A nominal syntax for internal parametricity

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Introduction

Parametricity and univalence

- Parametricity says that terms respect logical relations.
  - Two functions are related if they map related inputs to related outputs.
  - Two pairs are related if they are componentwise related.
  - Two types are related if there is a relation between them.
- In homotopy type theory, terms respect equality.
  - Two functions are equal if they map equal inputs to equal outputs (function extensionality).
  - Two pairs are equal if they are componentwise equal.
  - Two types are equal if there is a relation between them and this relation is the graph of an equivalence (univalence).
- Our goal is to replace intensional equality in type theory by one which is defined recursively over the type structure.
- Inspiration:
  - Bernardy-Moulin: Internal parametricity, 2012
  - Bezem-Coquand-Huber: The cubical sets model of type theory, 2013
  - Altenkirch-McBride-Swierstra: Observational type theory, 2007
  - Martin-Löf: An intuitionistic theory of types, 1972
Parametricity

- Parametricity says that terms respect logical relations.

\[ A : U, u : A, s : A \rightarrow A \vdash t : A \]

\[ \rho_0 \equiv (A \mapsto \mathbb{N}, u \mapsto \text{zero}, s \mapsto \text{suc}) \]
\[ \rho_1 \equiv (A \mapsto \text{Bool}, u \mapsto \text{true}, s \mapsto \text{not}) \]

- If \( \rho_0 \) and \( \rho_1 \) are related, then \( t[\rho_0] \) is related to \( t[\rho_1] \).
- It can’t happen that \( t[\rho_0] \equiv \text{suc (suc zero)} \) and \( t[\rho_1] \equiv \text{false} \).

\[ \sim_A : \mathbb{N} \rightarrow \text{Bool} \rightarrow U \]
\[ x \sim_A b \equiv (x \text{ even}) \text{ is } b \]

- A simpler example:

\[ A : U, x : A \vdash t : A \]
Specifying a logical relation

- The logical relation for a type $A$:

\[
\frac{\Gamma \vdash A : U}{\Gamma = \vdash \sim_{A} : A[0] \to A[1] \to U} \quad \frac{\Gamma \vdash}{\Gamma = \vdash} \quad 0_{\Gamma}, 1_{\Gamma} : \Gamma = \Rightarrow \Gamma
\]

- The context of related elements:

\[
\equiv \quad (\Gamma, x : A) = \equiv \cdot \quad \Gamma = , x_{0} : A[0], x_{1} : A[1], x_{2} : x_{0} \sim_{A} x_{1}
\]

- Substitutions 0, 1 project out the corresponding components:

\[
i. \equiv () \quad : \cdot \Rightarrow \emptyset \quad i_{\Gamma, x : A} \equiv (i_{\Gamma}, x \mapsto x_{i}) : (\Gamma, x : A) = \Rightarrow \Gamma, x : A
\]
Defining the logical relation

\[
\begin{align*}
\Gamma \vdash A : U \\
\therefore \Gamma^= \vdash \sim_A : A[0] \rightarrow A[1] \rightarrow U
\end{align*}
\]

\[
f_0 \sim_{\Pi(x:A).B} f_1 \equiv \Pi(x_0 : A[0], x_1 : A[1], x_2 : x_0 \sim_A x_1).f_0 x_0 \sim_B f_1 x_1
\]

\[
(a, b) \sim_{\Sigma(x:A).B} (a', b') \equiv \Sigma(x_2 : a \sim_A a').b \sim_B [x_0 \mapsto a, x_1 \mapsto a'] b'
\]

\[
A \sim_u B \equiv A \rightarrow B \rightarrow U \text{ (parametricity)}
\]

\[
A \sim_u B \equiv A \simeq B \text{ (later)}
\]
Parametricity

\[
\Gamma \vdash t : A \\
\Gamma^= \vdash t^= : t[0] \sim_A t[1]
\]

This can be proven by induction on the term \( t \):

\[
\begin{align*}
(f \ u)^= & \equiv f^= \ u[0] \ u[1] \ u^= \\
(\lambda x. t)^= & \equiv \lambda x_0, x_1, x_2. t^= \\
x^= & \equiv x_2 \\
\text{U}^= & \equiv \sim\text{U} \\
(t[\rho])^= & \equiv t^= [\rho^=] \quad (\rho^= \text{ is pointwise})
\end{align*}
\]

The previous example:

\[
(A : U, u : A, s : A \rightarrow A)^= \vdash t^= : t[0] \sim_A t[1]
\]
Internal parametricity

Internalisation

The type of \texttt{param} is not well-formed:

\[
\cdot \vdash \texttt{param} : \Pi(A : U, t : A). t \sim_A t
\]

We need a substitution from \((A : U, t : A)\) to \((A : U, t : A)^\equiv\).

