \subseteq \mathcal{L}(A) \times \mathcal{L}(A)$; this R satisfies

$$R(L,K) \implies \begin{cases} \langle\rangle \in L \text{ iff } \langle\rangle \in K \\ R(L_a, K_a), \text{ for each } a \in A \end{cases}$$

Regular languages example I

Recall that the subset $\mathcal{R}(A) \subseteq \mathcal{L}(A)$ of regular languages is the smallest one with:

- $0 = \emptyset \in \mathcal{R}(A)$
- $1 = \{\langle\rangle\} \in \mathcal{R}(A)$
- $\{a\} \in \mathcal{R}(A)$, for each $a \in A$ — often simply: $a \in \mathcal{R}(A)$
- $L + K = L \cup K \in \mathcal{R}(A)$, for $L, K \in \mathcal{R}(A)$
- $LK = \{\sigma; \tau \mid \sigma \in L, \tau \in K\} \in \mathcal{R}(A)$, for $L, K \in \mathcal{R}(A)$
- $L^* = \bigcup_{n \in \mathbb{N}} L^n \in \mathcal{R}(A)$, for $L \in \mathcal{R}(A)$

Examples: a^*, $a^* b^*$, $(ab)^*$, $(1 + a)^*$ etc.

Regular languages form a coalgebra

Regular languages are closed under the Brzozowski derivative, via:

$$0_a = 0 \qquad \langle\rangle \notin 0$$

$$1_a = 0 \qquad \langle\rangle \in 1$$

$$b_a = \begin{cases} 1 \text{ if } b = a \\ 0 \text{ otherwise} \end{cases} \qquad \langle\rangle \notin b$$

$$(K + L)_a = K_a + L_a \qquad \langle\rangle \in K + L \text{ iff } \langle\rangle \in K \text{ or } \langle\rangle \in L$$

$$(KL)_a = \begin{cases} K_a L + L_a & \text{if } \langle\rangle \in K \\ K_a L & \text{otherwise} \end{cases} \qquad \langle\rangle \in KL \text{ iff } \langle\rangle \in K \text{ and } \langle\rangle \in L$$

$$\left(K^*\right)_a = K_a K^* \qquad \langle\rangle \in K^*.$$

This yields a coalgebra / automaton

$$\mathcal{R}(A) \xrightarrow{\langle \delta, \epsilon \rangle} \mathcal{R}(A)^A \times 2$$

Equality of regular languages via coinduction

- Now that regular language $\mathcal{R}(A)$ carry a coalgebra, we know what it means that languages $L, K \in \mathcal{R}(A)$ are bisimular
- namely: they are equal, when mapped to the final coalgebra
- But the finality map $\mathcal{R}(A) \longrightarrow \mathcal{L}(A)$ is inclusion, in:

$$\begin{array}{ccc} \mathcal{R}(A)^A \times 2 & \dashrightarrow & \mathcal{L}(A)^A \times 2 \\ \uparrow & & \uparrow\cong \\ \mathcal{R}(A) & \dashrightarrow & \mathcal{L}(A) = \mathcal{P}(A^\star) \end{array}$$

- Thus, for regular languages $L, K \in \mathcal{R}(A)$,

$$L \leftrightarrows R \quad \text{iff} \quad L = K$$

- a new proof method for equality of regular languages
  - the traditional one is cumbersome: it involves either algebraic reasoning, or turning expressions into machines, and then minimalising them

Equality of regular languages, example I

- Assume we wish to prove $(1 + a)^* = a^*$, over alphabet $A = \{a, b\}$
- We need $((1 + a)^*, a^*) \in R$ for a bisimulation $R \subseteq \mathcal{R}(A) \times \mathcal{R}(A)$
- Take the subset relation consisting of 2 pairs only:

$$R = \{((1 + a)^*, a^*)\} \cup \{(0, 0)\}$$

- Assume $(L, K) \in R$;
  - clearly, $\langle\rangle \in L$ iff $\langle\rangle \in K$
  - R is also closed under derivatives

  Obviously, for $L = 0 = K$, so take $L = (1 + a)^*, K = a^*$,

$$\begin{aligned}\big((1 + a)^*\big)_a &= (1 + a)_a(1 + a)^* = (1_a + a_a)(1 + a)^* \\ &= (0 + 1)(1 + a)^* = (1 + a)^*\end{aligned}$$

$$\big(a^*\big)_a = a_a a^* = 1a^* = a^*$$

$$\big((1 + a)^*\big)_b = 0 = \big(a^*\big)_b$$

Equality of regular languages, example II

- Aim: $(a+b)^* = (a^*b)^*a^*$, over alphabet $A = \{a, b\}$
- Take as relation $R = \{(\,(a+b)^*, (a^*b)^*a^*\,)\}$
- Clearly, for $(L, K) \in R$, $\langle\rangle \in L \Leftrightarrow \text{true} \Leftrightarrow \langle\rangle \in K$
- R is also closed under derivatives. Eg. for $(-)_b$:

$$\begin{aligned}\big((a+b)^*\big)_b &= (a+b)_b(a+b)^* \;=\; (a_b + b_b)(a+b)^* \\ &= (0+1)(a+b)^* \;=\; (a+b)^*\end{aligned}$$

$$\begin{aligned}\big((a^*b)^*a^*\big)_b &= \big((a^*b)^*\big)_b a^* + \big(a^*\big)_b \;=\; (a^*b)_b(a^*b)^*a^* + a_b a^* \\ &= ((a^*)_b b + b_b)(a^*b)^*a^* + 0a^* \\ &= (0+1)(a^*b)^*a^* + 0 \;=\; (a^*b)^*a^*.\end{aligned}$$

- Similarly for the derivative $(-)_a$. Hence R is a bisimulation, and the goal is proven.

