Two-dimensional proof-relevant parametricity

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Part I

Proof Irrelevant Parametricity
Parametricity

- **Ad hoc polymorphism**
  Defined in a different way depending on the type.
  Example:
  \[
  h : \forall X. X \rightarrow X \rightarrow X
  \]
  sum natural number ≠ concatenation lists.

- **Parametric polymorphism**
  Defined in the same way for any type.
  Example:
  \[
  \text{rev} : \forall X. \text{list}(X) \rightarrow \text{list}(X)
  \]
Usefulness of parametricity

- Extract properties from types.

Example: every parametric function

\[ h : \forall X. \text{list}(X) \to \text{list}(X) \]

satisfies

\[ h \left( \text{map } f \right) \, xs = \left( \text{map } f \right) \, h \, xs. \]

- Parametric terms give dinatural transformations.

Thanks to Reynolds’ relational interpretation.
Reynolds’ relational model of system F

- **Relations**
  - Relations are subsets $R \subseteq A \times B$ (Rel).
  - Equality $Eq: \text{Set} \rightarrow \text{Rel}$ plays special role.

- **Relational semantics for types**
  - $\llbracket T \rrbracket_0: |\text{Set}|^n \rightarrow \text{Set}$.
  - $\llbracket T \rrbracket_1: |\text{Rel}|^n \rightarrow \text{Rel}$.

- **Relational semantics for terms**
  - Natural transformation $\llbracket t \rrbracket_0$ between functors $|\text{Set}|^n \rightarrow \text{Set}$.
  - Natural transformation $\llbracket t \rrbracket_1$ between functors $|\text{Rel}|^n \rightarrow \text{Rel}$.
Reynolds’ key Theorems for parametricity

Theorem (Identity Extension Lemma (IEL))
*Functorial types interpretation commutes with equality*

\[ [T]_1 Eq = Eq [T]_0.\]

Theorem (Abstraction Theorem (AT))
*Term interpretation \([t]_0\) can be lifted to relational level \([t]_1\).*

Set level and relational level are mutually dependent.
Categorical semantics

The categorical semantics consists of:

- a category $\mathcal{B}$;
- a category $\mathcal{R}$ of binary relations $R_{(A,B)}$ between $A$ and $B$ in $\mathcal{B}$;
- equality functor $Eq: \mathcal{B} \rightarrow \mathcal{R}$.

The category $\mathcal{R}$ lives over $\mathcal{B}$. Some possible representations:

We start with reflexive graphs: more concrete.
In the 2D-setting, reflexive graphs don’t work while fibrations do.
Categorical model

Given a judgment $\Gamma, \Delta \vdash t : T$

- **Types**: equality preserving lifted functors.
- **Terms**: lifted natural transformations.
Interpretation of forall types

Start with all polymorphic functions.

- Set:

\[ [\forall X. T]_0 \bar{A} = \{ f : (S : \text{Set}) \rightarrow [T]_0(\bar{A}, S) \mid \} \]
Interpretation of forall types

Remove non parametric terms using uniformity condition.

Set:

\[
\forall X. T \bar{A} = \{ f : (S : \text{Set}) \rightarrow [T]_0(\bar{A}, S) \mid R_{(A,B)} \in \text{Rel} \Rightarrow (fA, fB) \in [T]_1(Eq\, \bar{A}, R_{(A,B)}) \}\]
Interpretation of forall types

Define relation $[\forall X. T]_1$: maps are related iff they map related inputs to related outputs

- Set:

$$[\forall X. T]_0 \bar{A} = \{ f : (S : \text{Set}) \rightarrow [T]_0(\bar{A}, S) \mid R_{(A,B)} \in \text{Rel} \Rightarrow (fA, fB) \in [T]_1(\text{Eq } \bar{A}, R_{(A,B)}) \}$$

- Rel:

$$[\forall X. T]_1 \bar{R} = \{(f, g) \mid R_{(A,B)} \in \text{Rel} \Rightarrow (fA, gB) \in [T]_1(\bar{R}, R_{(A,B)}) \}$$

