A proof with side effects of Gödel’s completeness theorem suitable for semantic normalisation

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TYPES

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Preliminary: proving with side effects

- Classical logic seen as a side effect:
  - Direct style = a control operator (e.g. callcc of type Peirce’s law) [Griffin 90]
  - Indirect style = continuation-passing-style/double-negation translation within intuitionistic logic

- This talk:
  - Interpreting Kripke forcing translation as indirect style for what is in direct style a monotonic memory update
  - Applying this to obtain a proof with side-effect of Gödel's completeness theorem as direct-style presentation of a proof of completeness w.r.t. Kripke semantics
Kripke forcing translation as an environment monad

Let $\geq$ be a partial order. A key clause of Kripke forcing is the interpretation of implication:

$$w \Vdash A \rightarrow B \triangleq \forall w' \geq w [ (w' \Vdash A) \rightarrow (w' \Vdash B) ]$$

The transformation

$$\Box_w A(w) \triangleq \forall w' \geq w A(w')$$

can be seen as a dependent environment monad, i.e. as indirect style for a monotonic memory update effect.
Direct-style for Kripke forcing

A rule for initialising the use of Kripke forcing:

\[
\begin{align*}
\Gamma, [b : x \geq t] & \vdash q : T(x) \\
\Gamma & \vdash r : \text{refl} \geq x' \\
\Gamma & \vdash s : \text{trans} \geq x' \text{ fresh in } \Gamma \text{ and } T(t) \\
\hline
\Gamma & \vdash \text{set } x := t \text{ as } b/_{(r,s)} \text{ in } q : T(t) \quad \text{SETEFF}
\end{align*}
\]

A rule for updating:

\[
\begin{align*}
\Gamma, [b : x \geq t(x')] & \vdash q : T(x) \\
\Gamma & \vdash r : t(x') \geq x' \\
[x \geq u] & \in \Gamma \text{ for some } u \\
x' & \text{ fresh in } \Gamma \\
\hline
\Gamma & \vdash \text{update } x := t(x) \text{ of } x' \text{ as } b \text{ by } r \text{ in } q : T(t(x)) \quad \text{UPDATE}
\end{align*}
\]

where we wrote \( T, U \) for \( \rightarrow, \forall \)-free formulas (= intuitively \( \Sigma_1^0 \)-formulas = base types)
Gödel’s completeness
Object language

We consider here the negative fragment of predicate logic as an object language (we consider \( \bot \) to be an arbitrary atom and abbreviate \( \neg A \equiv A \rightarrow \bot \)).

\[
\begin{align*}
t & \triangleq x \mid f(t_1, \ldots, t_n) \\
F, G & \triangleq \bot \mid \dot{P}(t_1, \ldots, t_n) \mid F \rightarrow G \mid \forall x F \\
\Gamma & \triangleq \epsilon \mid \Gamma, F
\end{align*}
\]

We take the following inference rules:

\[
\begin{align*}
\text{Ax}^{\Gamma, F, \Gamma'} & : (\Gamma, F \subset \Gamma') \rightarrow (\Gamma' \vdash F) \\
\text{App}^{\Gamma, F, G} & : (\Gamma \vdash F \rightarrow G) \rightarrow (\Gamma \vdash F) \rightarrow (\Gamma \vdash G) \\
\text{Abs}^{\Gamma, F, G} & : (\Gamma, F \vdash G) \rightarrow (\Gamma \vdash F \rightarrow G) \\
\text{Abs}_\forall^{\Gamma, x, F} & : (\Gamma \vdash F) \rightarrow (x \not\in FV(\Gamma)) \rightarrow (\Gamma \vdash \forall x F(x)) \\
\text{App}_\forall^{\Gamma, x, t, F} & : (\Gamma \vdash \forall x F) \rightarrow (\Gamma \vdash F[t/x])
\end{align*}
\]

Moreover, the following is admissible:

\[
\text{weak}^{\Gamma'}_{\Gamma, F} : (\Gamma \subset \Gamma') \rightarrow (\Gamma \vdash F) \rightarrow (\Gamma' \vdash F)
\]

We shall also write \( r^\Gamma_F \) for a proof of \( \Gamma \subset (\Gamma, F) \).
Tarskian models

A Tarskian model $\mathcal{M}$ is made of a domain $\mathcal{D}_M$ for interpreting terms, of an interpretation of function symbols $\mathcal{F}_M(f) : \mathcal{D}^{a_f} \rightarrow \mathcal{D}$ and of an interpretation of atoms $\mathcal{P}_M(\dot{P}) \subset \mathcal{D}^{a_{\dot{P}}}$ (for $a_f$, $a_{\dot{P}}$ the arity of $f$, $\dot{P}$ resp.).

