Online Algorithms
Lecture 4

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A puzzle

Move boxes within their ranges.

- Move boxes within their ranges.
A puzzle

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A puzzle

- Move boxes within their ranges.
- Align them so that they do not overlap vertically.
- Is this easy (in P) or difficult (NP-hard)?

What if there are only two (or 1000) different sizes of boxes?

Solved Elffers, de Weerdt

Easy for two sizes 1 and $p$

NP-hard for any other pair of sizes (including 2 and 4!)
A puzzle

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Easy for two sizes 1 and $p$
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Throughput scheduling

- Environment: One or more machines.
- Input: Jobs with length $p_j$, release time $r_j$, deadline $d_j$, and weight $w_j$. (Parameters are integers.)
- Output: Each job is assigned to a machine for a subinterval of $[r_j, d_j)$ of length $p_j$ or rejected. No overlaps.
- Objective: Maximize the number (weight) of the completed jobs.

This talk

- Online algorithms.
- Usually a single machine.
- Either jobs of equal length ($p_j = p$) and no weights
- or jobs of unit length ($p_j = 1$) with weights.
At time $r_j$, the other parameters of the job become known. At each time, if a machine is idle, the algorithm may decide to start a job.

**Competitive ratio**

An algorithm $A$ is $R$-competitive if for every instance $I$
- $OPT(I) \leq R \cdot A(I)$ for a deterministic algorithm, or
- $OPT(I) \leq R \cdot E[A(I)]$ for a randomized algorithm.
Jobs of equal length

Setting

- Equal lengths of jobs ($p_j = p$).
- No weights.
- Single machine.
Jobs of equal length

Setting
- Equal lengths of jobs \((p_j = p)\).
- No weights.
- Single machine.

Outline
1. Offline problem is polynomial.
2. Greedy algorithms are 2-competitive.
3. Lower bounds.
4. A better randomized algorithm.
5. Generalizations, variants.
GREEDY: If idle, start an arbitrary job.
Greedy algorithms

**GREEDY**: If idle, start an arbitrary job.

Any such algorithm is \( 2 \)-competitive.
Greedy algorithms

**GREEDY:** If idle, start an arbitrary job.

**Charging scheme – GREEDY is 2-competitive**

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
Greedy algorithms

GREEDY: If idle, start an arbitrary job.

Charging scheme – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
- Otherwise charge a job that OPT starts at $t$ to the job GREEDY runs at $t$. 
Lower bounds

No deterministic algorithm is better than 2-competitive. No randomized algorithm is better than $\frac{4}{3}$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)
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Lower bounds

- No deterministic algorithm is better than 2-competitive.
- No randomized algorithm is better than 4/3-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)
Generate two schedules, A and B. Flip a coin to choose one of them.
A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.
- A and B are produced by two identical processes using a common lock.

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A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.
- A and B are produced by two identical processes using a common lock.

- If the machine is idle (in A or B) and the set of pending jobs is not flexible (idling for time $p$ would lose some job), start the most urgent job.
- If the machine is idle (in A or B) and the set of pending jobs is flexible (idling for time $p$ does no harm):
  - If the lock is available, acquire it, start the most urgent job and release the lock after the job is completed.
  - Otherwise stay idle.
A barely random algorithm II
A barely random algorithm II

LOCK, A

B

LOCK, B

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A barely random algorithm II

A

B

A

B

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A barely random algorithm II

A

LOCK, B

A

LOCK, B
A barely random algorithm II

LOCK, B

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A barely random algorithm II

A
B

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A barely random algorithm II

\[ \begin{array}{c}
  \text{A} \\
  \text{B}
\end{array} \]
A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $\frac{5}{6}$.
A barely random algorithm III

- Analyzed by a more complex charging scheme.
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**Theorem**

*This algorithm is $5/3$-competitive.*
A barely random algorithm III

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- Each job in OPT charges $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$ to itself or to the job running at the same time in A and B.
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**Theorem**

This algorithm is $\frac{5}{3}$-competitive.