Sound in the Calculus of Constructions with impredicative Set.
Part II

Proof Relevant Parametricity
via
Two-dimensional Relations
Related work

- Reynolds “Types, abstraction and parametric polymorphism”
- Benton, Hofmann, Nigam “Proof-relevant logical relations for name generation”
- Bernardy and Moulin “A computational interpretation of parametricity”
- Bezem, Coquand and Huber “A model of type theory in cubical sets”
- Grandis “The role of symmetries in cubical sets and cubical categories (on weak cubical categories, I)”
Proof irrelevant vs proof relevant relations

- **Proof irrelevant relations**

There is an unique proof which relates two elements. For $A$ and $B$ sets, a relation $R$ is a subset

$$R \subseteq A \times B.$$ 

- **Proof relevant relations**

There are different proofs which relate two elements. For $A$ and $B$ sets, a relation $R$ is a map

$$R : A \times B \rightarrow \text{Set}.$$ 

$R(a, b)$ is the set of proofs which relate $a$ and $b$. 
∀ types with proof relevant relations is non trivial

Proof Irrelevant Uniformity Condition \( \cong \)

\[ R_{(A,B)} \in \text{Rel} \Rightarrow (fA, fB) \in [T]_1(\widehat{Eq \ A}, R_{(A,B)}) \]

- Existentially quantify over the uniformity condition.
  - Some parametric terms are missing

- Require a proof relevant uniformity condition.
  - Some non parametric terms are included

- Solution: a **parametric** proof relevant uniformity condition.
Implementing the solution

We add a new layer of relations: $2\text{Rel}$. 

First approach: globular. 
$2$-relation $Q$ is a relation between two relations $N$ and $S$ 

Exponentiation fails in preserving equalities. 

Correct approach: cubical.
The category of 2Rel

A 2-relation is of the form

\[
\begin{array}{c}
A & \xleftarrow{N} & B \\
\uparrow W & & \uparrow E \\
C & \xleftarrow{S} & D
\end{array}
\]

where edges are proof relevant binary relations

\[
N : A \times B \to \text{Set} \quad W : A \times C \to \text{Set} \\
S : C \times D \to \text{Set} \quad E : B \times D \to \text{Set}
\]

and \(Q\) is a proof irrelevant relation over the proofs

\[
Q(a, b, c, d) \subseteq N(a, b) \times W(a, c) \times S(c, d) \times E(b, d).
\]
The structure of 2Rel

Every 2-relation lives over four relations.

\[
\begin{array}{c}
\partial_1 \quad \partial_2 \quad \partial_3 \quad \partial_4 \\
\downarrow & \downarrow \quad \quad \downarrow & \downarrow \\
\partial_1 \quad \partial_2 \\
\text{Rel} \quad \text{Rel} \\
\downarrow \quad \downarrow \\
\text{Set} \\
\end{array}
\]

There is an equality functor \( Eq_2 : \text{Set} \to 2\text{Rel} \) given by

\[
\begin{align*}
A & \equiv A \\
Eq_2(A) : & \equiv \equiv \\
A & \equiv A
\end{align*}
\]

Is there an equality functor \( \hat{Eq} : \text{Rel} \to 2\text{Rel} \)?
More equalities

There are four different ways to lift a relation $R_{(A,B)}$ to a 2-relation

They all take two proofs $p$ and $q$ in $R$ and ask $p = q$.

These maps define functors which we consider as new equalities.

Some of them look almost the same.
Structure

These maps do not form a reflexive graph. However they naturally form a fibration:

\[
\begin{array}{ccc}
2\text{Rel} & \xrightarrow{U_2} & \square\text{-Rel} \\
\end{array}
\]

Where \( U_2 = \langle \partial_1, \partial_2, \partial_3, \partial_4 \rangle \) and \( \square\text{-Rel} \) is the category given by 2-relations removing the filling of the square.
Parametrical model for proof relevant relations

Three levels interpretation: $\llbracket \_ \rrbracket_0$, $\llbracket \_ \rrbracket_1$ and $\llbracket \_ \rrbracket_2$. Again we interpret

- **types** as lifted functors;
- **terms** as lifted natural transformations.
Equality preservation

New equalities ⇒ new statement of IEL.

