A Presheaf Model of Parametric Type Theory

Jean-Philippe Bernardy, Thierry Coquand and Guilhem Moulin

Dept. of Comput. Sci and Engin., Chalmers University of Technology and University of Gothenburg, Sweden
{bernardy,coquand,mouling}@chalmers.se

Abstract

We propose a new type theory with internalized parametricity. Compared to previous similar proposals, this version comes with a denotational semantics which is a refinement of the standard presheaf semantics of dependent type theory. Further, this presheaf semantics is a refinement of the one used to interpret nominal sets with restriction. The present calculus is a candidate for the core of a proof assistant with internalized parametricity.

Reynolds's abstraction theorem can be stated in a purely syntactical way: for instance, if a function f has type $(A : \star) \to A \to A$ — the type of the polymorphic identity — then the proposition $(A : \star) \to (P : A \to \star) \to (x : A) \to P\,x \to P\,(f\,A\,x)$ holds. However this result is not provable internally, *i.e.*, $(f : (A : \star) \to A \to A) \to (A : \star) \to (P : A \to \star) \to (x : A) \to P\,x \to P\,(f\,A\,x)$ is not provable. Several attempts have been made for designing an extension of dependent type theory in which such an internal form of parametricity holds. We propose another such system here. Our technical contributions are as follows:

- We present a type theory which internalizes parametricity and can be seen as a simplification and generalization of the systems of [1, 2]
- We provide a *denotational* semantics, in the form of a presheaf model, for this type theory. This model is a refinement of the presheaf semantics used to interpret nominal sets with restrictions [3, 4].

Syntax

We assume a special symbol '0', and a countable infinite set $\mathbb{I}$ of other symbols, called *colors*. The metasyntactic variables $i, j, \ldots$ range over colors, while φ range over $\mathbb{I} \cup \{0\}$. The main innovation of the type theory presented here is that terms may depend on (a finite number of) colors. We add the following constructions to the usual syntax of lambda calculi:

$$a, p, t, A, P, T \coloneqq \ldots \mid (a,_i\, p) \mid (x : A) \times_i P \mid A \ni_i a \mid a{\cdot}i$$

Remark. Here is some intuition for these new constructions:

- Any type is associated with a predicate for every color. The type $A \ni_i a$ expresses that a satisfies the parametricity predicate associated with the type A on color i. For each term a and color i, the term $a\,(i\,0)$ is the erasure of i in a. It is defined by induction on a and can be understood as a realizer of a.
- The term $a{\cdot}i$ yields a proof of $A \ni_i a\,(i\,0)$.
- The forms $(a,_i\, p)$ and $(x : A) \times_i P$ allow to locally associate parametricity proofs with a given realizer.

We index typing judgements with the set of free colors; our new constructions are typed and converted as follows:

$$\textsc{In-Abs}\ \frac{\Gamma \vdash a : A\,(i\,0) \qquad \Gamma \vdash p : A \ni_i a}{\Gamma, i : \mathbb{I} \vdash (a,_i p) : A} \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash P}{\Gamma, i : \mathbb{I} \vdash (x : A) \times_i P}\ \textsc{In-Pred}$$

$$\textsc{Out}\ \frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash a : A\,(i\,0)}{\Gamma \vdash A \ni_i a} \qquad\qquad \frac{\Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash a{\cdot}i : A \ni_i a\,(i\,0)}\ \textsc{Color-Elim}$$

$$(a,_i p){\cdot}i = p \qquad\qquad ((x : A) \times_i P[x]) \ni_i a = P[a]$$

$$t = (t\,(i\,0),_i t{\cdot}i) \qquad\qquad T = (x : T\,(i\,0)) \times_i (T \ni_i x)$$

Unlike previous type theories with internalized parametricity, the types $\star \ni_i A$ and $A \to \star$ are not convertible but *isomorphic*. The same goes for $((x : A) \to B[x]) \ni_i f$ and $(x : A) \to (x' : A \ni_i x) \to B[(x,_i x')] \ni_i (f\,x)$. However, in our system one can use parametricity generically via $\lambda A.\, \lambda a.\, a \cdot i \;:\; (A : \star) \to (x : A) \to A \ni_i x$. In particular, the proposition given in the introduction is provable by $\lambda f.\, \lambda A.\, \lambda P.\, \lambda a.\, \lambda p.\, (f\,(A \times_i P)\,(a,_i p)){\cdot}i$.

Presheaf model

We say that a function $f : I \to J \cup \{0\}$ is a *color map*, and note $f : I \to J$, if $i_1 = i_2$ for any $i_1, i_2 \in I$ with $f(i_1) = f(i_2) \in J$. We consider the category $\mathbf{pI}$ of finite color sets and color maps. We use a refined presheaf on $\mathbf{pI}^{\mathrm{op}}$ by requiring two further conditions (without this refinement, it is not clear how to validate the equality $((x : A) \times_i P[x]) \ni_i a = P[a]$):

1. for any object I, $F(I)$ is a set of I-elements, *i.e.*, of tuples indexed by the subsets of I;
2. for any projection map $\alpha : I \to I_\alpha$, the restriction map $F(I) \to F(I_\alpha)$, $u \mapsto u\alpha$ is the projection operation, *i.e.*, $u\alpha_J = u_J$ for any $J \subseteq I$.

A context $\Gamma \vdash$ is interpreted by a (usual) presheaf on $\mathbf{pI}^{\mathrm{op}}$.

A type $\Gamma \vdash A$ is interpreted by an I-set $A\rho$ for each object I and $\rho \in \Gamma(I)$, together with restriction maps $A\rho \to A(\rho f)$, $u \mapsto uf$ if $f : I \to J$ satisfying $u1 = u$ and $(uf)g = u(fg)$ for any $g : J \to K$. Furthermore the map $A\rho \to A(\rho\alpha)$, $u \mapsto u\alpha$ is the projection operation.

A term $\Gamma \vdash a : A$ is interpreted by a I-element $a\rho \in A\rho$ for each object I and $\rho \in \Gamma(I)$, such that $a\rho f = a(\rho f)$ for any $f : I \to J$.

If $\Gamma \vdash$ and $\Gamma \vdash A$ we define the interpretation of $\Delta = \Gamma, x : A$ by taking $\langle \rho, x = u \rangle \in \Delta(I)$ to mean $\rho \in \Gamma(I)$ and $u \in A\rho$. The restriction map is defined by $\langle \rho, x = u \rangle f = \langle \rho f, x = uf \rangle$.

If $\Gamma \vdash$ we define the interpretation of $\Delta = \Gamma, i : \mathbb{I}$ by taking $[\rho, i = \varphi] \in \Delta(I)$ to mean either $\varphi = 0$ and $\rho \in \Gamma(I)$, or $\varphi = j \in I$ and $\rho \in \Gamma(I \backslash \{j\})$.

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