We define \(R_\Gamma : \Gamma \Rightarrow \Gamma^\equiv:\)

\[
R. \equiv () \\
R_{\Gamma.x:A} \equiv (R_\Gamma, x, x, \texttt{refl}\ x)
\]

\[
\frac{\Gamma \vdash a : A}{\Gamma \vdash \texttt{refl}\ a \equiv (a^\equiv)[R_\Gamma] : a \sim_A [R_\Gamma] a}
\]

Now we can define \texttt{param}:

\[
\cdot \vdash \lambda A, t.\texttt{refl}\ t : \Pi(A : U, t : A). t \sim_A [R_{A:U}] t
\]

Before adding \texttt{refl} we didn’t extend the theory. Now \((\texttt{refl}\ x)\) is a new normal form if \(x\) is a variable.
Parametricity of parametricity

- Given \((\text{refl } x)\) as a new normal form, we need to say what \((\text{refl } x)\) is, i.e. \(\text{refl } x = x_0 x_1 x_2\).

- We could add a new term former \(\text{refl }:\)

\[
\Gamma \vdash A : U \\
\Gamma^= \vdash a_0 : A[0] \\
\Gamma^= \vdash a_1 : A[1] \\
\Gamma^= \vdash a_2 : a_0 \sim_A a_1 \\
\Gamma^= \vdash \text{refl } a_0 a_1 a_2 : A^=\{R\} a_0 a_1 a_2 a_0 a_1 a_2 \ (\text{refl } a_0) \ (\text{refl } a_1)
\]

\[
\Gamma^= \vdash \text{refl } a_2 : A^=\{R\} a_0 a_0 (\text{refl } a_0) a_1 a_1 (\text{refl } a_1) a_2 a_2
\]

- \(\text{refl}\) and \(\text{refl }\) ways correspond to the two ways of degenerating a line into a square.

- We have \(\text{refl } a \equiv a^=\{R\}\).

- We don’t know how to compute \(\text{refl } a_0 a_1 a_2\).
Higher dimensions

\((x : A)^{≡\equiv}\) can be viewed as a context of squares:

\[
\begin{align*}
(x : A)^{≡\equiv} & \equiv (x_0 : A[0]) \cdot x_1 : A[1] \cdot x_2 : x_0 \sim A x_1)^{≡\equiv} \\
\equiv & \quad x_{00} : A[00] \cdot x_{01} : A[01] \cdot x_{02} : x_{00} \sim A[0] x_{01} \\
& \quad \cdot x_{10} : A[10] \cdot x_{11} : A[11] \cdot x_{12} : x_{10} \sim A[1] x_{11} \\
& \quad \cdot x_{20} : x_{00} \sim_A [0] x_{10} \cdot x_{21} : x_{01} \sim_A [1] x_{11} \cdot x_{22} : x_{20} \sim x_{0} \sim_A x_{1} x_{21}
\end{align*}
\]

This can be drawn as:

The result of \(R(x:A)^{≡}\) and \((R_{x:A})^{≡}\):

\[
\begin{align*}
x_{01} \xrightarrow{x_{21}} x_{11} & \quad x_{1} \xrightarrow{\text{refl } x_{1}} x_{1} & \quad x_{0} \xrightarrow{x_{2}} x_{1} \\
x_{02} \xrightarrow{x_{22}} x_{12} & \quad x_{2} \xrightarrow{\text{refl } x_{2}} x_{2} & \quad x_{2} \xrightarrow{\text{refl } x_{0}} x_{0} \xrightarrow{\text{refl } x_{1}} x_{0} x_{1} x_{2} \xrightarrow{\text{refl } x_{1}} x_{0} x_{1} x_{2} \\
x_{00} \xrightarrow{x_{20}} x_{10} & \quad x_{0} \xrightarrow{\text{refl } x_{0}} x_{0} & \quad x_{0} \xrightarrow{x_{2}} x_{1}
\end{align*}
\]
Adding dimension names

First we add more information to the contexts: for each usage of $\equiv$ we will use a new dimension name, so $(x : A)\equiv$ will become $(x : A)^{ij}$:

$$(\Gamma, x : A)^i \equiv^i \Gamma^i, x_{i0} : A[0^i], x_{i1} : A[1^i], x_{i2} : A^i x_{i0} x_{i1}$$

This refines the types of $R_{(x:A)=}$ and $R_{\equiv x:A}$:

$$R_{i(x:A)^j} : (x : A)^j \Rightarrow (x : A)^{ji}$$

$$(R_{i(x:A)})^j : (x : A)^j \Rightarrow (x : A)^{ji}$$

Their targets are different contexts, with dimensions swapped.
A definitional quotient

We would like to equate \((x : A)^{ij}\) and \((x : A)^{ji}\). These have different variable names, but they contain the same information: eg. \(x_{i1j2}\) corresponds to \(x_{j2i1}\). We add the following quotients:

\[
\Gamma^{ij} \equiv \Gamma^{ji} \\
\rho : \Delta \Rightarrow \Gamma \\
\rho^{ij} \equiv \rho^{ji} : \Delta^{ij} \Rightarrow \Gamma^{ij} \\
\Gamma^{ij} \vdash t : A \\
\Gamma^{ij} \vdash t^{ij} \equiv t^{ji} : A^{ij}\{x \mapsto t\}^{ij}
\]

\(\{x \mapsto t\}^{ij}\) is \((x \mapsto t)^{ij}\) omitting the last element.

