Fixpoint logics and automata: a coalgebraic approach

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Logic & Automata

- Research area interfacing Logic and Theoretical Computer Science:

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- applications in program specification and verification

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- theory with long tradition:
  - Büchi
  - Rabin
  - . . .

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- automata are handy, non-linear generalizations of formulas

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- apply automata theory to prove results about logic

Classification of Automata

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- acceptance condition: Büchi/Muller/parity/. . .

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- Examples:

Name	Type	Acceptance condition
Büchi	$F \subseteq A$	$F \cap Inf(\rho) \neq \varnothing$
Muller	$\mathcal{M} \subseteq \wp(A)$	$Inf(\rho) \in \mathcal{M}$
parity	$\Omega : A \to \omega$	$\max\{\Omega(a) \mid a \in Inf(\rho)\}$ is even

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Theorem (Janin & Walukiewicz)
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. . . generalizes results on stream & tree automata.

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Main message

Key results in automata theory are part of Universal Coalgebra

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- Coalgebra
- Coalgebra Automata
- Towards Universal Automata Theory
- Final remarks

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- A pointed F-coalgebra is a pair $(\mathbb{S}, s_0)$, with s_0 a state in $\mathbb{S}$.

Examples

- flows (eg streams/infinite words): $\mathsf{F}S = C \times S$
- bi-flows (eg infinite trees): $\mathsf{F}S = C \times S \times S$
- Kripke frames: $\mathsf{F}S = \wp(S)$
- Kripke models: $\mathsf{F}S = \wp(\mathsf{Prop}) \times \wp(S)$

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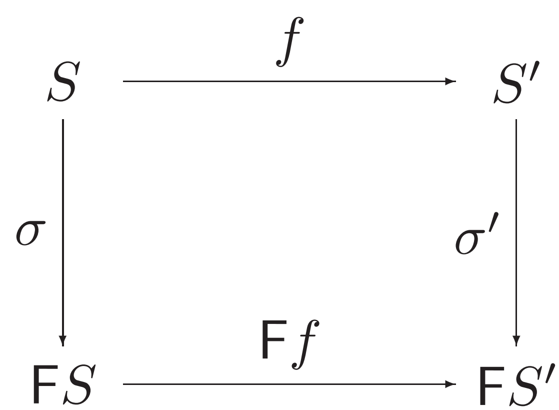
- A coalgebra homomorphism between two coalgebras $\mathbb{S}$ and $\mathbb{S}'$ is a map $f : S \to S'$ such that $\sigma' \circ f = \mathsf{F}f \circ \sigma$:



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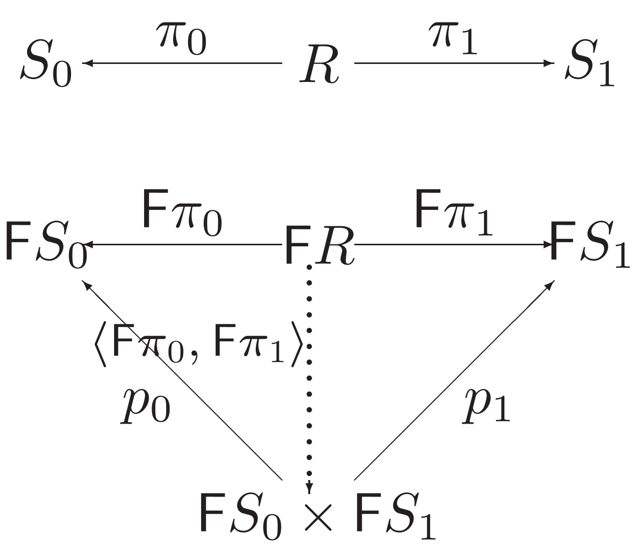

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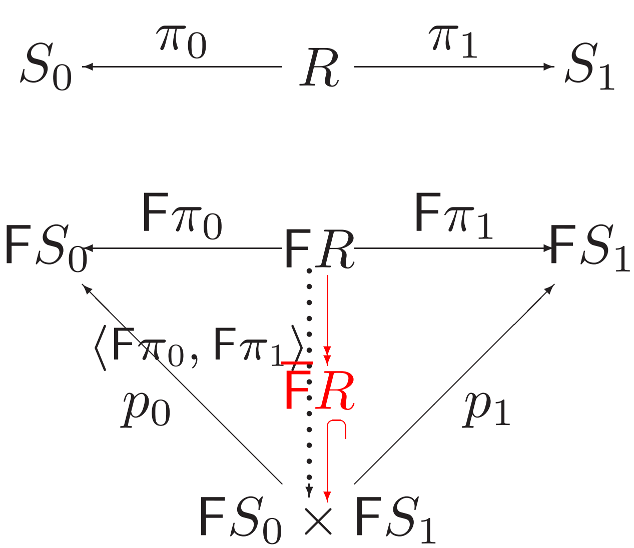

$\overline{\mathsf{F}}R := \{(\mathsf{F}\pi_0\rho, \mathsf{F}\pi_1\rho) \mid \rho \in \mathsf{F}R\}$.

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- C-labelled binary trees:

$$(c, s_l, s_r)\overline{\mathsf{B}}R(c', s'_l, s'_r) \text{ iff } c = c', s_l R s'_l \text{ and } s_r R s'_r.$$

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- Kripke models:

$$(\pi, X)\overline{\mathsf{K}}R(\pi', X') \text{ iff } \pi = \pi' \text{ and } X\overline{\mathsf{P}}RX'$$

Bisimulations via Relation Lifting

Definition: With $\mathbb{S} = (S, \sigma : S \to \mathsf{F}S)$ and $\mathbb{S}' = (S', \sigma' : S \to \mathsf{F}S)$ two F-coalgebras,

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Proof: Bisimulations are postfixpoints of the monotone operation

$$R \mapsto \{(s_0, s_1) \mid (\sigma_0 s_0, \sigma_1 s_1) \in \overline{\mathsf{F}}R\}$$

The relation $\leftrightarroweq$ is its greatest fixpoint.