Equality of regular languages, example III

- Non-trivial example:

$$a^* + a^* b(a + ba^* b)^* ba^* = \{\sigma \in A^\star \mid \sigma \text{ contains an even number of } b\text{'s}\}$$

- The proof is non-trivial, but essentially straightforward via bisimulations
- This technique is now incorporated in a tool (CIRC, based on Maude)
- Many more examples of such equality proof via bisimulations in the work of Jan Rutten

Prerequisites for quantum computation

- Do you need to be a quantum physicist to understand quantum computation?

- NO!

  > *One can be a masterful practioner of computer science without having the foggiest notion of what a transistor is, not to mention how it works*

  (David Mermin, Quantum Computer Science. An Introduction.)

- Here, as usual, we only deal with the abstract, mathematical model for quantum computation, not with its possible future realisation — which is work for physicists.

What is needed?

Main prerequisites for quantum computation

- Linear algebra (over complex numbers $\mathbb{C}$)
- Hilbert spaces (esp. finite dimensional ones)
- Tensors and spectral decomposition

Prerequisites for this talk

- Basic linear algebra
- Basic category theory

What is quantum computing about?

- Letting nature do matrix multiplications for us
- Computations are divided over multiple parallel worlds ("quantum parallellism")
- A new physical basis for computation
- Much focus so far on algorithms and complexity
- More recently also on semantics and quantum languages (Abramsky, Coeke, Panangaden, Selinger, ...)
- Actual, physical realisation of quantum computer still embryonic (in the order of 10 qubits)
- Quantum key distribution more developed (but also under attack, see *quantum hackers*)

Quantum mechanics/computation is strange

The four strangest phenomena

- Superposition: linear combinations of basic states
- Entanglement of states: explained via tensors $\otimes$
- Computation is reversible: explained via unitary maps
  (via daggers $(-)^\dagger$ ie. adjoints)
- Measurement
  - side-effect: the state changes to the result of the measurement
  - entangled objects are both changed when only one is measured

  Mathematical explanation via diagonalisation of operators
  (using eigenvectors & eigenvalues)

Quantum computation is powerful

- More efficient in certain tasks, via a new form of parallellism
  - Notably factorisation of numbers (Shor); threatening for RSA;
- New levels of security, e.g. with key exchange via entanglement

Attractions

- New area inbetween computer science, physics and logic
  - John Baez: category theory forms *Rosetta Stone*
- Great potential for both theory and applications
  - Chance to get theory in place before (programming) languages emerge
  - Early in CS: languages came before theory
- Issues are both familiar and new
  - See later discussion of 'walks'

Dirac notation for vectors

- Following Dirac, physicists use the notation $|\, x \,\rangle$ to denote an arbitrary vector. This is convenient with inner/outer products.
- Binary numbers inside $|\, - \,\rangle$ are often used for base vectors.
- For example,
  - In $\mathbb{C}^2$ one may read:

$$|\, 0 \,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\, 1 \,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  - And in $\mathbb{C}^4$,

$$|\, 00 \,\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\, 01 \,\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\, 10 \,\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\, 11 \,\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

  - Etc.

Quantum bits and classical bits

Quantum bit (or qubit)

A Qubit is a unit vector in $\mathbb{C}^2$, written as:

$$\alpha|0\rangle + \beta|1\rangle \quad \text{with} \quad \alpha, \beta \in \mathbb{C} \quad \text{satisfying} \quad |\alpha|^2 + |\beta|^2 = 1.$$

Thus, a qubit is a superposition of $|0\rangle$ and $|1\rangle$.
The α, β are called probability amplitudes.

Classical bit (or cbit)

A cbit is either 0 or 1.

More formally, it is a unit vector over Booleans:

$$\alpha|0\rangle + \beta|1\rangle \quad \text{with} \quad \alpha, \beta \in \mathbb{B} \quad \text{satisfying} \quad \alpha \text{ XOR } \beta = 1.$$

Measurement, informally

Given a qubit in state $\alpha|0\rangle + \beta|1\rangle$ one can measure it, and:

- obtain either $|0\rangle$ or $|1\rangle$ as successor state (side effect)
- the probability of getting $|0\rangle$ is $|\alpha|^2 \in [0, 1]$
- the probability of getting $|1\rangle$ is $|\beta|^2 \in [0, 1]$

  (Recall: $|\alpha|^2 + |\beta|^2 = 1$.)