**Open problem**

Find a randomized algorithm with the optimal competitive ratio.
Parallel machines make the problem easier!

Results

- For 2 machines, there is a $3/2$-competitive deterministic algorithm and this is optimal.
- For $m$ machines, there is an $R$-competitive deterministic algorithm with $R \rightarrow e/(e - 1) \approx 1.58$ for $m \rightarrow \infty$.
- The lower bound approaches $6/5$ for $m \rightarrow \infty$. 
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**Results**

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- For $m$ machines, there is an $R$-competitive deterministic algorithm with $R \to e/(e - 1) \approx 1.58$ for $m \to \infty$.
- The lower bound approaches $6/5$ for $m \to \infty$.

**Open problem**

Decrease the gap for $m \to \infty$. 
Unit time jobs with weights

Setting
- Unit length of jobs ($p_j = 1$).
- General weights.
- Single machine.
Unit time jobs with weights

Setting
- Unit length of jobs ($p_j = 1$).
- General weights.
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Outline
1. Offline problem is easy (matching).
2. Greedy algorithm is 2-competitive.
3. A better randomized algorithm.
4. A better deterministic algorithm.
5. Generalizations, variants.
Forwarding packets in network switches

Restricted scenarios

2-bounded: Some packets may wait a single step, some packets not at all. ($d_j \leq r_j + 2$)

Agreeable deadlines: $r_j < r_k$ implies $d_j \leq d_k$.

Weighted queues: The deadlines are not known, only their order. Limited number of weights.
Motivation and variants

Forwarding packets in network switches

Restricted scenarios

- 2-bounded: Some packets may wait a single step, some packets not at all. \((d_j \leq r_j + 2)\)
- Agreeable deadlines: \(r_j < r_k\) implies \(d_j \leq d_k\).
- Weighted queues: The deadlines are not known, only their order.
- Limited number of weights.
Greedy algorithm

GREEDY: If idle, start a pending job with the maximal weight.
Greedy algorithm

**GREEDY:** If idle, start a pending job with the maximal weight.

<table>
<thead>
<tr>
<th>Job</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPT</td>
<td>20, 10, 11, 5, 21</td>
</tr>
<tr>
<td>GREEDY</td>
<td>21, 11, 5</td>
</tr>
</tbody>
</table>

Charging scheme – GREEDY is 2-competitive

Charge (map) a job in OPT to itself in GREEDY, if scheduled. Otherwise charge a job in OPT to the job GREEDY runs at the same time.
Greedy algorithm

**GREEDY**: If idle, start a pending job with the maximal weight.

Charging scheme – Greedy is 2-competitive

- Charge (map) a job in OPT to itself in Greedy, if scheduled.
- Otherwise charge a job in OPT to the job Greedy runs at the same time.
A randomized algorithm

- At each time, pick uniformly random real $x \in (-1, 0)$.
- Let $h$ be the largest weight of a pending job.
- Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earliest deadline.

**Theorem**

*This algorithm is $e/(e - 1) \approx 1.58$-competitive.*
How much “money” we need at a given time and configuration?

We earn $R \cdot w_j$ for running a job and pay $w_j$ if OPT runs a job.
How much “money” we need at a given time and configuration?

We earn $R \cdot w_j$ for running a job and pay $w_j$ if OPT runs a job.

Let $\Phi = \sum_{j \in X} w_j$, where $X$ are the jobs that the algorithm completed but the adversary will schedule in the future.
A potential function

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To prove that ON is $R$-competitive, we show that in each step

$$\Phi_{old} + R \cdot w_{ON} - w_{OPT} \geq \Phi_{new}$$
A potential function

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We earn $R \cdot w_j$ for running a job and pay $w_j$ if OPT runs a job.