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Theorem: For all a, s: $(a, s) \in Win_{\exists}(\mathsf{B})$ iff $\mathbb{A}, a \leftrightarrows \mathbb{S}, s$.

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- $\Omega : A \to \omega$ is the parity acceptance condition

Acceptance Game

Acceptance game $\mathsf{B}(\mathbb{A}, \mathbb{S})$ of F-automaton $\mathbb{A} = \langle A, a_I \Delta, \Omega \rangle$ on F-coalgebra $\mathbb{S} = \langle S, \sigma \rangle$:

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Definition: $\mathbb{A}$ accepts $(\mathbb{S}, s)$ if $(a_I, s) \in Win_\exists(\mathcal{B}(\mathbb{A}, \mathbb{S}))$.

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Standard automata on words, trees, transition systems, etc, are all special instances of coalgebra automata

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- Acceptance generalizes bisimilarity.
- Separate the combinatorics (Acc) from the dynamics (Δ).

Results in Universal Coalgebra

(The following are joint results with Clemens Kupke)

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- New! The recognizable languages are closed under complementation.

(This is a joint result with Christian Kissig).

Results in Fixpoint Logic

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The above results have various corollaries in fixpoint logics.

Overview

- Logic & Automata
- Coalgebra
- Coalgebra Automata
- Towards Universal Automata Theory
- Final remarks

Predicate Liftings

Definition: A predicate lifting for a set functor F is a natural transformation

$$\lambda : \breve{\mathsf{P}} \longrightarrow \breve{\mathsf{P}} \circ \mathsf{F}$$

where $\breve{\mathsf{P}}$ is the contravariant power set functor.

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Define a modal operator $\heartsuit_\lambda$:

$$\begin{array}{lll} \mathbb{S}, s \Vdash \heartsuit_\lambda \varphi & \text{iff} & \sigma(s) \in \lambda_S([\![\varphi]\!]). \\ \mathbb{S}, s \Vdash \heartsuit_\lambda (\varphi_1, \ldots, \varphi_n) & \text{iff} & \sigma(s) \in \lambda_S([\![\varphi_1, \ldots, [\![\varphi_n]\!]\!]). \end{array}$$

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Naturality of λ guarantees bisimulation invariance of the modalities $\heartsuit_\lambda$.

Example: Probabilistic Modal Logic

Let D_ω be the finitary distribution functor given by

$$\begin{aligned}
\mathsf{D}_\omega S &:= \left\{\mu : S \to [0,1] \mid \sum_{s\in S}\mu(s) = 1 \ \&\ |Supp(\mu)| < \omega\right\} \\
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This gives rise to probabilistic modal logic:

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Coalgebraic Modal (Fixpoint) Logic

ML_Λ For a family Λ of predicate liftings, use $\{\heartsuit_\lambda \mid \lambda \in \Lambda\}$ as modalities:

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For a family Λ of such liftings, introduce fixpoint extension $\mu\mathrm{ML}_\Lambda$ of ML_Λ.

Theorem (Cîrstea, Kupke & Pattinson [CSL'09]))
If a family Λ of monotone liftings admits a 'reasonable' one-step tableau, the satisfiability problem for $\mu\mathrm{ML}_\Lambda$ is in EXPTIME.

One-step semantics

Definition: $\Lambda^{\heartsuit} A := \{\heartsuit_\lambda(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_i \in A\}$.

$\mathcal{L}_0 B$ is the set of lattice terms over B:

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Associate a notion of one-step satisfaction $\models^v \subseteq \mathsf{F}S \times \mathcal{L}_0 \Lambda^{\heartsuit} A$:

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Note: With $v : A \to \mathcal{P}(S)$, associate a relation $Z_v := \{(a, s) \mid s \in v(a)\}$.

Λ-Automata

Definition: Let Λ be a family of predicate liftings for a set functor F.

A Λ-automaton is a quadruple $\mathbb{A} = \langle A, a_I, \delta, \Omega \rangle$ with

- A a finite set of states, of which a_I is the starting state
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Definition: Acceptance game of a Λ-automaton $\mathbb{A}$ and a T-coalgebra $\mathbb{S}$:

Position	Player	Admissible moves	Priority
$(a, s) \in A \times S$	$\exists$	$\{v : A \to \mathcal{P}(S) \mid TS, \sigma(s) \models^v \delta(a)\}$	$\Omega(a)$
$v : A \to \mathcal{P}(S)$	$\forall$	$Z_v = \{\{(a', s') \mid s' \in v(a')\}$	0

Bounded Model Property

(The following is joint work with Gaëlle Fontaine and Raul Leal)

Theorem: Let $\mathbb{A}$ be some Λ-automaton such that $L(\mathbb{A}) \neq \varnothing$.
Then $\mathbb{A}$ accepts some pointed coalgebra $(\mathbb{S}, s)$ of size exponential in $|A|$.

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In fact, our proof provides a direct construction of $(\mathbb{S}, s)$ from $\mathbb{A}$.

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Further Coalgebraic Inspiration

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- Model theory of modal μ-calculus (joint work with Gaëlle Fontaine)

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- Axiomatics of modal fixpoint logics (joint work with Luigi Santocanale)

Conclusions

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- Our understanding of modal fixpoint logics may benefit from a coalgebraic perspective.

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- completeness theory for coalgebraic fixpoint logics.
- automata and fixpoints in algebra|coalgebra duality.

References

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