> Note The state (α, β) itself remains hidden

Measurement can be done with respect to any basis.

Two products

Given vector spaces V, W we need:

- The biproduct $V \times W$, with elements (v, w)
- The tensor product $V \otimes W$, with elements $v \otimes w$

Tensor distributes over biproducts:

$$W \otimes (V \times U) \cong (W \otimes V) \times (W \otimes U).$$

In the finite dimensional case:

$$dim(V \times W) = dim(V) + dim(W) \qquad dim(V \otimes W) = dim(V) \cdot dim(W)$$

Biproducts

The biproduct $V \times W$ of vector/Hilbert spaces is both a categorical product and coproduct (sum), with projections:

$$\begin{array}{ccccc} V & \xleftarrow{\pi_1} & V \times W & \xrightarrow{\pi_2} & W \\ v & \longleftarrow\!\shortmid & (v, w) & \longmapsto & w \end{array}$$

and coprojections (injections):

$$\begin{array}{ccccc} V & \xrightarrow{\kappa_2} & V \times W & \xleftarrow{\kappa_2} & W \\ v & \longmapsto & (v, 0) & & \\ & & (0, w) & \longleftarrow\!\shortmid & w \end{array}$$

The singleton space $\{0\}$ is unit element for $\times$, ie. $\{0\} \times V \cong V$.

Tensor products

The tensor product $V \otimes W$ contains linear combinations of pairs $v \otimes w$. The mapping $\otimes\colon V \times W \to V \otimes W$ is "bilinear" (and universal).

$$\text{If} \begin{cases} V \text{ has basis } |\, a_1 \,\rangle, \ldots, |\, a_n \,\rangle \\ W \text{ has basis } |\, b_1 \,\rangle, \ldots, |\, b_m \,\rangle, \end{cases}$$

then $V \otimes W$ has a basis given by:

$$|\, a_i b_j \,\rangle = |\, a_i \,\rangle \otimes |\, b_j \,\rangle \qquad \text{for } \; i \leq n, j \leq m.$$

The scalars $\mathbb{C}$ are unit element for $\otimes$, ie. $\mathbb{C} \otimes V \cong V$.
$\otimes$ is used for parallel composition, possibly with entanglement.

Tensor example

- A single qubit lives in $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$, with basis $|0\rangle, |1\rangle$
- Two qubits in parallel live in $\mathbb{C}^2 \otimes \mathbb{C}^2$, for which:

$$\begin{aligned}\mathbb{C}^2 \otimes \mathbb{C}^2 &= \big(\mathbb{C} \times \mathbb{C}\big) \otimes \big(\mathbb{C} \times \mathbb{C}\big) \\ &\cong \big(\mathbb{C} \otimes \mathbb{C}\big) \times \big(\mathbb{C} \otimes \mathbb{C}\big) \times \big(\mathbb{C} \otimes \mathbb{C}\big) \times \big(\mathbb{C} \otimes \mathbb{C}\big) \\ &\cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \\ &= \mathbb{C}^4\end{aligned}$$

  with basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, where $|00\rangle = |0\rangle \otimes |0\rangle$ etc.
- n qubits live in $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^n}$, with base vectors

$$|\, b_{n-1} \ldots b_0 \,\rangle \qquad \text{for} \quad b_i \in \{0, 1\}.$$

Putting qubits together

Two qubits $\alpha_0|0\rangle + \alpha_1|1\rangle$ and $\beta_0|0\rangle + \beta_1|1\rangle$ can be put in parallel, in $\mathbb{C}^2 \otimes \mathbb{C}^2$, as:

$$\begin{aligned}&\big(\alpha_0|0\rangle + \alpha_1|1\rangle\big) \otimes \big(\beta_0|0\rangle + \beta_1|1\rangle\big)\\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle.\end{aligned}$$

However, an arbitrary element in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is of the form:

$$\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$$

It captures two entangled qubits, since in general such separate α's and β's cannot be recovered from γ's.

Two basic results

LEMMA (separability/disentanglement condition)

An arbitrary element

$$\gamma_{00}|\,00\,\rangle + \gamma_{01}|\,01\,\rangle + \gamma_{10}|\,10\,\rangle + \gamma_{11}|\,11\,\rangle \;\in\; \mathbb{C}^2 \otimes \mathbb{C}^2$$

with $|\gamma_{00}|^2 + |\gamma_{01}|^2 + |\gamma_{10}|^2 + |\gamma_{11}|^2 = 1$ is of non-entangled form:

$$\big(\alpha_0|\,0\,\rangle + \alpha_1|\,1\,\rangle\big) \otimes \big(\beta_0|\,0\,\rangle + \beta_1|\,1\,\rangle\big)$$

if and only if

$$\gamma_{00}\gamma_{11} = \gamma_{01}\gamma_{10}.$$

LEMMA Classical bits can always be separated:

(Since classically, only one of the γ_{ij} is 1; the others are 0)

