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Priestley Duality for Strong Proximity Lattices

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Abstract

In 1937 Marshall Stone extended his celebrated representation theorem for Boolean algebras to distributive lattices. In modern terminology, the representing topological spaces are zero-dimensional stably compact, but typically not Hausdorff. In 1970, Hilary Priestley realised that Stone's topology could be enriched to yield order-disconnected compact ordered spaces.
In the present paper, we generalise Priestley duality to a representation theorem for strong proximity lattices. For these a "Stone-type" duality was given in 1995 in joint work between Philipp Sünderhauf and the second author, which established a close link between these algebraic structures and the class of all stably compact spaces. The feature which distinguishes the present work from this duality is that the proximity relation of strong proximity lattices is "preserved" in the dual, where it manifests itself as a form of "apartness." This suggests a link with constructive mathematics which in this paper we can only hint at. Apartness seems particularly attractive in view of potential applications of the theory in areas of semantics where continuous phenomena play a role; there, it is the distinctness between different states which is observable, not equality.
The idea of separating states is also taken up in our discussion of possible morphisms for which the representation theorem extends to an equivalence of categories.

Keywords: strong proximity lattices, totally order-disconnected spaces, Priestley duality, stably compact spaces, apartness relations.

1 Introduction

Philipp Sünderhauf and the second author in their paper [7] introduced a class of bounded distributive lattices enriched with additional structure as follows:

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Definition 1.1 A binary relation $\prec$ on a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ is called a *proximity* if, for every $a, x, y \in L$ and $M \subseteq_{fin} L$,

$$
\begin{aligned}
&(\prec\prec)\ \prec \circ \prec \,=\, \prec,\\
&(\vee\!-\prec)\ M \prec a \Longleftrightarrow \bigvee M \prec a,\\
&(\prec-\wedge)\ a \prec M \Longleftrightarrow a \prec \bigwedge M,\\
&(\prec-\vee)\ a \prec x \vee y \Longrightarrow (\exists\, x', y' \in L)\ x' \prec x,\ y' \prec y \text{ and } a \prec x' \vee y',\\
&(\wedge\!-\prec)\ x \wedge y \prec a \Longrightarrow (\exists\, x', y' \in L)\ x \prec x',\ y \prec y' \text{ and } x' \wedge y' \prec a.
\end{aligned}
$$

$M \prec a$ and $a \prec M$, respectively, stand for $(\forall m \in M)\ m \prec a$ and $(\forall m \in M)\ a \prec m$. A *strong proximity lattice* is a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ together with a proximity relation $\prec$ on L. [3]

The notion of strong proximity lattice subsumes that of bounded distributive lattice as the lattice order $\leq$ is always a proximity.

The objective in [15] and [7] was to define a duality for *stably compact spaces*, which are those topological spaces that are sober, compact, locally compact, and for which the collection of compact saturated subsets is closed under finite intersections where a saturated set is an intersection of open sets. A reason why stably compact spaces are interesting for computer scientists is that this notion captures by topological means most semantic domains in the mathematical theory of computation [13,14]. The duality established in [7], then, is the basis for a logical description (expounded in [5,6]) of these spaces similar to Samson Abramsky's *domain theory in logical form*, [1].

While stably compact spaces are typically T_0, *Priestley duality*, [9,10], associates with a bounded distributive lattice a *Hausdorff* space. The resulting spaces are defined as follows:

Definition 1.2 A *Priestley space* is a compact ordered space $\langle X; \mathcal{T}, \leq\rangle$ such that for every $x, y \in X$, if $x \not\geq y$ then there exists a clopen lower set U such that $x \in U$ and $y \notin U$.

The question that this paper is answering is the following:

How can Priestley's representation theorem for bounded distributive lattices be extended to strong proximity lattices?

From a representation point of view the various dualities can be classified as follows:

[3] The qualifier "strong" distinguishes the concept from its precursor in [15], where $(\wedge\!-\prec)$ was not a requirement.

	T_0	T_2
strong proximity lattices	Jung & Sünderhauf, [7]	present paper
distributive lattices	Stone, [18]	Priestley, [9,10]
Boolean algebras	Stone, [16,17]	

Our interest is partly driven by mathematical systematics but there is also a story to be told from a semantics point of view. In [6] the argument was made that the proximity $\prec$ relates two logical propositions ϕ and ψ if the *observation* of ϕ always implies that ψ is *actually true*. Consequently, the logical system does not necessarily satisfy the identity axiom $\phi \vdash \phi$, and while the paper [6] demonstrates that a satisfactory and even elegant logical apparatus can still be built, the lack of this basic law of logic may feel strange. In the present paper, the view is that the proximity is *additional structure*, over and above the lattice operations, and that for the latter the usual axioms of logic are still valid. Consequently, a *model* of the logic is given by a prime ideal, prime filter pair, as it is usually. The additional structure on the logic then gives rise to *additional structure* on the space of all models, which we read as *apartness* information.[4] The intuition is that two states of affair (i.e., models) can be observably separated if and only if they are "sufficiently apart." To give an example, consider the real numbers represented in their usual decimal representation. Mathematically, we deem $a = 1.000\ldots$ and $b = 0.999\ldots$ equal; constructively, the concrete presentation of a number is important, and in our example one would find that a and b can not be told apart in finite time but their equality can also not be established in finite time (if our only access to the numbers is by successively reading digits).

