Introduction to algorithmic mechanism design

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Profit maximization in auctions

\[ \nu_1 = 12 \]
\[ \nu_2 = 7 \]
\[ \nu_3 = 6 \]

Objective: Maximize revenue
Vickrey auction: Give the item to the highest bidder (bidder 1). The winner pays the second highest bid (payment=7).

\[ v_1 = 12 \]
\[ v_2 = 7 \]
\[ v_3 = 6 \]
A better auction for maximizing revenue?

- The revenue of the Vickrey auction may be far from optimal
  - For example, when $v_1 = 1000$, $v_2 = 10$
- On the other hand, first-price auction can be strategically manipulated
  - In the example, the first player will bid $\tilde{v}_1 = 11$ (assuming complete information for the bidders)
Because the auctioneer has incomplete information, it is not even clear how to formulate the optimization problem of maximizing revenue.

There are two approaches to address this issue:

1. **Worst-case approach**
   - We make as few assumptions as necessary and compute the optimal solutions assuming an adversarial setting.
   - **Advantage:** Few assumptions
   - **Disadvantage:** Pessimistic

2. **Bayesian approach**
   - We assume that the unknown parameters are drawn from publicly-known probability distributions.
   - **Advantage:** Reasonable and “practical”
   - **Disadvantage:** Based on arbitrary strong assumptions
The Bayesian setting for single item auctions

- The value of bidder $i$ comes from a probability distribution $F_i$.
- All the distributions are known by the bidders and the auction designer.
- For example, there are $n$ bidders with values independently drawn from the uniform distribution $U[0, 1]$. 
One bidder, single item

- Assume that we have a single bidder with value $v \sim F$
- Deterministic auctions: Post a price $p$
  - If $v \geq p$, the bidder gets the item and pays $p$
  - Otherwise, the bidder does not get the item and pays 0
- What is the optimal $p$?
  - $p$ maximizes $p(1 - F(p))$
  - For example, for the uniform distribution $U[0,1]$: $p = 1/2$ and the expected revenue is $1/4$
First price auction?

- Assume \( n \) bidders with values drawn from distributions \( F_i \)
- What is a **Bayesian Nash equilibrium** for the first-price auction?
- Not an easy answer
- For example, with 2 bidders with \( v_1, v_2 \sim U[0, 1] \):
  - Each bidder lies and declares half of her actual value:
    \[ b_i(v_i) = \frac{v_i}{2} \]
  - Even with lies, the highest bid gets the item!
Revenue equivalence principle

- Assume 2 bidders with values drawn from distribution $U[0, 1]$
- With the first-price auction the expected revenue is $1/3$
  - equal to $E_{v_1, v_2 \sim U[0,1]}[\max(v_1, v_2)/2]$, because bidders declare half of their actual values
- With the second-price (Vickrey) auction the expected revenue is also $1/3$
  - equal to $E_{v_1, v_2 \sim U[0,1]}[\min(v_1, v_2)]$

**Proposition**

*All single-item auctions that allocate the item to the player with highest value have the same expected revenue.*

(Another version of: the payments are determined by the allocation)
Better than Vickrey?

- Assume bidders with values drawn from distribution $U[0, 1]$.
- Is there an auction with higher revenue than the Vickrey auction?
- Yes: Vickrey auction with reserve price $1/2$.
  - It gives the item to the highest bidder only if her value exceeds the reserve price.
  - She pays the maximum of the second bid and $1/2$.
  - Equivalent to: the auctioneer participates in the auction as a bidder with value equal to the reserve price.
- Revenue: $5/12$ (better than $1/3$).

**Problem:** Given the probability distributions of the values, find the auction which maximizes revenue.
There are \( n \) bidders with values \( v_i \) drawn from distributions \( F_i \).

We want to design a truthful auction which maximizes revenue.

The revelation principle holds: every auction can be transformed into an equivalent truthful one.
Optimal auctions for single item

- $v_i$: value of bidder $i$
- $F_i$: cumulative probability distribution of $v_i$, $F_i(x) = \Pr(v_i \leq x)$
- $f_i$: probability density function $f_i(x) = F_i'(x)$
- $a_i(v)$: allocation probability, i.e. probability of bidder $i$ getting the item
- $p_i(v)$: payment of bidder $i$

**Theorem**

An auction is truthful if and only if the allocation probability $a_i(v)$ is non-decreasing in $v_i$. The payment for player $i$ is given by

$$p_i(v) = a_i(v) \cdot v_i - \int_0^{v_i} a_i(z) \, dz$$
Virtual valuations

Definition
The virtual valuation of bidder $i$ is

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Theorem (Myerson)
The expected profit of a truthful mechanism is equal to

$$E_v \left[ \sum_i \varphi_i(v_i) a_i(v) \right]$$

Therefore, the optimal auction is **VCG applied to virtual values**!

Caution! Virtual values can be negative.
Myerson’s optimal auction

- The bidders provide their bids \( v_i \)
- The auctioneer computes the virtual valuations \( \varphi_i(v_i) \)
- She runs VCG on the virtual valuations (discarding negative valuations)
  - She gives the item to the bidder with highest virtual valuation (if it is non-negative)
  - Let \( p'_i \) be the maximum of the second highest valuation and 0
  - The winner pays \( p_i = \varphi_i^{-1}(p'_i) \)
Example

- $n$ bidders with values in $U[0, 1]$

- The virtual values are

  $$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = v_i - \frac{1 - v_i}{1} = 2v_i - 1$$

- Notice that $\varphi_i(1/2) = 0$; $1/2$ is a reserve price!

- Myerson's auction gives the item to the highest bidder $i$ only if $v_i \geq 1/2$

- The winner pays the maximum of the second highest bid and the reserve price
Ironing for irregular distributions

- The auction computes payments $p_i = \varphi_i^{-1}(p'_i)$
- This works only if the inverse function $\varphi_i^{-1}$ is defined, or equivalently if $\varphi_i$ is a monotone function.
- Otherwise, we have to do some “ironing” of the functions
Proof of Myreson’s theorem

The payment of bidder $i$ when her value is $v_i$ is given by

$$p_i(v) = a_i(v) \cdot v_i - \int_0^{v_i} a_i(z) \, dz$$

For simplicity, assume that the values are in $[0, h]$. The expected revenue from bidder $i$

$$R_i = \int_0^h p_i(v) f_i(v_i) \, dv_i$$

$$= \int_0^h a_i(v) \cdot v_i \cdot f_i(v_i) \, dv_i - \int_0^h \int_0^{v_i} a_i(z) f_i(v_i) \, dz \, dv_i$$

The whole trick is to inverse the order of the two integrals in the above expression. By doing this we get

$$R_i = \int_0^h \left( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) a_i(v_i) f_i(v_i) \, dv_i = E_{v_i} [\varphi_i(v_i) a_i(v)]$$

Adding the revenue from all bidders, we get the theorem.
Auctions with 2 or more items are notoriously hard.

We don’t even know the optimal auction for the special case of

- uniform distribution, additive valuations
- 1 bidder!
- 3 or more items (recently, with my student Yiannis Giannakopoulos, we determined the optimal auction for up to 6 items)