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To prove that ON is $R$-competitive, we show that in each step

$$\Phi_{old} + R \cdot E[w_{ON}] - w_{OPT} \geq E[\Phi_{new}]$$
At each time, pick uniformly random real $x \in (-1, 0)$.
Let $h$ be the largest weight of a pending job.
Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earliest deadline.

**Theorem**

This algorithm is $e/(e - 1) \approx 1.58$-competitive. This is optimal against the adaptive online adversary. I.e., it is optimal among the algorithms analyzed using a potential.
Charging scheme

Alternating heavy and urgent packets eventually leads to a 1.939-competitive algorithm.
Deterministic algorithms I

Charging scheme

Alternating heavy and urgent packets eventually leads to a 1.939-competitive algorithm.

Potential function

Can be used to give a 1.828-competitive algorithm.
Deterministic algorithms II

Modifying the optimal schedule

At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
- Increase the weight or deadline of some pending job.

Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.
Deterministic algorithms II

Modifying the optimal schedule
At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
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Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.

Weighted queues
There exists a 1.897-competitive algorithm.
2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a 1.25-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.
Lower bounds

2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a $1.25$-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.

Open problem

Is the general problem harder than the 2-bounded case?
Algorithm \textsc{LessGreedy}(\phi)

- Schedule packet \( p \in P \) maximizing \( \phi \cdot w_p + w(Q_p) \)
  - \( P \): the optimal current schedule
  - \( Q_p \): the optimal schedule after \( p \) scheduled and time incremented (\( p \notin Q_p \))
New deterministic algorithm

Algorithm **LESSGREEDY**($\phi$)

- Schedule packet $p \in \mathcal{P}$ maximizing $\phi \cdot w_p + w(Q_p)$
  - $\mathcal{P}$: the optimal current schedule
  - $Q_p$: the optimal schedule after $p$ scheduled and time incremented ($p \not\in Q_p$)
  - $w_p$ is the immediate gain
  - $w(Q_p)$ is the optimal *future* profit unless new packets arrive
New deterministic algorithm

**Algorithm LESSGREEDY(\(\phi\))**

- Schedule packet \(p \in \mathcal{P}\) maximizing \(\phi \cdot w_p + w(Q_p)\)
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- Very elegant algorithm . . . but *not* \(\phi\)-competitive
Amortization Techniques

1. Increasing weights in the algorithm’s plan
   - Algorithm’s future profit *may* get higher
   - Decrease algorithm’s current profit by weight increase
Optimal deterministic algorithm

Amortization Techniques

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2. Modifications of the adversary “plan” ADV
   - ADV: already released packets from OPT in future slots
   - Some packets in ADV replaced by lighter ones
   - The adversary appropriately compensated

Theorem

There is a $\phi$-competitive deterministic algorithm.

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Optimal deterministic algorithm

Amortization Techniques

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3. Potential function
Optimal deterministic algorithm

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**Open Problems**

<table>
<thead>
<tr>
<th>$m \geq 1$ packets sent in each step</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Our algorithm is $\phi \approx 1.618$-competitive for any $m \geq 1$</td>
</tr>
<tr>
<td>• Best upper bound tends to $\frac{e}{e-1} \approx 1.58$</td>
</tr>
</tbody>
</table>

**Randomized algorithms**

<table>
<thead>
<tr>
<th>Gap between $1.25$ and $\frac{e}{e-1} \approx 1.58$</th>
</tr>
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<tbody>
<tr>
<td>• Algorithms that see only an ordering of the weights, not their values</td>
</tr>
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</table>

**Memoryless algorithms**

<table>
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<tr>
<th>Is there a lower bound $&gt; \phi$ for memoryless algorithms?</th>
</tr>
</thead>
<tbody>
<tr>
<td>• What is the ratio of LESSGREEDY($\alpha$)? (Schedule $p \in P$ max. $\alpha \cdot w_p + w(Q_p)$)</td>
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