The following definition attempts to capture the intuitive notion of apartness on a Priestley space:

Definition 1.3 A binary relation $\propto$ on a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ is called an *apartness* if, for every $a, c, d, e \in X$,

$$
\begin{aligned}
&(\propto\mathcal{T})\ \propto \text{ is open in } \langle X;\mathcal{T}\rangle \times \langle X;\mathcal{T}\rangle\\
&(\downarrow\!\propto\!\uparrow)\ a \leq c \propto d \leq e \Longrightarrow a \propto e,\\
&(\propto\forall)\ a \propto c \Longleftrightarrow (\forall b \in X)\ a \propto b \text{ or } b \propto c,\\
&(\propto\uparrow\uparrow)\ a \propto ({\uparrow}c \cap {\uparrow}d) \Longrightarrow (\forall b \in X)\ a \propto b,\ b \propto c \text{ or } b \propto d,
\end{aligned}
$$

[4] Indeed, there is a rather conventional way to fill in the right upper position in the table above. For this one equips the collection of *round* prime filters of L with the topology generated by all $U_x := \{F \mid x \in F\}$, and all $O_x := \{F \mid \exists y \notin F.x \prec y\}$, $x \in L$. This yields the *patch topology* of a stably compact space which is already obtainable from the Jung-Sünderhauf dual.

$(\downarrow\downarrow\propto)$ $(\downarrow c \cap \downarrow d) \propto a \Longrightarrow (\forall b \in X)\ d \propto b,\ c \propto b$ or $b \propto a$.

where $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

Remark 1.4 (i) For any Priestley space $\langle X; \leq, \mathcal{T} \rangle$, $\not\geq$ is an apartness (because the order is required to be closed for ordered spaces).

(ii) $\propto$ is an apartness on $\langle X; \leq, \mathcal{T} \rangle$ if and only if $\propto^{-1}$ is an apartness on $\langle X; \geq, \mathcal{T} \rangle$.

(iii) It aids the intuition to assume that an element can not be apart from itself but as a matter of fact our results do not rely on this assumption.

(iv) If we were to axiomatise *indistinguishability* instead of apartness, then $(\propto\forall)$ would express the transitivity of this relation. Axiom $(\downarrow\propto\uparrow)$, however, would not have a simple formulation.

(v) On the real line, axioms $(\propto\uparrow\uparrow)$ and $(\downarrow\downarrow\propto)$ are the same as $(\propto\forall)$.

Our question above is answered as follows:

The dual of a strong proximity lattice L is the corresponding Priestley space of prime ideals, equipped with the apartness,

$$I \propto_{\prec} J \stackrel{\text{def}}{\Longleftrightarrow} (\exists\ x \notin I)(\exists\ y \in J)\ x \prec y.$$

Vice versa, the dual of a Priestley space X with apartness $\propto$ is the lattice of clopen downsets equipped with the proximity,

$$A \prec_{\propto} B \stackrel{\text{def}}{\Longleftrightarrow} A \propto (X \setminus B).$$

Up to isomorphism, the correspondence is one-to-one.

We will show that the action of Priestley duality on morphisms can also be adapted to the current setting.

Continuous order-preserving maps that reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms that preserve the proximity relation.

While Priestley maps are the correct choice for establishing the duality, they are too specialised from a computational point of view; if we consider their manifestation on semantic domains then we recognise them as (order-preserving) Lawson continuous functions. This does not cover the computable maps, however, which typically are only Scott-continuous. Nevertheless, the situation here is no different from domain theory where also more than one kind of map is studied on a fixed class of spaces, for example, embedding-projection pairs, Scott-continuous function, strict Scott-continuous function, stable function, etc.

We consider two more general notions of morphism in order to capture more computable functions, and study their transformation under the duality. On the side of strong proximity lattices we replace homomorphisms with *approximable relations*; this approach goes back to Scott's morphisms for *information systems* in [14], and was adapted to strong proximity lattices in [7]. The definition is as follows:

Definition 1.5 Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be a binary relation from L_1 to L_2. The relation $\vdash$ is called *approximable* if for every $a \in L_1, b \in L_2$, $M_1 \subseteq_{fin} L_1$ and $M_2 \subseteq_{fin} L_2$,

$$
\begin{aligned}
&(\vdash - \prec_2)\ \vdash \circ \prec_2 = \vdash,\\
&(\prec_1 - \vdash)\ \prec_1 \circ \vdash = \vdash,\\
&\quad(\vee - \vdash)\ M_1 \vdash b \iff \bigvee M_1 \vdash b,\\
&\quad(\vdash - \wedge)\ a \vdash M_2 \iff a \vdash \bigwedge M_2,\\
&\quad(\vdash - \vee)\ a \vdash \bigvee M_2 \Longrightarrow (\exists\ N \subseteq_{fin} L_1)\ a \prec_1 \bigvee N \text{ and } (\forall n \in N)\\
&\qquad\qquad(\exists\ m \in M_2)\ n \vdash m.
\end{aligned}
$$

The relation $\vdash$ is called *weakly approximable* if it satisfies all of the above conditions but not necessarily $(\vdash - \vee)$.

Note that since we are dealing with a relation, rather than a function, we are free to turn around the direction. Thus we will arrive at an equivalence of categories rather than a duality. The relationship between proximity homomorphisms and approximable relations is then seen to be analogous to that between Dijkstra's *weakest preconditions* and *Hoare logic*: A homomorphism h from L_2 to L_1 gives the weakest precondition $h(\phi)$ that needs to be satisfied for ϕ to hold at the end of the computation. An approximable relation $\vdash$ from L_1 to L_2, on the other hand, links all propositions ϕ, ψ where ϕ (before the computation) entails ψ afterwards.