Every term former needs to be symmetric, eg. \(\Pi\) types need to know which argument corresponds to which index, because we need

\[(\Pi(x : A).B)^{ij} \equiv (\Pi(x : A).B)^{ji}.
\]

i.e.

\[
\Pi(x_{i0j0} : A[0i0j], x_{i0j1} : A[0i1j], x_{i0j1} : A[0i]^j x_{i0j0} x_{i0j1}, \ldots) . B^{ij} \ldots
\]

\(\equiv \Pi(x_{i0j0} : A[0i0j], x_{i0j1} : A[0i1j], \ldots, x_{j1i0} : (A^j[0i] x_{j0i0} x_{j1i0}) . B^{ji} \ldots.
\)
New rules for \( \Pi \) types

New rules for full \( \Pi \) types and relations (\( I \) is a set of dimension names):

\[
\begin{align*}
\xi : \Gamma \Rightarrow (X : U)^I & \quad \Gamma.(x : X)^I[\xi] \vdash B : U \\
\Gamma \vdash \Pi(x : X)^I[\xi].B : U
\end{align*}
\]

\[
\begin{align*}
\xi : \Gamma \Rightarrow \{X : U\}_I & \\
\Gamma \vdash \Pi\{x : X\}_I[\xi].U : U
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \{x : X\}_I[\xi] & \vdash t : B \\
\Gamma \vdash \lambda\{x : X\}_I[\xi].t : \Pi\{x : X\}_I[\xi].B
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash f : \Pi\{x : X\}_I[\xi].B & \quad \Gamma \vdash \omega : \{y : X\}_I[\xi] \\
\Gamma \vdash \text{app}(f, \omega) : B[\text{id} \# \omega]
\end{align*}
\]
The computation rule of refl

We generalise the rule for parametricity from adding one dimension to adding a set of dimensions at a time. The parametricity rule becomes:

\[
\Gamma \vdash t : A \\
\Gamma^I \vdash t^I : \text{app}_I(A^I, \{x \mapsto t\}_I)
\]

The order of dimensions does not matter.

Lifting of the universe becomes:

\[
U^I \equiv \lambda\{X : U\}_I.\Pi\{x : X\}_I.U
\]

Now we have \((\text{refl}_i a)^j \equiv (a^i[R_{i\Gamma}])^j \equiv a^{ji}[R_{i\Gamma j}] \equiv a^{ji}[R_{i\Gamma j}] \equiv \text{refl}_i a^j\), so the problem with the computation rule of refl

\[\equiv\]

disappears.
Operational semantics

To rigorously define the theory, we need telescope contexts and substitutions. We defined a call-by-name operational semantics for this theory where the weak-head normal forms are:

\[
t, A ::= \ldots
\]

\[
\nu ::= () | (\nu, x \mapsto g) | (\nu, x \mapsto t[\nu])
\]

\[
\nu ::= \text{U} | (\prod\{x : X\}I[\rho].A)[\nu] | (\lambda\{x\}I.t)[\nu] | n
\]

\[
n ::= g | \text{app}(n, \nu)
\]

\[
g ::= x | \text{refl}_i g
\]

terms

environments

values (whnfs)

neutral values

generic values
Capturing univalence

Now we would like to replace the definition of $U^i$ to be a relation which is a graph of an equivalence.

We use the following notion of equivalence:

\[
(A \sim_{U^i} B) = \Sigma(\sim: A \to B \to U)
\]
\[
. \Pi(x : A).\text{isContr}(\Sigma(y : B).x \sim y)
\]
\[
\times \Pi(y : B).\text{isContr}(\Sigma(x : A).x \sim y).
\]

The new rule for parametricity expresses that terms respect (cubical) equality (we need to project out the relation):

\[
\Gamma \vdash t : A
\]
\[
\Gamma' \vdash t' : \text{app}_I(\sim_{A^I}, \{x \mapsto t\}_I)
\]
\[
\Gamma' \vdash A^I : A[0_I] \sim_{U^i} A[1_I]
\]

However we don’t have $U^{i\bar{j}} \equiv U^{j\bar{i}}$, so we need to refine this definition.
Conclusions

- We defined a theory for internal parametricity with an operational semantics.
- We still need to prove termination and completeness of the semantics.
- It’s not clear which definition of equivalence to use to capture univalence in a symmetric way.
- The final goal is a type theory with internal parametricity and univalence where some dimensions are parametricity dimensions and some dimensions are equalities (1-to-1 relations, having Kan fillers).

Thank you for your attention!