The Next 700 Modal Type Assignment Systems

Andreas Abel
Talk presented by Andrea Vezzosi

Department of Computer Science and Engineering
Chalmers and Gothenburg University, Sweden

Types for Proofs and Programs Annual Meeting 2015
Tallinn, Estonia
18 May 2015
Motivation

- Grand goal: integrate analyses (positivity, termination, strictness) into dependent type theory.
- Reason: with meta variables, type checking not cleanly separable from other analyses.
- A lot of type-based analyses out there:
  - Strictness.
  - Relevance.
  - Linearity.
  - Positivity.
- Can be formulated as non-standard type systems.
- Sometimes a bit ad-hoc.
- Can we systematize them?
Simply Typed Lambda-Calculus With Largest Type

- Terms \( t, u ::= x \mid \lambda x t \mid t \; u \) (untyped).
- Type assignment \( \Gamma \vdash t : A \).
- Types \( A, B ::= \top \mid A \to B \).
- Contexts \( \Gamma \) are total functions from term variables to types.
  - Empty context \( \varepsilon \) is constant function \( \varepsilon(x) = \top \).
  - Domain \( \text{dom}(\Gamma) = \{ x \mid \Gamma(x) \neq \top \} \).
  - Update/extension \( \Gamma' = (\Gamma, x:A) \) is \( \Gamma'(y) = \begin{cases} U & \text{if } y = x \\ \Gamma(y) & \text{otherwise} \end{cases} \)
  - Singleton context \( x:A \) is \( (\varepsilon, x:A) \).
Trivial subtyping

- Subtyping $A \leq B$ iff $A = B$ or $B = \top$.

- Greatest lower bound $A \land B$ is undefined except that

$$A \land \top = \top \land A = A \land A = A.$$
Context subsumption = Weakening

- Subsumption $\Gamma \leq \Delta$ is pointwise: $\forall x. \Gamma(x) \leq \Delta(x)$.

- Joining contexts $\Gamma \land \Delta$ is pointwise (undef. if undef. at some $x$):
  $$(\Gamma \land \Delta)(x) = \Gamma(x) \land \Delta(x)$$

Note: $\varepsilon \land \Gamma = \Gamma \land \varepsilon = \Gamma \land \Gamma = \Gamma$. 
Simple Type Assignment

\[
\begin{align*}
\text{hyp} & : x : A \vdash x : A \\
\text{abs} & : \Gamma, x : A \vdash t : B \\
\text{app} & : \Gamma \vdash t : A \rightarrow B \quad \Delta \vdash u : A \\
\text{sub} & : \Gamma \leq \Delta \quad \Delta \vdash t : A \quad A \leq B
\end{align*}
\]

- Looks *linear*.
- But *app* allows contraction.
- And *sub* allows weakening.
Exact Quantitative Typing

- Quantifies resource use exactly.
- Quantified type $Q ::= qA$, quantity $p, q \in \mathbb{N}$.
- Types $A, B ::= \top \mid Q \rightarrow B$.
- E.g. $5A \rightarrow B$: to produce one $B$, we need exactly $5$ $A$.
- Context $\Gamma$ maps variables $x$ to quantified types $Q$.
- Scaling $p(qA) = (pq)A$ and $(p\Gamma)(x) = p(\Gamma(x))$.
- Partial sum $pA + qB = (p + q)(A \land B)$.
- Pointwise context sum $(\Gamma + \Delta)(x) = \Gamma(x) + \Delta(x)$. 
**Exact Quantitative Type Assignment**

- \( x : 1A \vdash x : A \)  \( \text{hyp} \)
- \( \Gamma, x : qA \vdash t : B \)  \( \text{abs} \)
- \( \Gamma \vdash \lambda xt : qA \to B \)
- \( \Gamma \vdash t : qA \to B \)
- \( \Delta \vdash u : qA \)
- \( \Gamma + \Delta \vdash t u : B \)  \( \text{app} \)
- \( q\Gamma \vdash t : qA \)  \( \text{mod} \)

- **mod** introduces quantified types.
- **app** splits resources between function and argument.
Wasting (Weakening)

- Subtyping quantified types.

\[
q \geq q' \quad A \leq A' \\
\quad qA \leq q'A'
\]

- Weakening.

\[
\Gamma \leq \Delta \quad \Delta \vdash t : A \quad A \leq B \\
\quad \Gamma \vdash t : B \quad \text{sub}
\]
Typical Quantitative Type Systems

- **Strictness**: Is a variable used *at least once*?  
  Quantities: 0.. and 1..

- **Relevance**: Is a variable used *never*?  
  Quantities: 0 and 1..

- **Linear typing**: Is a variable used *exactly once or unrestrictedly*?  
  Quantities: 1 and 0..
Quantitative Typing, Revisited

- Quantities $q \in P$ are sets of natural numbers.
- $P \subseteq \mathcal{P}(\mathbb{N})$ is partially ordered by
  $$p \leq q \iff p \supseteq q$$
- $P$ should form a monoid with composition
  $pq = \{mn \mid m \in p \text{ and } n \in q\}$ and a suitable unit $1$.
  $$\underline{x : 1A \vdash x : A \quad \text{hyp}}$$
- Sum $p + q$ is smallest set $r \in P$ such that $m + n \in r$ for all $m \in p$ and $n \in q$ (might not exists).
- Default element $p_0$ for empty context $\varepsilon(x) = p_0^\top$ (controls weakening).
Example 1: Linear typing