Let us now look on the side of Priestley spaces; whereas in [7] we identified the corresponding morphisms as those functions that are continuous with respect to the upper topology, now we can not expect functionality at all. The reason is that the Priestley dual contains more points than the *spectrum* considered in [7] and there is no reason to assume that the process acts functionally on the additional elements. In keeping with the spirit of the present paper, we instead consider *relations* between Priestley spaces which relate those pairs of elements that are "observably unrelated" by the computational process. Here is the definition:

Definition 1.6 Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces with apartness relations $\propto_1$ and $\propto_2$, respectively, and let $\ltimes$ be a binary relation from X_1 to X_2. The relation $\ltimes$ is called *separating* (or a *separator*) if it is open in $\mathcal{T}_1 \times \mathcal{T}_2$ and if, for every $a, b \in X_1, d, e \in X_2$ and $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$,

$(\downarrow_1\ltimes\uparrow_2)$ $a \leq_1 b \ltimes d \leq_2 e \Longrightarrow a \ltimes e$,

$(\forall\ltimes)$ $b \ltimes d \Longleftrightarrow (\forall c \in X_1)\ b \propto_1 c$ or $c \ltimes d$,

$(\ltimes\forall)$ $b \ltimes d \Longleftrightarrow (\forall c \in X_2)\ b \ltimes c$ or $c \propto_2 d$,

$(\ltimes n\uparrow)$ $b \ltimes \bigcap \uparrow d_i \Longrightarrow (\forall c \in X_1)\ b \propto_1 c$ or $(\exists\ i)\ c \ltimes d_i$.

The relation $\ltimes$ is called *weakly separating* if it is open and satisfies all of the above conditions, but not necessarily $(\ltimes n\uparrow)$.

Some effort is required to show that we get a category this way (see Section 4.3) but the expected equivalence does hold:

Let X_1 and X_2 be Priestley spaces with apartness relation. Then (weakly) separating relations from X_1 to X_2 are in one-to-one correspondence with (weakly) approximable relations from the dual of X_1 to the dual of X_2.

We state two more results which cannot be fully proved here because of lack of space. The first concerns the connection between the T_0 dualities and the T_2 dualities tabulated above. It is well known that the Stone dual of a distributive lattice is strongly related to the Priestley dual [4]: In one direction one takes the patch topology, in the other one restricts to upper open sets. We expect to be able to construct the Jung-Sünderhauf duals of strong proximity lattices (which are precisely the stably compact spaces) from Priestley spaces with apartness, and vice versa. This is indeed possible. To state the precise result we make the following definitions.

Definition 1.7 Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$ and $A, B \subseteq X$. Then

(i) $\propto [A] = \{x \in X \mid x \propto A\}$ and $[A] \propto = \{x \in X \mid A \propto x\}$.

(ii) An open upper set O in X is said to be *isolated* if $O = [X \setminus O] \propto$. The set of all isolated subsets of X is denoted by $\mathit{iso}(X)$.

(iii) $\mathit{core}(X) = \{x \in X \mid [x] \propto = X \setminus \downarrow x\} = \{x \in X \mid X \setminus \downarrow x$ is isolated$\}$.

Theorem 1.8 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle \mathit{core}(X), \mathcal{T}'\rangle$, where*

$$\mathcal{T}' = \{O \cap \mathit{core}(X) \mid O \text{ is an open lower subset of } X\},$$

is a stably compact space. Moreover every stably compact space can be obtained in this way and is a retract of a Priestley space with apartness.

Our terminology suggests that apartness relations on Priestley spaces are related to Giuseppe Sambin's *pre-topologies*, [11,12,2]; this is indeed the case:

Theorem 1.9 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then $\mathit{iso}(X)$ is closed under finite intersections. Therefore $\langle \mathit{iso}(X); \cap, X\rangle$ is*

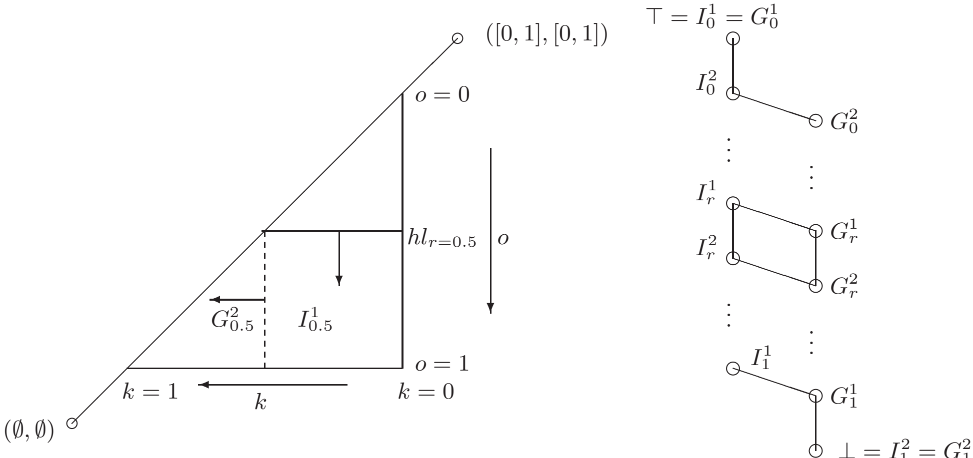

Fig. 1. The lattice $\mathcal{B}$ on the left together with its representing Priestley space X.