Truth is defined by

\[
\begin{align*}
[x]_\mathcal{M} & \triangleq \sigma(x) \\
[ft_1 \ldots t_{af}]_\mathcal{M} & \triangleq \mathcal{F}_M(f)([t_1]_\mathcal{M}, \ldots, [t_{af}]_\mathcal{M}) \\
\models_\mathcal{M} \dot{P}(t_1, \ldots, t_{a_{\dot{P}}}) & \triangleq \mathcal{P}_M(\dot{P})([t_1]_\mathcal{M}, \ldots, [t_{a_{\dot{P}}}]_\mathcal{M}) \\
\models_\mathcal{M} \bot & \triangleq \mathcal{P}_M(\bot) \\
\models_\mathcal{M} F \rightarrow G & \triangleq \models_\mathcal{M} F \rightarrow \models_\mathcal{M} G \\
\models_\mathcal{M} \forall x F & \triangleq \forall t \in \mathcal{M}_D \models_\mathcal{M}^{\sigma[x\leftarrow t]} F
\end{align*}
\]
Completeness w.r.t Tarskian models

Let $Clas$ be the theory containing $\neg\neg F \rightarrow F$ for all formulas $F$ (atoms are enough).

We define $\vdash_{C} F$ to be $Clas \vdash_{M} F$ in minimal logic.

A Tarskian model $\mathcal{M}$ for classical logic is a Tarskian model which satisfies $\models_{\mathcal{M}} Clas$ (in a classical meta-language, all Tarskian models are classical, but not in an intuitionistic meta-language).

The statement of completeness w.r.t Tarskian models for classical logic is:

$$\forall \mathcal{M} \forall \sigma (\models^{\sigma}_{\mathcal{M}} Clas \rightarrow \models^{\sigma}_{\mathcal{M}} F) \rightarrow Clas \vdash_{M} F$$

The usual proof is by contradiction, building a saturated counter-model by enumeration of the formulas.

The proof with effects we shall consider actually works for arbitrary theories, so that we shall consider instead the following statement:

$$(\forall \mathcal{M} \forall \sigma \models^{\sigma}_{\mathcal{M}} F) \rightarrow \vdash_{M} F$$
Completeness w.r.t. Kripke models
Kripke models

A Kripke model $\mathcal{K}$ is an increasing family of Tarskian models indexed over a set of worlds $\mathcal{W}_\mathcal{K}$ ordered by $\geq_\mathcal{K}$. In the absence of $\lor$ and $\exists$, it is enough to take $\mathcal{D}_\mathcal{K}$ constant.

Truth relatively to $\mathcal{K}$ at world $w$ is defined by:

$$[x]_\mathcal{K}^\sigma \triangleq \sigma(x)$$

$$[ft_1 \ldots t_{af}]_\mathcal{K}^\sigma \triangleq \mathcal{F}_\mathcal{K}(f)([t_1]_\mathcal{K}^\sigma, \ldots, [t_{af}]_\mathcal{K}^\sigma)$$

$$w \models_\mathcal{K} \dot{P}(t_1 \ldots t_{af}) \triangleq \mathcal{P}_\mathcal{K}(\dot{P})([t_1]_\mathcal{K}^\sigma, \ldots, [t_{af}]_\mathcal{K}^\sigma)$$

$$w \models_\mathcal{K} \perp \triangleq \mathcal{P}_\mathcal{K}(\perp)_w$$

$$w \models_\mathcal{K} F \rightarrow G \triangleq \forall w' \geq_\mathcal{K} w \left( w' \models_\mathcal{K} F \rightarrow w' \models_\mathcal{K} G \right)$$

$$w \models_\mathcal{K} \forall x F \triangleq \forall t \in \mathcal{K}_D \ w \models_\mathcal{K}^{[x \leftarrow t]} F$$

The statement of completeness w.r.t. Kripke models is:

$$(\forall \mathcal{K} \forall \sigma \forall w \in \mathcal{W}_\mathcal{K} \ w \models_\mathcal{K}^\sigma F') \rightarrow \vdash_M F$$
Completeness w.r.t Kripke models

The “standard” proof works by building the canonical model $K_0$ defined by taking $W_{K_0}$ to be the typing contexts ordered by inclusion, $D_{K_0}$ to be the terms, $K_{\mathcal{F}}(f)$ to be the syntactic application of $f$, and $K_{\mathcal{P}}(\hat{P})(\Gamma)(t_1, \ldots, t_{a_{\hat{P}}})$ to be $\Gamma \vdash_M \hat{P}(t_1, \ldots, t_{a_{\hat{P}}})$.