Recap on coalgebras

- Mathematical models for state-based computation:
  - state space, say X
  - transition map $X \to F(X)$
  - F is a functor that fixes the type of computation
- A deterministic automaton involves a pair of maps:

$$X \xrightarrow{\langle \text{step}, \text{final?} \rangle} X^A \times 2$$

  where $2 = \{0, 1\}$.
- A non-deterministic automaton, or transition system, with input actions A is a coalgebra:

$$X \xrightarrow{\text{succs}} \mathcal{P}(X)^A$$

- By replacing powerset $\mathcal{P}$ by distribution functor $\mathcal{D}$ one obtains probabilistic automata (aka. Markov chain)

Coalgebras of monads

- In this context we use coalgebras $X \to T(X)$ where T is a monad: a special functor with *unit* and *mulitplication*
- As a result, coalgebras can be composed
  - More formally: the set of coalgebras $(TX)^X$ forms a monoid
  - Even more formally: a monoid in the category of T-algebras
- Main monad examples:
  - Powerset $\mathcal{P}$
  - Distribution $\mathcal{D}$
  - Multiset $\mathcal{M}$

Distribution monad $\mathcal{D}$

For a set X, define

$$\mathcal{D}(X) \;=\; \{\varphi \colon X \to [0,1] \;\big|\; \text{support}(\varphi) \text{ is finite, and } \textstyle\sum_x \varphi(x) = 1\}$$

Such $\varphi \in \mathcal{D}(X)$ is a formal convex combination:

$$r_1 x_1 + \cdots + r_n x_n \quad \text{where} \quad \begin{cases} \text{support}(\varphi) = \{x_1, \ldots, x_n\} \\ r_i = \varphi(x_i) > 0 \\ r_1 + \cdots + r_n = 1 \end{cases}$$

Coalgebras $X \to \mathcal{D}(X)$ are Markov chains, giving probabilistic transitions:

$$x \xrightarrow{r_i} x_i \quad \text{with} \quad \textstyle\sum_i r_i = 1.$$

Multiset monad $\mathcal{M}$

The mulitset monad $\mathcal{M}$ over complex numbers is similar to, but simpler than, the distribution monad $\mathcal{D}$. For a set X, now define

$$\mathcal{M}(X) \;=\; \{\varphi \colon X \to \mathbb{C} \;\big|\; \text{support}(\varphi) \text{ is finite } \}$$

Such $\varphi \in \mathcal{M}(X)$ is a formal linear combination:

$$z_1 x_1 + \cdots + z_n x_n \quad \text{where} \quad \begin{cases} \text{support}(\varphi) = \{x_1, \ldots, x_n\} \\ z_i = \varphi(x_i) \in \mathbb{C} \end{cases}$$

Such formal linear combinations form the free vector space on X.

Coalgebras $X \to \mathcal{M}(X)$ are like weighted automata, adding weights/resources in $\mathbb{C}$ as labels to transitions.

Starting to walk (for the classically educated)

Tales from the Ministry of Coalgebraic Walks

Walk the walk

Consider a line of integer points $\ldots, -2, -1, 0, 1, 2, \ldots \in \mathbb{Z}$. Different styles of walks, say of a drunkard, on this line will be described next:

- non-deterministic
- probabilistic
- quantum

All three computational styles can & will be represented coalgebraically.

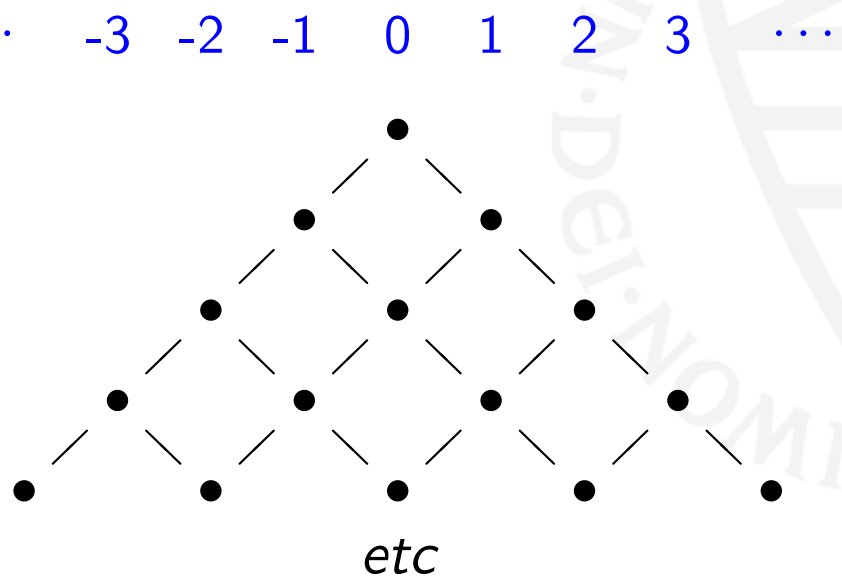
Non-deterministic walks: definition

Coalgebraic represenation, of possible left-or-right stepping:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad s \quad} & \mathcal{P}(\mathbb{Z}) \\ k & \longmapsto & \{k-1, k+1\} \end{array}$$