a commutative monoid. The relation $\lhd$ *on* $iso(X)$*, defined by*

$$O_1 \lhd O_2 \overset{\text{def}}{\iff} O_1 \propto (X \setminus O_2),$$

satisfies the requirements for a precover *in the sense of [12].*

We conclude our overview with two examples. The first concerns the stably compact space that consists of a single element $*$. Its strong proximity lattice is a chain of three elements $0 < e < 1$ with $0 \prec x$ and $x \prec 1$ for all three choices of x. Our Priestley dual has *two* elements; besides $* = \{0\}$ there is now also $n = \{0, e\}$ which is "indistinguishable" from $*$, that is, the apartness relation is empty. A case can be made that this is in analogy to an automated theorem prover which can answer either with a proof or a counterexample for a given statement. In the denotational semantics, value $*$ would be assigned if the process stops with a proof, and value n is assigned if the process is trapped in an infinite search for a counterexample. In finite time we are not able to distinguish the validity of the two propositions that the theorem prover is working on.[5]

For a more elaborate example consider the strong proximity lattice $\mathcal{B} = \{((o,1],[k,1]) \mid 0 \le k \le o \le 1\}$ where

$$((o,1],[k,1]) \prec ((o',1],[k',1]) \overset{\text{def}}{\iff} o' < k.$$

It represents the unit interval (with the Scott topology) as a stably compact space under the Jung-Sünderhauf duality. Figure 1 gives a pictorial description of the lattice $\mathcal{B}$. For a given $r \in [0,1]$, we define a horizontal line hl_r and a

[5] A more complete model of this situation would also have values for the case that a counterexample is produced, and for the case that the process searches forever for a proof. These two values are clearly apart from $*$ and n but indistinguishable from each other.

vertical line vl_r of the lattice $\mathcal{B}$ as follows

$$hl_r = \{((r,1],[k,1]) \mid 0 \leq k \leq r\},\ vl_r = \{((o,1],[r,1]) \mid r \leq o \leq 1\}.$$

The following facts describe the Priestley space $\langle X; \mathcal{T}, \leq, \propto\rangle$ which represents $\mathcal{B}$. A pictorial presentation is given in Figure 1.

(i) X is the set of prime ideals of $\mathcal{B}$ which can be described concretely as:

$$X = \{I_r^1, I_r^2, G_r^1, G_r^2 \mid 0 \leq r \leq 1\},$$

where

$$I_r^1 = \downarrow hl_r,\ I_r^2 = \downarrow hl_r \setminus hl_r,\ G_r^1 = \downarrow vl_r \text{ and } G_r^2 = \downarrow vl_r \setminus vl_r.$$

Pictorial descriptions of $I_{0.5}^1$ and $G_{0.5}^2$ are also given in Figure 1.

For a computational interpretation, assume some concrete representation of real numbers as finite and infinite streams of digits. A stream that begins with 0.5 and then stops explicitly (indicating that all following digits are zero) corresponds to value $G_{0.5}^2$ in that it validates all tests $x < 0.5 + \epsilon$ with $\epsilon > 0$. On the other hand, a stream that begins with 0.4 and then produces 9's forever, corresponds to $G_{0.5}^1$ in that the test $x < 0.5$ does not produce "false" in finite time. No test can distinguish between the two streams by looking at a finite initial segment of digits; the most we have is a test ($x < 0.5$) which terminates for one and never answers for the other.

(ii) $\leq = \subseteq$.

(iii) The following collection is a sub-basis for $\mathcal{T}$,

$$\mathcal{S}(\mathcal{T}) = \{\downarrow I_r^2,\ \downarrow G_r^2,\ \uparrow I_r^1,\ \uparrow G_r^1 \mid 0 \leq r \leq 1\}.$$

(iv) $(\forall A_r, A_{r'} \in X)\ A_r \propto A_{r'} \iff r > r'$.

(v) Let $0 \leq r \leq 1$. Then $A = \uparrow G_r^1$ and $B_r = \bigcup_{r'<r} \uparrow G_{r'}^1$ are open upper subsets of X. We note that A is not isolated because $G_r^2 \in X \setminus A$ and $G_r^2 \not\propto G_r^1$ which means that $A \neq [X \setminus A] \propto$. On the other hand, B_r is isolated because, clearly, $B_r = [X \setminus B_r] \propto$. Moreover $\{B_r \mid 0 \leq r \leq 1\}$ is the set of isolated proper subsets of X.

2 From Priestley spaces with apartness to strong proximity lattices

In the following we denote the sets of clopen lower and upper sets of a Priestley space X by $\mathcal{O}^{\mathcal{T}}(X)$ and $\mathcal{U}^{\mathcal{T}}(X)$, respectively. We begin with two preparatory technical results (where the first is well known from the theory of compact ordered spaces, [8]).

Lemma 2.1 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space. For closed upper subsets $A, B \subseteq X$ and $O \in \mathcal{T}$, if $A \cap B \subseteq O$, then there exist $V_1, V_2 \in \mathcal{U}^{\mathcal{T}}(X)$ such that $A \subseteq V_1, B \subseteq V_2$ and $V_1 \cap V_2 \subseteq O$.*

Proof. In a Priestley space every closed upper subset is the intersection of clopen upper subsets containing it. Therefore

$$\begin{aligned} A \cap B &= \bigcap\{W \in \mathcal{U}^{\mathcal{T}}(X) \mid A \subseteq W\} \cap \bigcap\{W' \in \mathcal{U}^{\mathcal{T}}(X) \mid B \subseteq W'\} \\ &= \bigcap\{W \cap W' \mid W, W' \in \mathcal{U}^{\mathcal{T}}(X), A \subseteq W \text{ and } B \subseteq W'\}. \end{aligned}$$