The main lemma proves $\Gamma \vdash_M F \iff \Gamma \models_{K_0} F$ by induction on $F$.

\[
\begin{align*}
\uparrow^\Gamma_F & \quad \Gamma \vdash_M F \quad \rightarrow \quad \Gamma \models_{K_0} F \\
\uparrow^\Gamma_{\hat{P}(\vec{t})} & \quad p \quad \triangleq \quad p \\
\uparrow^\Gamma_{F \to G} & \quad p \quad \triangleq \quad \Gamma' \mapsto h \mapsto m \mapsto \uparrow^\Gamma_{G} \cdot \text{App}^{\Gamma',F,G}(\text{weak}_{\Gamma,F}(h,p), \downarrow_{F} m) \\
\uparrow^\Gamma_{\forall x F} & \quad p \quad \triangleq \quad t \mapsto \uparrow^\Gamma_{F[t/x]} \cdot \text{App}_\forall^{\Gamma,x,F}(p,t) \\
\downarrow^\Gamma_F & \quad \Gamma \models_{K_0} F \quad \rightarrow \quad \Gamma \vdash_M F \\
\downarrow^\Gamma_{\hat{P}(\vec{t})} & \quad m \quad \triangleq \quad m \\
\downarrow^\Gamma_{F \to G} & \quad m \quad \triangleq \quad \text{Abs}_{\rightarrow}^{\Gamma,F,G}(\downarrow^\Gamma_{G} m (\Gamma, F) r^\Gamma_{F} (\uparrow^\Gamma_{F} \text{Ax}_{1,F,\Gamma}(b_{F}))) \\
\downarrow^\Gamma_{\forall x F} & \quad m \quad \triangleq \quad \text{Abs}_\forall^{\Gamma,x,F}(\dot{y}, \downarrow^\Gamma_{F[z/x]}(m \dot{y})) \quad \quad \dot{y} \text{ fresh in } \Gamma
\end{align*}
\]

And finally:

\[
\text{compl} \triangleq \nu \mapsto \downarrow^\epsilon_A (\nu K_0 \emptyset \epsilon) : (\forall \mathcal{K} \forall \sigma \forall w \in W_{\mathcal{K}} w \models_{\mathcal{K}} F) \rightarrow \vdash_M F
\]
Completeness w.r.t. Kripke models in direct-style
Kripke forcing translation for second-order arithmetic

We consider a second-order arithmetic multi-sorted over first-order datatypes such as $\mathbb{N}$, lists, formulas, etc., and with primitive recursive atoms written $P(t_1, \ldots, t_{aP})$.

$$A, B \triangleq X(t_1, \ldots, t_{aX}) \ | \ P(t_1, \ldots, t_{aP}) \ | \ A \land B \ | \ A \rightarrow B \ | \ \forall x \ A \ | \ \forall X \ A$$

Let $\geq$ be a partial order. We extend Kripke forcing to second order quantification.

$$
\begin{align*}
    w \models X(t_1, \ldots, t_{aX}) & \triangleq X(w, t_1, \ldots, t_{aX}) \\
    w \models P(t_1, \ldots, t_{aP}) & \triangleq P(t_1, \ldots, t_{aP}) \\
    w \models A \land B & \triangleq (w \models A) \land (w \models B) \\
    w \models A \rightarrow B & \triangleq \forall w' \geq w [(w' \models A) \rightarrow (w' \models B)] \\
    w \models \forall x \ A & \triangleq \forall x \ w \models A \\
    w \models \forall X \ A & \triangleq \forall X (\text{mon}(X) \rightarrow w \models A)
\end{align*}
$$

where $\text{mon}(X) \triangleq \forall w \forall w' \geq w (X(w, t_1, \ldots, t_{aX}) \rightarrow X(w', t_1, \ldots, t_{aX}))$
Relating completeness w.r.t Tarskian models to completeness w.r.t. Kripke models

We get a stronger statement of completeness by considering completeness w.r.t Kripke models by specifically instantiating $\mathcal{W}_K$ to be the typing contexts and $\geq$ to be inclusion of contexts.

$$(\forall (\mathcal{D}_K, \mathcal{F}_K, \mathcal{P}_K) \forall \sigma [\epsilon \models_{(\mathcal{W}_K, \mathcal{D}_K, \mathcal{F}_K, \mathcal{P}_K)} F]) \rightarrow \vdash_M F$$

