The Decision Problem for Linear Tree Constraints

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We present new results on a constraint satisfaction problem arising from the inference of resource types.

Linear constraints were introduced by Hofmann and Jost in the context of type-based amortized resource analysis by the potential method [5] where it was applied to functional programs. The constraint systems appearing in this system and subsequent ones has finitely many variables and can be reduced to linear programming. Later, Hofmann and Rodriguez extended type-based amortized resource analysis to object-oriented programs [8],[6] which led to constraints involving infinite lists or trees whose entries are numerical variables. Therefore, a straightforward reduction to linear programming is no longer an option. However, the constraint systems exhibit enough regularity that a heuristic procedure developed by Hofmann and Rodriguez allowed to find solutions in many cases.

In the case of infinite lists the constraint systems can be simply described as follows. One has finitely many unknowns ranging over infinite lists (sequences) of nonnegative rational numbers including $\infty$. Terms and constraints are built from those by addition (+) and comparison ($\leq$) understood pointwise, and the tail function ($tl$) that removes the first element of a sequence. Furthermore, constraint systems may contain ordinary linear arithmetic constraints over the nonnegative rationals including infinity where the head function ($hd$) maps sequences to numbers. For example, the following constraint system is solvable and has the (not unique) solution $y$ a constant list and $x$ an exponentially decreasing list and $z$ a Fibonacci list with an additional linear summand. We have the (arithmetic and list) constraints

\[
\begin{align*}
hd(x) &= 2, \\
hd(tl(x)) &= 5, \\
hd(y) &\geq 1, \\
hd(z) &\geq 2,
\end{align*}
\]

\[
\begin{align*}
tl(y) &\geq y, \\
z + z &\leq tl(z), \\
tl(tl(x)) &\geq tl(x) + x + tl(y) + tl(y).
\end{align*}
\]

and the solutions

\[
\begin{align*}
y &= 1,1,..., \\
x &= 2,5,9,16,27,55,\ldots,n-1,n,n(n-1)+2,\ldots, \\
z &= 2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots,\frac{1}{2^n},\ldots
\end{align*}
\]

In this paper we report progress on the general question of solvability and decidability of these constraint systems. First, in the case of infinite lists of numerical variables we can in many cases draw a connection to rational generating functions and in particular see that certain resource behaviours cannot arise as solutions to type inference problems. A concrete example is the harmonic series because it has a logarithmic generating function [4].

We also show that the general constraint satisfaction problem as formulated by Hofmann and Rodriguez admits a reduction from the Skolem-Mahler-Lech problem [7],[2] whose decidability status is as yet unknown but at least NP-hard. Recall that the Skolem-Mahler-Lech problem
The decision problem for linear tree constraints asks whether a sequence defined by a linear recurrence with integer coefficients (like Fibonacci) contains zero.

However, we were able to better delineate the image of the translation from type inference to the constraint satisfaction problem and thus identified a subproblem which—if solved—can still be used for type-based resource inference but has better algorithmic properties.

More precisely, we can show that the characteristic polynomials of the linear recurrences corresponding to the identified fragment always have a dominating zero (eigenvalue) which allows for a decision procedure based on comparison of growth rates. There are still several fine points to overcome which result from the fact that we are dealing with inequations rather than equations.

We then also give a heuristic reduction from the case of general trees to infinite lists which carries type-based amortized resource analysis considerably further than it was previously the case. In particular, we are able to analyse programs with superpolynomial resource consumption and also have a more intrinsic characterisation of the limitations of the analysis.

In currently ongoing work, we try to go beyond the question of mere satisfiability of constraint systems and to extract minimal solutions and also to treat trees in general not merely by translating them to lists.

References