By the compactness of $(X \setminus O)$ and the closedness of sets $W \cap W'$, there exists a finite set

$$\{W_i \cap W'_i \mid W_i, W'_i \in \mathcal{U}^{\mathcal{T}}(X),\ A \subseteq W_i, B \subseteq W'_i \text{ and } 1 \leq i \leq n\}$$

such that $\bigcap_{1 \leq i \leq n}(W_i \cap W'_i) \subseteq O$, so we can set $V_1 = \bigcap_i W_i$ and $V_2 = \bigcap_i W'_i$. □

Lemma 2.2 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. For closed subsets $A, B \subseteq X$, if $A \propto B$ then there exist $U \in \mathcal{O}^{\mathcal{T}}(X)$ and $V \in \mathcal{U}^{\mathcal{T}}(X)$ such that $A \subseteq U$, $B \subseteq V$ and $U \propto V$.*

Proof. By $(\downarrow\!\propto\!\uparrow)$, $\downarrow A \propto \uparrow B$. Recall that $\downarrow A$ and $\uparrow B$ are closed subsets of X because $\mathcal{T}$ is a Priestly topology. We first show that there exist open sets O_1 and O_2 such that $\downarrow A \subseteq O_1$, $\uparrow B \subseteq O_2$ and $O_1 \propto O_2$. Fix $x \in \downarrow A$. Then for every $y \in \uparrow B$, by openness of $\propto$, there exist O_{1y}, $O_{2y} \in \mathcal{T}$ such that $x \in O_{1y}$, $y \in O_{2y}$ and $O_{1y} \propto O_{2y}$. The set $\{O_{2y} \mid y \in \uparrow B\}$ is an open cover of $\uparrow B$, and so a finite sub-cover $\{O_{2y_i} \mid 1 \leq i \leq n\}$ exists. Set $O_{1x} = \bigcap_i O_{1y_i}$ and $O_{2x} = \bigcup_i O_{2y_i}$. Then O_{1x} and O_{2x} are open sets with $x \in O_{1x}, \uparrow B \subseteq O_{2x}$ and $O_{1x} \propto O_{2x}$. Now, the set $\{O_{1x} \mid x \in \downarrow A\}$ is an open cover of $\downarrow A$ and so a finite sub-cover $\{O_{1x_i} \mid 1 \leq i \leq m\}$ exists. Set $O_1 = \bigcup_i O_{1x_i}$ and $O_2 = \bigcap_i O_{2x_i}$. Then O_1 and O_2 are open sets with $\downarrow A \subseteq O_1$, $\uparrow B \subseteq O_2$ and $O_1 \propto O_2$.

Now as $(X, \leq, \mathcal{T})$ is a Priestley space, $\downarrow A = \bigcap\{U \in \mathcal{O}^{\mathcal{T}}(X) \mid \downarrow A \subseteq U\} \subseteq O_1$, and because $X \setminus O_2$ is a compact subset of X, there exists a finite set of clopen-lowers $\{U_i \in \mathcal{O}^{\mathcal{T}}(X) \mid 1 \leq i \leq n\}$ such that $\downarrow A \subseteq \bigcap_i U_i \subseteq O_1$. Set $U = \bigcap_i U_i$. Then U is a clopen-lower subset of X with $\downarrow A \subseteq U \subseteq O_1$. Similarly there exists a clopen-upper subset V of X with $\uparrow B \subseteq V \subseteq O_2$. Therefore $U \propto V$, which completes the proof. □

Remark 2.3 Singletons are closed subsets in a Priestley space. Therefore, as a special case of Lemma 2.2 we have that for every $a, b \in X$, if $a \propto b$ then there exists $U \in \mathcal{O}^{\mathcal{T}}(X)$ and $V \in \mathcal{U}^{\mathcal{T}}(X)$ such that $a \in U$, $b \in V$ and $U \propto V$.

Remark 2.4 Our proof of Lemma 2.2 uses only the openness of $\propto$ and $(\downarrow\!\propto\!\uparrow)$. Conversely, considering Remark 2.3, the lemma implies $(\downarrow\!\propto\!\uparrow)$ and the openness of $\propto$.

We define the dual for a Priestley space with apartness as follows:

Definition 2.5 Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then

$$Prox(X) = \langle \mathcal{O}^{\mathcal{T}}(X); \cup, \cap, \emptyset, X; \prec_{\propto}\rangle,$$

where $\prec_{\propto}$ is the binary relation defined on $\mathcal{O}^{\mathcal{T}}(X)$ as:

$$A \prec_{\propto} B \overset{\text{def}}{\iff} A \propto (X \setminus B).$$

Remark 2.6 Note that if X carries the trivial apartness $\not\geq$ then the lattice $\mathcal{O}^{\mathcal{T}}(X)$ will be equipped with the trivial proximity $\prec_{\propto} = \subseteq$. In fact, the converse is also true: if $\prec_{\propto} = \subseteq$ then $\propto = \not\geq$.

Lemma 2.7 *The relation $\prec_{\propto}$ satisfies ($\prec\prec$).*

Proof. Suppose $A \prec_{\propto} C$ and $D = X \setminus C$. So $A \propto D$. Fix $a \in A$ and set $O_a = \{x \in X \mid a \propto x\}$. Then $O_a \in \mathcal{T}$ by openness of $\propto$, $a \propto O_a$ and $(X \setminus O_a) \propto D$ by ($\propto\forall$) and the fact that $a \propto D$.