Now, applying forcing shows that

$$\epsilon \models_{x} (\forall (\mathcal{D}_M, \mathcal{F}_M, \mathcal{P}_M) \forall \sigma \models_{(\mathcal{D}_M, \mathcal{F}_M, \mathcal{P}_M)} F')$$

is equivalent to

$$\forall (\mathcal{D}_K, \mathcal{F}_K, \mathcal{P}_K) \forall \sigma (\epsilon \models_{(\mathcal{W}_K, \mathcal{D}_K, \mathcal{F}_K, \mathcal{P}_K)} F')$$

and hence that forcing over the statement of completeness w.r.t. Tarskian models is equivalent to the instantiation of the set of worlds to typing contexts of completeness w.r.t. Kripke models.
Excerpt of our meta-language with effects

\[ \Gamma \vdash p : A(y) \quad y \text{ fresh in } \Gamma \]
\[ \Gamma \vdash \lambda y.p : \forall y A(y) \]
\[ \forall_I \]

\[ \Gamma \vdash p : \forall x A(x) \quad t \text{ updatable-variable-free or } t \text{ an updatable variable and } A(x) \text{ of type } 1 \]
\[ \Gamma \vdash pt : A(t) \]
\[ \forall_E \]

\[ \forall^2 \]

\[ \forall I \]
\[ \forall^2 E \]

\[ \Gamma, [b : x \geq t] \vdash q : T(x) \quad \Gamma \vdash r : refl \geq \quad \Gamma \vdash s : trans \geq \quad x \text{ fresh in } \Gamma \text{ and } T(t) \]
\[ \Gamma \vdash \text{set } x := t \text{ as } b/(r,s) \text{ in } q : T(t) \]
\[ \text{SETEFF} \]

\[ \Gamma, [b : x \geq t(x')] \vdash q : T(x) \quad \Gamma \vdash r : t(x') \geq x' \quad [x \geq u] \in \Gamma \text{ for some } u \quad x' \text{ fresh in } \Gamma \]
\[ \Gamma \vdash \text{update } x := t(x) \text{ of } x' \text{ as } b \text{ by } r \text{ in } q : T(t(x)) \]
\[ \text{UPDATE} \]

where \( C \) of type 1 means in the grammar \( C ::= P(t_1, \ldots, t_{a_P}) \mid P(t_1, \ldots, t_{a_P}) \to C \mid \forall x C \) and \( \text{mon}_\Gamma B \) means \( B \) monotonic for all updatable variables in \( \Gamma \)
The completeness proof in direct-style

In direct style, $\mathcal{K}_0$ is the model $\mathcal{M}_0$ defined by $\mathcal{P}_{\mathcal{M}}(\dot{P})(t_1, ..., t_{a_\dot{P}}) \triangleq \Gamma \vdash \dot{P}(t_1, ..., t_{a_\dot{P}})$ for $\Gamma$ a given updatable variable.

\[
\begin{align*}
\uparrow_F & \quad \Gamma \vdash_M F \quad \longrightarrow \quad \models_{\mathcal{M}_0} F \\
\uparrow_{P(t)} & \quad g \quad \triangleq \quad g \\
\uparrow_{F \rightarrow G} & \quad g \quad \triangleq \quad m \mapsto \uparrow_G \text{App}_{\Gamma,F,G}(g, \downarrow_F m) \\
\uparrow_{\forall x F} & \quad g \quad \triangleq \quad t \mapsto \uparrow_{F[t/x]} \text{App}_{\forall,F}(g, t)
\end{align*}
\]

\[
\begin{align*}
\downarrow_F & \quad \models_{\mathcal{M}_0} F \quad \longrightarrow \quad \Gamma \vdash_M F \\
\downarrow_{P(t)} & \quad m \quad \triangleq \quad m \\
\downarrow_{F \rightarrow G} & \quad m \quad \triangleq \quad \text{Abs}_{\Gamma,F,G} (\text{update } \Gamma := (\Gamma, F) \text{ of } \Gamma_1 \text{ as } b_F \text{ by } r_{\Gamma_1}^F \text{ in } \downarrow_G (m (\uparrow_F \text{Ax}_{\Gamma_1,F,G}^F(b_F)))) \\
\downarrow_{\forall x F} & \quad m \quad \triangleq \quad \text{Abs}^F_{\forall,F}(\dot{y}, \downarrow_{F[z/x]} (m \dot{y}))
\end{align*}
\]

\[
\text{compl} \triangleq v \mapsto \text{set } \Gamma := \epsilon \text{ as } b/(r,s) \text{ in } \downarrow^F (v \mathcal{M}_0 \emptyset)
\]

Obviously, the resulting proof in the object language is a reification of the proof of validity as in Normalisation-by-Evaluation / semantic normalisation [C. Coquand 93, Danvy 96, Altenkirch-Hofmann 96, Okada 99, ...]
Properties of the meta-language with update effect

- Can be equipped with a reduction semantics (derived from the forcing interpretation)
- Consistent because interpretable by forcing in pure intuitionistic logic
- Inconsistent with any non-intuitionistic assumption (like classical logic)
- However, variants (under investigation) are possible:
  - Local use of classical reasoning providing Markov’s principle and Double Negation Shift are possible using Ilik’s variant of Kripke forcing
  - Full compatibility with classical logic using Cohen forcing to be investigated
  - Preservation of consistency when mixing several uses of forcing on functional “conditions” to be investigated