Iteration, starting in $0 \in \mathbb{Z}$, yields:

$$\begin{array}{rl} 0 & \mapsto \{-1, 1\} \\ & \mapsto \{-2, 0, 2\} \\ & \mapsto \{-3, -1, 1, 3\} \\ & \cdots \{-n, -n+2, \cdots n-2, n\} \end{array}$$



Non-deterministic walks: iteration

Formally, iteration is done via the Kleisli extension endomap:

$$\begin{array}{ccc} \mathcal{P}(\mathbb{Z}) & \xrightarrow{\quad s^{\#} \quad} & \mathcal{P}(\mathbb{Z}) \\ U & \longmapsto & \bigcup_{k \in U} \{k-1, k+1\} \end{array}$$

The subset of successors of $0 \in \mathbb{Z}$, after n steps, is obtained as the n-th iterate:

$$\left(s^{\#}\right)^{n}(\{0\}) \quad = \quad \{-n, -n+2, \cdots n-2, n\}.$$

Aside: categorically, this can be described directly as iteration in the Kleisli category of the powerset monad $\mathcal{P}$.

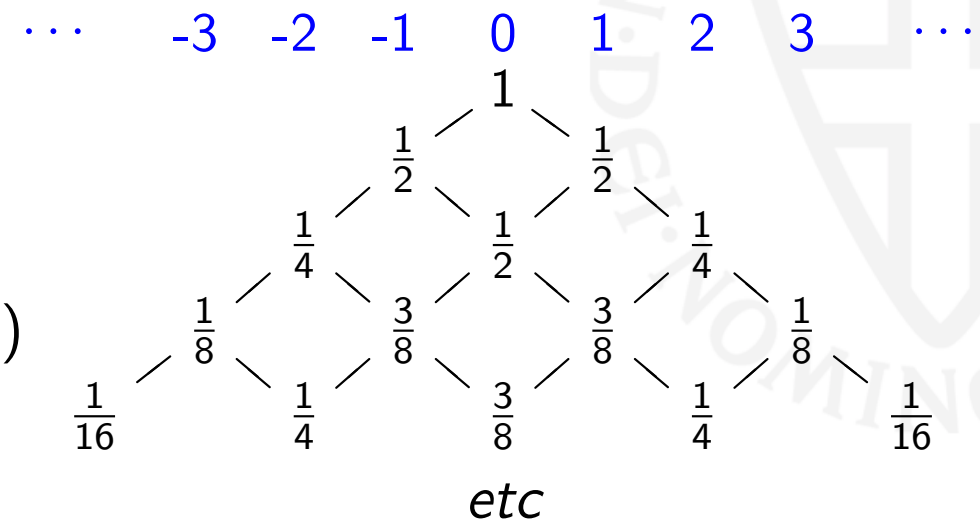
Probabilistic walks: definition

Probabilistic left-or-right stepping, each with chance $\frac{1}{2}$ is expressed via a formal convex sum / distribution, as:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad d \quad} & \mathcal{D}(\mathbb{Z}) \\ k & \longmapsto & \frac{1}{2}(k-1) + \frac{1}{2}(k+1) \end{array}$$

Iteration, starting in $0 \in \mathbb{Z}$, now yields:

$$\begin{aligned} 0 &\mapsto \tfrac{1}{2}(\text{-}1) + \tfrac{1}{2}(1) \\ &\mapsto \tfrac{1}{4}(\text{-}2) + \tfrac{1}{2}(0) + \tfrac{1}{4}(2) \\ &\mapsto \tfrac{1}{8}(\text{-}3) + \tfrac{3}{8}(\text{-}1) + \tfrac{3}{8}(1) + \tfrac{1}{8}(3) \end{aligned}$$



Probabilistic walks: the general formula

This tree of probabilities involves Pascal's triangle.

Starting in $k \in \mathbb{Z}$, after n iterations one obtains the formal convex sum:

$$\frac{\binom{n}{0}}{2^n}(k-n)+\frac{\binom{n}{1}}{2^n}(k-n+2)+\frac{\binom{n}{2}}{2^n}(k-n+4)+\ldots+\frac{\binom{n}{n-1}}{2^n}(k+n-2)+\frac{\binom{n}{n}}{2^n}(k+n)$$

Using the sum formula for binomial coefficients:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Probabilistic walks: iteration

Again there is a Kleisli extension endomap $d^{\#} : \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$

$$\begin{array}{l} \Big(r_1 k_1 + \cdots + r_n k_n\Big) \\ \longmapsto \Big(\frac{1}{2} r_1(k_1 - 1) + \frac{1}{2} r_1(k_1 + 1) + \cdots + \frac{1}{2} r_n(k_n - 1) + \frac{1}{2} r_n(k_n + 1)\Big) \end{array}$$

The subset of successors of $0 \in \mathbb{Z}$, after n steps, is obtained as the n-th iterate:

$$\Big(d^{\#}\Big)^n(1\,0) \;=\; \frac{\binom{n}{0}}{2^n}(-n) + \frac{\binom{n}{1}}{2^n}(-n+2) + \ldots + \frac{\binom{n}{n-1}}{2^n}(n-2) + \frac{\binom{n}{n}}{2^n}(n)$$

It is iteration in the Kleisli category of the distribution monad $\mathcal{D}$.