Now by Lemma 2.2, there exists a clopen-lower subset B_a of X such that $(X \setminus O_a) \subseteq B_a$ and $B_a \propto D$. Therefore $a \propto (X \setminus B_a)$. Using Lemma 2.2 again, there exists $U_a \in \mathcal{O}^{\mathcal{T}}(X)$ such that $a \in U_a$ and $U_a \propto (X \setminus B_a)$. The set $\{U_a \mid a \in U\}$ is an open cover of A which is compact as it is closed. Hence a finite sub-cover $\{U_{a_i}\}_{1\leq i\leq n}$ exists. Set $U = \bigcup_{1\leq i\leq n} U_{a_i}$ and $B = \bigcup_{1\leq i\leq n} B_{a_i}$. Therefore U and B are clopen-lower subsets of X with $A \subseteq U \propto (X \setminus B)$ and $B \propto D$ which implies $A \prec_{\propto} B$ and $B \prec_{\propto} C$. This proves $\prec_{\propto} \subseteq \prec_{\propto};\prec_{\propto}$.

For the other inclusion, suppose $A \prec_{\propto} B$, $B \prec_{\propto} C$ and $D = X \setminus C$. Then $A \propto (X \setminus B)$ and $B \propto D$. Pick any $a \in A$ and $d \in D$, then any $b \in X$ is either in B or in $X \setminus B$, so $a \propto b$ or $b \propto d$ from which $a \propto d$ follows by ($\propto\forall$). Therefore $A \propto D$ which implies $A \prec_{\propto} C$. □

Lemma 2.8 *The relation $\prec_{\propto}$ satisfies ($\prec -\vee$) and ($\wedge- \prec$).*

Proof. Suppose $A \prec_{\propto} U \cup V$, $C = X \setminus U$ and $D = X \setminus V$. Then $A \propto C \cap D$ implies

$$(\forall a \in A)(\forall c \in C)(\forall d \in D)\ a \propto (\uparrow c \cap \uparrow d).$$

Fix $c \in C$ and $d \in D$. Set $O_c = \{x \in X \mid x \propto c\}$ and $O_d = \{x \in X \mid x \propto d\}$. By ($\downarrow\propto\uparrow$) and openness of $\propto$, O_c and O_d are open lower subsets of X and clearly $O_c \propto c$, $O_d \propto d$. Moreover, by ($\propto\uparrow\uparrow$), $A \propto (X \setminus (O_c \cup O_d))$.

By Lemma 2.2 and Lemma 2.1,

$$\begin{aligned} A \propto (X \setminus O_c) \cap (X \setminus O_d) \text{ hence } (\exists V_c, V_d \in \mathcal{U}^{\mathcal{T}}(X))\ & (X \setminus O_c) \subseteq V_c, \\ & (X \setminus O_d) \subseteq V_d \text{ and } A \propto (V_c \cap V_d) \\ \text{and }\ & (X \setminus V_c) \propto c \text{ and } (X \setminus V_d) \propto d \end{aligned}$$

$$\text{hence } (\exists W_c, W_d \in \mathcal{U}^{\mathcal{T}}(X))\ c \in W_c, d \in W_d,$$
$$(X \setminus V_c) \propto W_c \text{ and } (X \setminus V_d) \propto W_d.$$

The sets $\{W_c \mid c \in C\}$ and $\{W_d \mid d \in D\}$ are open covers of compact subsets C and D, respectively. Therefore finite subcovers $\{W_{c_i} \mid c_i \in C \text{ and } 1 \leq i \leq n\}$ and $\{W_{d_i} \mid d_i \in D \text{ and } 1 \leq i \leq m\}$ exist.

Set $U' = \bigcap_i X \setminus V_{c_i}$ and $V' = \bigcap_i X \setminus V_{d_i}$. Then U' and V' are clopen lower subsets satisfying $U' \propto C \subseteq \bigcup_i W_{c_i}$ which implies $U' \prec_\propto U$. Equally, $V' \propto D \subseteq \bigcup_i W_{d_i}$ implies $V' \prec_\propto V$. And finally $A \propto X \setminus (U' \cup V')$ implies $A \prec_\propto U' \cup V'$.

The argument for $(\wedge\!-\prec)$ is dual to this. □

Theorem 2.9 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then $Prox(X) = \langle \mathcal{O}^{\mathcal{T}}(X); \cup, \cap, \emptyset, P; \prec_\propto\rangle$ is a strong proximity lattice.*

Proof. Clearly $\langle \mathcal{O}^{\mathcal{T}}(X); \cup, \cap, \emptyset, X\rangle$ is a bounded distributive lattice. Lemma 2.7 proves that $\prec_\propto$ satisfies $(\prec\prec)$. $(\vee\!-\prec)$ and $(\prec-\wedge)$ only require Boolean manipulation. Lemma 2.8 proves $(\prec-\vee)$ and $(\wedge\!-\prec)$. □

3 From strong proximity lattices to Priestley spaces with apartness

For a lattice L, the set of prime ideals of L is denoted by $\mathcal{I}_{\mathcal{P}}(L)$. This is ordered by inclusion and given the *Priestley topology* $\mathcal{T}_L$ generated by the collections $O_x = \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid x \notin I\}$ and $U_x = \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid x \in I\}$. Obviously, $U_x = \mathcal{I}_{\mathcal{P}}(L) \setminus O_x$ and so each O_x is a clopen lower, and each U_x a clopen upper set.