$\llbracket - \rrbracket_2$ should preserve every new equality functor:

$\llbracket T \rrbracket_2(Eq_{ns}) = Eq_{ns}(\llbracket T \rrbracket_1)$  $\llbracket T \rrbracket_2(Eq_{we}) = Eq_{we}(\llbracket T \rrbracket_1)$

$\llbracket T \rrbracket_2(Eq_{se}) = Eq_{se}(\llbracket T \rrbracket_1)$  $\llbracket T \rrbracket_2(Eq_{nw}) = Eq_{nw}(\llbracket T \rrbracket_1)$

We call IEL2 this generalization of the IEL to 2-relations.
For all types in proof relevant model

Start with all polymorphic functions

- Set:

\[
\begin{align*}
\llbracket \forall X. T \rrbracket_0 A &= \{ f_0 : (S : \text{Set}) \to \llbracket T \rrbracket_0 (\bar{A}, S) \\
&\quad f_1 : (R_{(A,B)} : \text{Rel}) \to \llbracket T \rrbracket_1 (\text{Eq} \, \bar{A}, R)(f_0 A, f_0 B) \}
\end{align*}
\]

- Rel:

\[
\begin{align*}
\llbracket \forall X. T \rrbracket_1 \bar{R}((f_0, f_1), (g_0, g_1)) &= \{ \phi : (R_{(A,B)} : \text{Rel}) \to \llbracket T \rrbracket_1 (\bar{R}, R_{(A,B)})(f_0 A, g_0 B) \}
\end{align*}
\]
For all types in proof relevant model

**Remove non parametric terms** using two-dimensional uniformity condition.

- **Set:**
  \[
  \left[\forall X. T\right]_0 \bar{A} = \left\{ f_0 : (S : Set) \rightarrow \left[ T \right]_0 (\bar{A}, S) \middle| f_1 : (R_{(A,B)} : \text{Rel}) \rightarrow \left[ T \right]_1 (\text{Eq } \bar{A}, R)(f_0 A, f_0 B) \right\}
  \]

  \( Q \in \text{2Rel} \Rightarrow (f_1 N, f_1 W, f_1 S, f_1 E) \in \left[ T \right]_2 (\text{Eq}_2 \bar{A}, Q)(f_0 A, f_0 B, f_0 C, f_0 D) \}

- **Rel:**
  \[
  \left[\forall X. T\right]_1 \bar{R}((f_0, f_1), (g_0, g_1)) = \left\{ \phi : (R_{(A,B)} : \text{Rel}) \rightarrow \left[ T \right]_1 (\bar{R}, R_{(A,B)})(f_0 A, g_0 B) \right\}
  \]
For all types in proof relevant model

Remove non parametric proofs using two-dimensional uniformity condition.

Set:
\[
\begin{align*}
\forall X. T]_0 \bar{A} & = \\
& \{ f_0 : (S : \text{Set}) \to [T]_0(\bar{A}, S) \\
& f_1 : (R_{(A,B)} : \text{Rel}) \to [T]_1(\bar{E}q \bar{A}, R)(f_0 A, f_0 B) | \\
Q \in 2\text{Rel} \Rightarrow (f_1 N, f_1 W, f_1 S, f_1 E) \in [T]_2(\bar{E}q_2 \bar{A}, Q)(f_0 A, f_0 B, f_0 C, f_0 D) \}
\end{align*}
\]

Rel:
\[
\begin{align*}
\forall X. T]_1 \bar{R}((f_0, f_1), (g_0, g_1)) & = \\
& \{ \phi : (R_{(A,B)} : \text{Rel}) \to [T]_1(\bar{R}, R_{(A,B)})(f_0 A, g_0 B) | \\
P(\bar{E}q_{ns}, \bar{E}q_{we}, \bar{E}q_{nw}, \bar{E}q_{se}) \}
\end{align*}
\]
For all types in proof relevant model