Quantum walks

Situation / plan

- Quantum walks are standardly described as endomap $S \to S$ (see eg. work of Julia Kempe)
- Here it will be shown that there is also an (equivalent) coalgebraic / monadic description.

Quantum walks: endomap definition I

Two vector spaces

- $\mathcal{M}(\mathbb{Z})$, the free vector space on $\mathbb{Z}$ (over $\mathbb{C}$), with base vectors written as: $|\, k \,\rangle \in \mathcal{M}(\mathbb{Z})$, for $k \in \mathbb{Z}$
- $\mathbb{C}^2$, the one qubit space, with base vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ (think of the direction of the walk)

The state space is $\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$, with basis $\left\{ \begin{array}{l} |\uparrow\rangle \otimes |\, k \,\rangle \\ |\downarrow\rangle \otimes |\, k \,\rangle \end{array} \right.$

Quantum walks: endomap definition II

The relevant endomap is:

$$\begin{array}{ccc} \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) & \xrightarrow{\quad q \quad} & \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \\ |\uparrow\rangle \otimes |k\rangle & \longmapsto & \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |k-1\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |k+1\rangle \\ |\downarrow\rangle \otimes |k\rangle & \longmapsto & \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |k-1\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |k+1\rangle \end{array}$$

Implicitly, the Hadamard operator $H \colon \mathbb{C}^2 \to \mathbb{C}^2$ is used:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Quantum walks: iteration and probabilities I

Obtaining probabilities in quantum walks

- We are interested in the probability of reaching $|\, k \,\rangle$ after n steps. Starting from 0 — more precisely, from $|\uparrow\rangle \otimes |\, 0 \,\rangle$.
- We do so by n-times iterating the endomap q, and then measuring in the basis $|\, k \,\rangle$.
- The sum of norms $|\alpha_i|^2$ of amplitudes α_i for $|\, k \,\rangle$ then gives the probability.

Quantum walks: iteration and probabilities II

$$|\uparrow\rangle \otimes |0\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}|\uparrow\rangle \otimes |\text{-}1\rangle + \tfrac{1}{\sqrt{2}}|\downarrow\rangle \otimes |1\rangle$$

$$\text{probabilities} \begin{cases} |\text{-}1\rangle & |\tfrac{1}{\sqrt{2}}|^2 = \tfrac{1}{2} \\ |1\rangle & |\tfrac{1}{\sqrt{2}}|^2 = \tfrac{1}{2} \end{cases}$$

$$\mapsto\; \tfrac{1}{2}|\uparrow\rangle \otimes |\text{-}2\rangle + \tfrac{1}{2}|\downarrow\rangle \otimes |0\rangle + \tfrac{1}{2}|\uparrow\rangle \otimes |0\rangle - \tfrac{1}{2}|\downarrow\rangle \otimes |2\rangle$$

$$\text{probabilities} \begin{cases} |\text{-}2\rangle & |\tfrac{1}{2}|^2 = \tfrac{1}{4} \\ |0\rangle & |\tfrac{1}{2}|^2 + |\tfrac{1}{2}|^2 = \tfrac{1}{2} \\ |2\rangle & |-\tfrac{1}{2}|^2 = \tfrac{1}{4} \end{cases}$$

$$\mapsto\; \cdots \text{(next page)}$$

Quantum walks: iteration and probabilities II

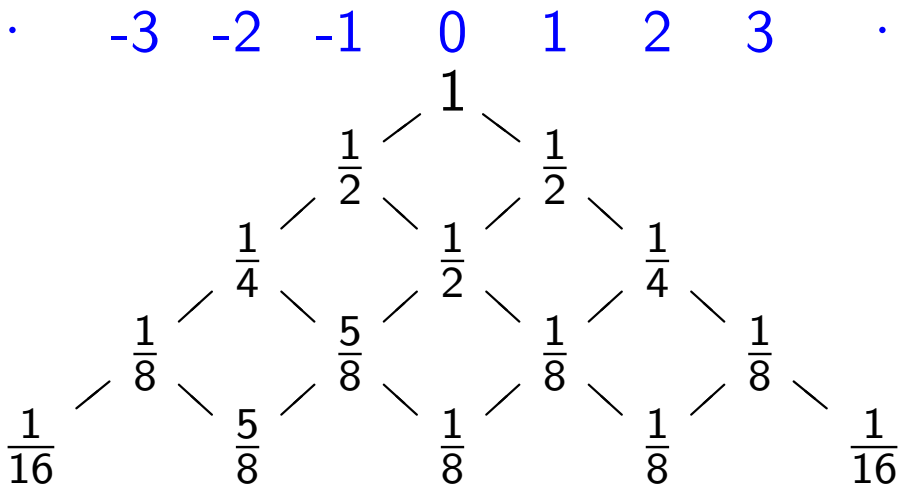
$$
\begin{aligned}
\cdots \mapsto\ & \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |\text{-}3\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |\text{-}1\rangle + \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |\text{-}1\rangle \\
& - \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |1\rangle + \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |\text{-}1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |1\rangle \\
& - \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |3\rangle \\
= \ & \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |\text{-}3\rangle + \tfrac{1}{\sqrt{2}}|\uparrow\rangle \otimes |\text{-}1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |\text{-}1\rangle \\
& - \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |3\rangle
\end{aligned}
$$