Definition 3.1 Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. We set

$$Pries(L) = \langle \mathcal{I}_{\mathcal{P}}(L); \subseteq, \mathcal{T}_L; \propto_\prec\rangle,$$

where $\propto_\prec$ is the binary relation defined on $\mathcal{I}_{\mathcal{P}}(L)$ as follows:

$$I \propto_\prec J \stackrel{\text{def}}{\Longleftrightarrow} (\exists\, x \notin I)(\exists\, y \in J)\ x \prec y.$$

Remark 3.2 $\prec\, = \,\leq \quad \Longrightarrow \quad \propto_\prec\, = \,\not\supseteq.$

We will now show that $\propto_\prec$ does indeed validate the requirements for an apartness. The following preparatory result extends the definition to the basic clopen sets O_x and U_x.

Lemma 3.3 *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice and $x, y \in L$. Then*

$$x \prec y \Longleftrightarrow O_x \propto_\prec U_y.$$

Proof. ($\Longrightarrow$) is clear. For the other direction, suppose $x, y \in L$ such that $x \not\prec y$. Set $\xi = \{I \in \mathcal{I}(L) \mid y \in I,\ (\forall t \in I)\ x \not\prec t\}$. $\xi \neq \emptyset$, because $\downarrow y \in \xi$ and $(\xi, \subseteq)$ is a poset. If $\{I_i\}$ is a non-empty chain in $(\xi, \subseteq)$ then clearly $\bigcup_i I_i \in \xi$. Therefore by Zorn's Lemma ξ has a maximal element J. We claim that J is prime. Suppose $a, b \in L \setminus J$ but $a \wedge b \in J$. $J_a = \downarrow\{a \vee c \mid c \in J\}$ is an ideal properly containing J. Because J is maximal in ξ, $J_a \notin \xi$. So there exists $c_a \in J$ such that $x \prec a \vee c_a$. Similarly, there exists $c_b \in J$ such that $x \prec b \vee c_b$. Now, by [7, Lemma 7], we note the following:

$$\begin{aligned} x \prec a \vee c_a \text{ and } x \prec b \vee c_b &\Longrightarrow x \prec (a \vee c_a) \vee c_b \text{ and } x \prec (b \vee c_b) \vee c_a \\ &\Longleftrightarrow x \prec ((a \vee c_a) \vee c_b) \wedge ((b \vee c_b) \vee c_a) \\ &\Longleftrightarrow x \prec (a \wedge b) \vee (c_a \vee c_b). \end{aligned}$$

This gives a contradiction, because $J \in \xi$ and $(a \wedge b) \vee (c_a \vee c_b) \in J$.

Let $\mathcal{F}(L)$ be the collection of filters in L, and set $\zeta = \{F \in \mathcal{F}(L) \mid x \in F \text{ and } (\forall a \in F)(\forall b \in J)\ a \not\prec b\}$. $\zeta \neq \emptyset$ because $\uparrow x \in \zeta$ and $(\zeta, \subseteq)$ is a poset. If $\{F_i\}$ is a non-empty chain in $(\zeta, \subseteq)$ Then clearly $\cup_i F_i \in \zeta$. Hence by Zorn's Lemma ζ has a maximal element F. We claim that F is prime. Suppose $a, b \in L \setminus F$ but $a \vee b \in F$. Then $F_a = \uparrow\{a \wedge c \mid c \in F\}$ is a filter properly containing F. Because F is maximal in ζ, $F_a \notin \zeta$. So there exists $c_a \in F$ and $d_a \in I$ such that $a \wedge c_a \prec d_a$. Similarly, there exists $c_b \in F$ and $d_b \in I$ such that $a \wedge c_b \prec d_b$. By [7, Lemma 7], we note the following:

$$\begin{aligned} a \wedge c_a \prec d_a \text{ and } b \wedge c_b \prec d_b &\Longrightarrow (a \wedge c_a) \wedge c_b \prec d_a \vee d_b \\ &\qquad \text{and } (b \wedge c_b) \wedge c_a \prec d_b \vee d_a \\ &\Longleftrightarrow ((a \wedge c_a) \wedge c_b) \vee ((b \wedge c_b) \wedge c_a) \prec d_a \vee d_b \\ &\Longleftrightarrow (a \vee b) \wedge (c_a \wedge c_b) \prec d_a \vee d_b. \end{aligned}$$

The last statement is a contradiction, because $(a \vee b) \wedge (c_a \wedge c_b) \in F$, $d_a \vee d_b \in I$ and $F \in \zeta$. Hence F is a prime filter. Set $I = B \setminus F$. Then I and J are prime ideals with

$$x \notin I,\ y \in J, \text{ and } I \not\propto_\prec J$$

which completes the proof. □

Lemma 3.4 *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the relation $\propto_\prec$ of Pries(L) is open in $\mathcal{T}_L \times \mathcal{T}_L$ and satisfies $(\downarrow\!\propto\!\uparrow)$.*

Proof. Clearly $\propto_\prec$ satisfies $(\downarrow\!\propto\!\uparrow)$. Now suppose $I \propto_\prec J$. Then there exist $x, y \in L$ such that $x \prec y$, $x \notin I$ and $y \in J$. Therefore

$$\{I \in \mathcal{I}_\mathcal{P}(L) \mid x \notin I\} \propto_\prec \{J \in \mathcal{I}_\mathcal{P}(L) \mid y \in J\}.$$

But these sets are open in the Priestley space which proves the openness of $\propto_\prec$. □

Remark 3.5 Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the relation $\propto_\prec$ of *Pries*(L) satisfies Lemma 2.2 by Remark 2.4 and Lemma 3.4.

