Relational parametricity is a fundamental concept within theoretical computer science and the foundations of programming languages, introduced by John Reynolds [6]. His fundamental insight was that types can be interpreted not just as functors on the category of sets, but also as equality preserving functors on the category of relations. This gives rise to a model where polymorphic functions are uniform in a suitable sense; this can be used to establish e.g. representation independence, equivalences between programs, or deriving useful theorems about programs from their type alone [7].

The relations Reynolds considered were proof-irrelevant, which from a type theoretic perspective is a little limited. As a result, one might like to extend his work to deal with proof-relevant, i.e. set-valued relations. However naive attempts to do this fail: the fundamental property of equality preservation cannot be established. Our insight is that just as one uses parametricity to restrict definable elements of a type, one can use parametricity to restrict definable proofs to ensure equality preservation in the proof-relevant setting.

Proof-relevant logical relations also appear in Benton, Hofmann and Nigam’s recent work [1], but for unary relations, and in a setting without ∀-types. Hence they do not need to consider equality preservation. In our work, we consider proof-relevant binary relations. A proof-relevant relation \( R \) over two sets \( A \) and \( B \) consists of a map \( R : A \times B \to \text{Set} \) which, for every two elements \((a, b) \in A \times B\), associates a set of proofs that they are related. On top of this, we have a new proof-irrelevant layer: a 2-dimensional relation \( Q \) is defined over four sets and four binary proof-relevant relations. It can be thought of as over a square, where the vertices are sets and the edges are the binary proof-relevant relations:

\[
\begin{array}{c}
A & \xrightarrow{R_1} & B \\
\downarrow_{R_2} & \quad & \downarrow_{R_4} \\
C & \xleftarrow{R_3} & D.
\end{array}
\]

The 2-dimensional relation \( Q \) is defined to be a proof-irrelevant relation over proofs agreeing on the vertices: \( Q(a, b, c, d) \subseteq R_1(a, b) \times R_2(a, c) \times R_3(c, d) \times R_4(b, d) \), where \((a, b, c, d) \in A \times B \times C \times D\).

Equality preservation, i.e. the Identity Extension Property, is fundamental for parametricity. The equality relation on a set gives a functor \( \text{Eq} \) from sets to relations, where \( \text{Eq}(a, b) = \{ \ast | a = b \} \).

Clearly, there is no unique way to define a corresponding map from binary relations to 2-dimensional relations. In the relational interpretation of ∀-types we instead use three different “equality” maps, represented by the following squares:

\[
\begin{align*}
& A \xrightarrow{\text{Eq}_1} A \\
& B \xrightarrow{\text{Eq}_1} B \\
& B \quad & A \xrightarrow{\text{Eq}_2} B \\
\text{γ}_1 & & A \xrightarrow{\text{Eq}_3} B \\
B & & B \xrightarrow{\text{Eq}_3} B.
\end{align*}
\]

Note that the structure involved is highly symmetric. In fact, \( \text{Eq}_2 \) arises from \( \text{Eq}_1 \) by reflection along the diagonal. Similarly, the same reflection along the diagonal applied to \( \text{Γ}_1 \) gives another
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notion of “equality”. We can easily define a group action on squares and binary proof-relevant relations, and it looks like the interpretation of types are invariant under this action, although we are in the middle of the proof. If this holds, it would simplify the presentation of the interpretation of \( \forall \)-types and make it easier to work with them.

So far we proved that each layer of sets, proof-relevant relations and 2-dimensional relations is fibred over the layer below. This structure gives a parametric 2-dimensional proof-relevant model for system \( F \), where types are interpreted as equality-preserving lifted functors and terms as lifted natural transformations. In this interpretation, the Identity Extension Lemma and the Abstraction Theorem hold. In particular we need to generalize the statement of the Identity Extension Lemma, since we now have more kinds of “equalities”. The (bi)fibrational structure of the model can be used to derive the standard consequences of parametricity \([5]\). For this, we need to generalize also the Graph Lemma, which is fundamental in the proofs of existence of initial algebras and terminal coalgebras. We are checking the details of the latter proofs, and the proof that parametricity implies (di)naturality.

Interestingly, these 2-dimensional relations have clear higher dimensional analogues where \((n+1)\)-relations are fibred over a \(n\)-cube of \(n\)-relations \([4]\). Thus the story of proof relevant logical relations quickly expands into one of higher dimensional structures similar to the cubical sets which arises in Homotopy Type Theory \([3]\). Of course, there are also connections to Bernardy and Moulin’s work on internal parametricity \([2]\).

References