**Define 2-relation**  $[\forall X. T]_2$: maps are related iff they map related inputs to related outputs

▶ **Set:**

$$\forall X. T]_0 \bar{A} = \{ f_0 : (S : \text{Set}) \to [T]_0(\bar{A}, S) \}
\quad f_1 : (R_{(A,B)} : \text{Rel}) \to [T]_1(\bar{E}q \bar{A}, R)(f_0 A, f_0 B) | Q \in 2\text{Rel} \Rightarrow (f_1 N, f_1 W, f_1 S, f_1 E) \in [T]_2(\bar{E}q_2 \bar{A}, Q)(f_0 A, f_0 B, f_0 C, f_0 D)$$

▶ **Rel:**

$$\forall X. T]_1 \bar{R}((f_0, f_1), (g_0, g_1)) = \{ \phi : (R_{(A,B)} : \text{Rel}) \to [T]_1(\bar{R}, R_{(A,B)})(f_0 A, g_0 B) | P(Eq_{ns}, Eq_{we}, Eq_{nw}, Eq_{se}) \}$$

▶ **2Rel:**

$$\forall X. T]_1 \bar{Q}((f_0, f_1), (g_0, g_1), (h_0, h_1), (l_0, l_1)) = \{ (\phi, \phi', \xi, \xi') | Q \in 2\text{Rel} \Rightarrow (\phi N, \phi' W, \xi S, \xi' E) \in [T]_2(\bar{Q}, Q)(f_0 A, g_0 B, h_0 C, l_0 D) \}$$
$P(Eq_{ns}, Eq_{we}, Eq_{nw}, Eq_{se})$: highly intricate

$\forall Q: 2\text{Rel}$

$$(f_1N, \phi'W, g_1S, \xi'E) \in [T]_2(\overline{Eq_{ns}} \overline{R}, Q)(f_0A, f_0B, g_0C, g_0D)$$

$$(\phi N, f_1W, \xi S, g_1E) \in [T]_2(\overline{Eq_{we}} \overline{R}, Q)(f_0A, g_0B, f_0C, g_0D)$$

$$(\phi N, f_1W, g_1S, g_1E) \in [T]_2(\overline{Eq_{se}} \overline{R}, Q)(f_0A, g_0B, g_0C, g_0D)$$

$$(f_1N, f_1W, \xi S, \xi'E) \in [T]_2(\overline{Eq_{nw}} \overline{R}, Q)(f_0A, f_0B, f_0C, g_0D)$$
Symmetric structure

Relations have the following symmetry

\[ A \xleftrightarrow{R} B = B \xleftrightarrow{R^{op}} A \]

In the case of set \( R^{op}(b, a) = R(a, b) \).

Similar symmetries for 2Rel transform equalities in other equalities:
Conjecture

**Conjecture**: functors invariant under the previous symmetries.

**Symmetries**

\[
\sigma_1 : \text{Rel} \rightarrow \text{Rel} \quad \sigma_2 : \text{2Rel} \rightarrow \text{2Rel}
\]

then

\[
\llbracket T \rrbracket_1(\sigma_1(R)) = \sigma_1(\llbracket T \rrbracket_1(R))
\]

and

\[
\llbracket T \rrbracket_2(\sigma_2(Q)) = \sigma_2(\llbracket T \rrbracket_2(Q)).
\]

**Simplification**: less requirements in the interpretation of for all types.

We checked it for Rel. The trick is that the functor are equality preserving and equality is symmetric.
AT2 and IEL2: we have proven

Theorem (AT2)

Every term $Γ, ∆ ⊢ t : T$ defines a lifted natural transformation

$$([t]_2, [t]_1, [t]_0) : ([Δ]_2, [Δ]_1, [Δ]_0) → ([T]_2, [T]_1, [T]_0).$$

Theorem (IEL2)

$$[T]_1(Eq) = Eq([T]_0)$$

$$[T]_2(Eq_{ns}) = Eq_{ns}([T]_1) \quad [T]_2(Eq_{we}) = Eq_{we}([T]_1)$$

$$[T]_2(Eq_{se}) = Eq_{se}([T]_1) \quad [T]_2(Eq_{nw}) = Eq_{nw}([T]_1)$$
Suggestions?

We are looking for some applications.

If you have in mind some possible applications, please let us know!
Thank you!