$$
\text{probabilities} \begin{cases} |\text{-}3\rangle & |\frac{1}{2\sqrt{2}}|^2 = \frac{1}{8} \\ |\text{-}1\rangle & |\frac{1}{\sqrt{2}}|^2 + |\frac{1}{2\sqrt{2}}|^2 = \frac{5}{8} \\ |1\rangle & |-\frac{1}{2\sqrt{2}}|^2 = \frac{1}{8} \\ |3\rangle & |\frac{1}{2\sqrt{2}}|^2 = \frac{1}{8} \end{cases}
$$

There is "drift to the left" due to interference.

Quantum walks: iteration and probabilities III

The resulting tree of quantum walk probabilities starts as:



The matrix involved — Hadamard's H in this case — determines the drifting, and thus how the tree is traversed. This may yield optimisations in data processing.

Basic isomorphism

LEMMA $\mathbb{C}^2 \otimes \mathcal{M}(X) \cong \mathcal{M}(X + X)$ where $+$ is disjoint union

Proof By the following chain of isomorphisms:

$$\begin{aligned}
\mathbb{C}^2 \otimes \mathcal{M}(X) &= (\mathbb{C} \oplus \mathbb{C}) \otimes \mathcal{M}(X) \\
&\cong C \otimes \mathcal{M}(X) \oplus \mathbb{C} \otimes \mathcal{M}(X) \\
&\qquad \text{by distributivity} \\
&\cong \mathcal{M}(X) \oplus \mathcal{M}(X) \\
&\qquad \text{since } \mathbb{C} \text{ is tensor unit} \\
&\cong \mathcal{M}(X + X) \\
&\qquad \oplus \text{ is also coproduct of spaces,} \\
&\qquad \text{and } \mathcal{M} \text{ is a free functor.}
\end{aligned}$$

Quantum endomorphism as coalgebra

THEOREM Linear maps (in $\mathbf{Vect}_{\mathbb{C}}$)

$$\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \longrightarrow \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$$

correspond bijectively to functions (in **Sets**)

$$\mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})^2$$

that is, to coalgebras with state space $\mathbb{Z}$ of the functor $\mathcal{M}(2 \cdot -)^2$.

Proof

$$\frac{\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \longrightarrow \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})}{\mathcal{M}(\mathbb{Z} + \mathbb{Z}) \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})} \quad \text{linear}$$

$$\frac{\mathcal{M}(\mathbb{Z} + \mathbb{Z}) \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})}{\mathbb{Z} + \mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})} \quad \text{linear}$$

$$\frac{\mathbb{Z} + \mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})}{\mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})^2}$$

Qubit action is monadic!

In the background, there is a new monad transformer result.

LEMMA If T is a monad, then so is $X \mapsto T(n \cdot X)^n$, for each $n \in \mathbb{N}$

For quantum walks we use $T = \mathcal{M}$ and $n = 2$.

Three categories for reversible computation

- Reversible computation will be described in three categories:

BifRel	sets with bifinite relations, for non-deteministic computation
dBisRel	sets with discrete bistochastic relations, for probabilistic computation
BifMRel	sets with bifinite multirelations, for quantum computation

- Each of the three walks will be described there
- Only the quantum walks turn out to be reversible

Bifinite relations

- The category **BifRel** is defined as follows
  - object are sets X, and
  - maps $X \to Y$ are relations $r \colon X \times Y \to 2 = \{0, 1\}$ such that:
    - for each $x \in X$ the set $\{y \in Y \mid r(x, y) \neq 0\}$ is finite;
    - also, for each $y \in Y$ the set $\{x \in X \mid r(x, y) \neq 0\}$ is finite.
- Thus: such r factors both as:

$$X \longrightarrow \mathcal{P}_{fin}(Y) \qquad \text{and as} \qquad Y \longrightarrow \mathcal{P}_{fin}(X)$$

- For a map $r \colon X \to Y$ there is $r^\dagger \colon Y \to X$ in the reverse direction, via argument swapping: $r^\dagger(y, x) = r(x, y)$.
- This makes **BifRel** a dagger category.