- $P = \{1, !\}$ with $1 = \{1\}$ (linear) and $! = \mathbb{N}$ (unrestricted).

\[
\begin{array}{ccc}
pq & 1 & N \\
1 & 1 & N \\
N & N & N \\
\end{array}
\quad
\begin{array}{ccc}
p + q & 1 & N \\
1 & / & N \\
N & N & N \\
\end{array}
\]

- $1 + 1$ undefined: no contraction for linear hypotheses!
- $p_0 = ! \leq 1$: weakening only with unrestricted hypotheses.

\[
(x : !A) \leq (x : !T) = \varepsilon
\]

- Hypothesis rule usable for $!$ via $\text{sub}:

\[
\begin{array}{c}
(x : !A) \leq (x : 1A) \\
x : 1A \vdash x : A
\end{array}
\quad
\begin{array}{c}
x : !A \vdash x : A
\end{array}
\]
Example 2: Strictness typing

- $P = \{\mathbb{L}, \mathbb{S}\}$ with $\mathbb{L} = \mathbb{N}$ (lazy) and $\mathbb{S} = \mathbb{N} \setminus \{0\} = 1$ (strict).

\[
\begin{array}{c|c|c}
pq & L & S \\
\hline
L & L & L \\
S & L & S \\
\end{array}
\quad
\begin{array}{c|c|c}
p + q & L & S \\
\hline
L & L & S \\
S & S & S \\
\end{array}
\]

- $p_0 = \mathbb{L} \leq \mathbb{S}$: weakening only with lazy hypotheses!

\[(x : \mathbb{L}A) \leq (x : \mathbb{S}T) = \varepsilon\]

- Hypothesis rule usable for $\mathbb{L}$ via $\text{sub}$:

\[
\frac{(x : \mathbb{L}A) \leq (x : \mathbb{S}A) \quad x : \mathbb{S}A \vdash x : A}{x : \mathbb{L}A \vdash x : A}
\]
Example 3: Relevance typing

- $P = \{0, 1\}$ with $0 = \{0\}$ (unused) and $1 = \mathbb{N} \setminus \{0\} = 1$ (used).
- $P$ is discrete: $0 \not\leq 1 \not\leq 0$.

<table>
<thead>
<tr>
<th>$pq$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$p + q$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- $p_0 = 0$: weakening only with unused hypotheses!

\[(x : \emptyset A) \leq (x : 1 \top) = \varepsilon\]

- Irrelevant hypotheses are unusable

\[x : \emptyset A \not\vdash x : A\]
Variance / Monotonicity / Polarity / Positivity

- Scala: type operator variance for subtyping.
- Coq/Agda: positivity checking for inductive/coinductive types.
- 4-point lattice:

<table>
<thead>
<tr>
<th></th>
<th>variance</th>
<th>monotonicity</th>
<th>occurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td>∅</td>
<td>invariant</td>
<td>constant</td>
<td>none</td>
</tr>
<tr>
<td>+</td>
<td>covariant</td>
<td>monotone</td>
<td>positive</td>
</tr>
<tr>
<td>−</td>
<td>contravariant</td>
<td>antitone</td>
<td>negative</td>
</tr>
<tr>
<td>±</td>
<td>mixed-variant</td>
<td>any function</td>
<td>both / don’t know</td>
</tr>
</tbody>
</table>

- Function composition is $pq$.
- Combine variable occurrences in subterms with $p + q$. 
Positivity Typing

- Partially ordered monoid \( P = \{\emptyset, +, -, \pm\} \).

\[
\begin{array}{c|cccc}
\emptyset & \pm & + & - & \emptyset \\
\pm & \pm & \pm & \pm & \emptyset \\
+ & \pm & + & - & \emptyset \\
- & \pm & - & + & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\end{array}
\]

- Encoding: \( p \leq q = p \supseteq q \) and \( p + q = p \cup q \) where

\[
\begin{array}{c|ccc}
\emptyset & + & - & \pm \\
\{\} & \{+1\} & \{-1\} & \{+1, -1\} \\
\end{array}
\]

- Default polarity \( p_0 = \emptyset \): weaken with anything.
Modal types

- Make $qA$ first class: $A, B ::= \top \mid qA \mid A \rightarrow B$.
- Back to lambda-calculus rules plus $mod$.

\[
\begin{align*}
\Gamma \vdash x : A & \quad \text{hyp} \quad \Gamma, x : A \vdash t : B & \quad \text{abs} \\
\Gamma \vdash t : A \rightarrow B & \quad \Delta \vdash u : A & \quad \text{app} \\
\Gamma \vdash t : A & \quad A \leq B & \quad \text{sub} \\
q\Gamma \vdash t : qA & \quad \text{mod}
\end{align*}
\]
Linear Types, Revisited

- \( P = \{1, !\} \), let \( 1_A = A \).
- Observe \( !!A = !A \).
- Promotion is an instance of \textit{mod}.

\[
\begin{align*}
!\Gamma \vdash t : A \\
!\Gamma \vdash t : !A \\
\text{mod}
\end{align*}
\]
Conclusions & Further work

- A fairly generic simple/modal type system parametrized over a partially ordered monoid $P$ with sum and default element $p_0$.
- Captures several well-known non-standard type system.
- Further work:
  - Nakano’s modality for recursion.
  - Semantics?
  - Connected to Kripke models?
- Recent related work: Conor McBride uses worlds to integrate linear and dependent types.