Non-deterministic walks in **BifRel**

- The non-deterministic walks $\mathbb{Z} \to \mathcal{P}_{fin}(\mathbb{Z})$ form an endomap:

$$\mathbb{Z} \xrightarrow{s} \mathbb{Z} \qquad \text{in } \mathbf{BifRel}$$

- Namely, $s(n, m) = 1$ iff $m = n - 1$ or $m = n + 1$
- Its reverse is: $s^\dagger(m, n) = 1 \iff s(n, m) = 1$
$$\iff m = n - 1 \text{ or } m = n + 1$$
$$\iff n = m - 1 \text{ or } n = m + 1$$
$$\iff s(n, m) = 1$$
- But s is not unitary: ie. $s^\dagger \neq s^{-1}$
- For instance:

$$(s \circ s^\dagger)(n, n') = 1 \iff n' = n - 2 \text{ or } n' = n \text{ or } n' = n + 2$$

(The identity $\mathbb{Z} \to \mathbb{Z}$ in **BifRel** is $\mathrm{id}(n, n') = 1$ iff $n = n'$)

Discrete bistochastic relations

- The category **dBisRel** is defined as follows
  - object are sets X, and
  - maps $X \to Y$ are functions $r \colon X \times Y \to [0,1]$ such that:
    - for each x there are finitely many y with $r(x,y) \neq 0$ and $\sum_y r(x,y) = 1$
    - also, for each $y \in Y$ the set $\{x \in X \mid r(x,y) \neq 0\}$ is finite and $\sum_x r(x,y) = 1$
- we get discrete probability distributions in two directions:

$$X \longrightarrow \mathcal{D}(Y) \qquad \text{and as} \qquad Y \longrightarrow \mathcal{D}(X)$$

- For a map $r \colon X \to Y$ in **dBisRel** there is again $r^\dagger \colon Y \to X$ via argument swapping: $r^\dagger(y,x) = r(x,y)$.
- This makes also **dBisRel** a dagger category.

Probabilistic walks in **dBisRel**

- The probabilistic walks $\mathbb{Z} \to \mathcal{D}(\mathbb{Z})$ form an endomap:

$$\mathbb{Z} \xrightarrow{d} \mathbb{Z} \qquad \text{in } \mathbf{dBisRel}$$

- Namely, $d(n,m) = \begin{cases} \frac{1}{2} & \text{if } m = n-1 \text{ or } m = n+1 \\ 0 & \text{otherwise.} \end{cases}$
- Its reverse is: $d^\dagger(m,n) = d(n,m)$
- Also d is not unitary: ie. $s^\dagger \neq s^{-1}$
- For instance:

$$(d \circ d^\dagger)(n,n') = \begin{cases} \frac{1}{4} & \text{if } n' = n-2 \text{ or } n' = n+2 \\ \frac{1}{2} & \text{if } n' = n \\ 0 & \text{otherwise.} \end{cases}$$

(The identity $\mathbb{Z} \to \mathbb{Z}$ in **dBisRel** is $\mathrm{id}(n,n') = 1$ iff $n = n'$)

Bifinite multirelations

- The category **BifMRel** is defined as follows
  - object are sets X, and
  - maps $X \to Y$ are relations $r \colon X \times Y \to \mathbb{C}$ such that:
    - for each $x \in X$ the set $\{y \in Y \mid r(x,y) \neq 0\}$ is finite;
    - also, for each $y \in Y$ the set $\{x \in X \mid r(x,y) \neq 0\}$ is finite.
- Thus: such r factors both as:

$$X \longrightarrow \mathcal{M}(Y) \qquad \text{and as} \qquad Y \longrightarrow \mathcal{M}(X)$$

- For a map $r \colon X \to Y$ there is $r^\dagger \colon Y \to X$ again via: $r^\dagger(y,x) = r(x,y)$.
- This makes **BifMRel** a dagger category.

Probabilistic walks in **dBisRel**

- The quantum walks $\mathbb{Z} \to \mathcal{M}(\mathbb{Z}+\mathbb{Z})^2$ form an endomap:

$$\mathbb{Z}+\mathbb{Z} \xrightarrow{q} \mathbb{Z}+\mathbb{Z} \qquad \text{in } \mathbf{BifMRel}$$

- Namely,

$$\begin{cases} q(\kappa_1 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_1 n, \kappa_2(n+1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_2(n+1)) = -\frac{1}{\sqrt{2}} \end{cases}$$

Recall Hadamard

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Its reverse is: $q^\dagger(u,v) = q(v,u)$
- This time: q is is unitary: ie. $q^\dagger \neq q^{-1}$

One of the four verifications

$$
\begin{aligned}
\big(q \circ q^\dagger\big)(\kappa_1 m, \kappa_1 n) &= \textstyle\sum_x q^\dagger(\kappa_1 m, x) \cdot q(x, \kappa_1 n) \\
&= q^\dagger(\kappa_1 m, \kappa_1(m+1)) \cdot q(\kappa_1(m+1), \kappa_1 n) \\
&\qquad + q^\dagger(\kappa_1 m, \kappa_2(m+1)) \cdot q(\kappa_2(m+1), \kappa_1 n) \\
&= \begin{cases} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \kappa_1 n = \kappa_1 m \\ 0 & \text{otherwise} \end{cases} \\
&= \mathrm{id}(\kappa_1 m, \kappa_1 n).
\end{aligned}
$$

Future work

Better understand the three categories:

$$\mathbf{BifRel} \longrightarrow \mathbf{dBisRel} \longrightarrow \mathbf{BifMRel}$$

and see what structure (categorical and logical) is typical for which kind of computing.

Thanks for your attention; hopefully you feel inspired