Lemma 3.6 *The relation* $\propto_\prec$ *satisfies* $(\propto\forall)$.

Proof. For any $I, J \in \mathcal{I}_\mathcal{P}(L)$,

$$\begin{aligned} I \propto_\prec J &\Longleftrightarrow (\exists\, a \in X \setminus I)(\exists\, c \in J)\ a \prec c \\ &\Longleftrightarrow (\exists\, b \in L)\ a \prec b \text{ and } b \prec c,\ \text{by } (\prec\prec) \\ &\Longleftrightarrow O_a \propto_\prec U_b \text{ and } O_b \propto_\prec U_c \text{ by Lemma } 3.3 \\ &\Longleftrightarrow (\forall K \in \mathcal{I}_\mathcal{P}(K))\ I \propto_\prec K \text{ or } K \propto_\prec J. \end{aligned}$$

The right-to-left direction of the last equivalence is proved as follows. Set $O = \{K \in \mathcal{I}_\mathcal{P}(L) \mid K \propto_\prec J\}$. Then O is an open set with $I \propto_\prec (\mathcal{I}_\mathcal{P}(L) \setminus O)$ and $O \propto_\prec J$. Finally apply Lemma 2.2 to get a clopen upset A around $\mathcal{I}_\mathcal{P}(L) \setminus O$ with $I \propto_\prec A$ and hence $\mathcal{I}_\mathcal{P}(L) \setminus A \propto_\prec J$. By the compactness of $\mathcal{I}_\mathcal{P}(L) \setminus O$ the set A can be chosen to be of the form U_b. $\square$

Lemma 3.7 *The relation* $\propto_\prec$ *satisfies* $(\propto\uparrow\uparrow)$ *and* $(\downarrow\downarrow\propto)$.

Proof. Let $I, J, K \in \mathcal{I}_\mathcal{P}(L)$ be such that $I \propto_\prec (\uparrow K \cap \uparrow J)$. Recall that $\uparrow K$ and $\uparrow J$ are closed subsets of $\mathcal{T}_L$. Recalling Remark 3.5, we can apply Lemma 2.2 and 2.1 to get $a, x, y \in L$ such that $I \in O_a$, $\uparrow K \subseteq U_x$, $\uparrow J \subseteq U_y$ and

$$O_a \propto_\prec U_x \cap U_y = U_{x \vee y}.$$

Hence, by Lemma 3.3, $a \prec x \vee y$. Now we have

$$\begin{aligned} a \prec x \vee y\ &\text{hence}\ (\exists\, x', y' \in L)\ x' \prec x,\ y' \prec y \text{ and } a \prec x' \vee y',\ \text{by } (\prec -\vee) \\ &\text{hence}\ O_{x'} \propto_\prec U_x,\ O_{y'} \propto_\prec U_y \text{ and } O_a \propto_\prec U_{x' \vee y'} \text{ by Lemma } 3.3 \\ &\text{hence}\ (\forall H \in \mathcal{I}_\mathcal{P}(L))\ H \propto_\prec K, H \propto_\prec J \text{ or } I \propto_\prec H. \end{aligned}$$

The argument for $(\downarrow\downarrow\propto)$ is dual. $\square$

Lemmas 3.4, 3.6, and 3.7 prove:

Theorem 3.8 *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the relation $\propto_\prec$ of Pries(L) is apartness on the Priestley space $\langle \mathcal{I}_\mathcal{P}(L); \subseteq, \mathcal{T}_L\rangle$.*

4 One duality and two equivalences

4.1 Objects

We show that the translations of the previous two sections are (essentially) inverses of each other. Since our theory is based on Priestley duality, only the behaviour of proximity and apartness need to be examined.

Definition 4.1 A lattice homomorphism (isomorphism) between strong proximity lattices is said to be a *proximity homomorphism (proximity isomorphism)* if it preserves (preserves in both directions) the proximity relation (relations).

Theorem 4.2 *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the map*

$$\eta_L : L \longrightarrow \mathcal{O}^{\mathcal{T}}(\mathcal{I}_{\mathcal{P}}(L)); x \longmapsto O_x,$$

is a proximity isomorphism from L to $Prox(Pries(L))$.

Proof. By Priestley duality, η_L is a lattice isomorphism. For every $x, y \in L$,

$$\begin{aligned} x \prec y &\Longleftrightarrow O_x \propto_{\prec} U_y = (\mathcal{I}_{\mathcal{P}}(L) \setminus O_y), \text{ by Lemma 3.3} \\ &\Longleftrightarrow O_x \prec_{\propto_\prec} O_y \\ &\Longleftrightarrow \eta_L(x) \prec_{\propto_\prec} \eta_L(y). \end{aligned}$$

□

Definition 4.3 Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. A map $f : X_1 \longrightarrow X_2$ is said to be:

- an *apartness map* from X_1 to X_2 if it is continuous, order-preserving, and for every $a, b \in X_1$,

$$f(a) \propto_2 f(b) \Longrightarrow a \propto_1 b,$$

- an *apartness homeomorphism* from X_1 to X_2 if it is an order-preserving homeomorphism and for every $a, b \in X_1$,

$$a \propto_1 b \Longleftrightarrow f(a) \propto_2 f(b).$$

Theorem 4.4 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the map*

$$\epsilon_X : X \longrightarrow \mathcal{I}_{\mathcal{P}}(\mathcal{O}^{\mathcal{T}}(X)); x \longmapsto \{U \in \mathcal{O}^{\mathcal{T}}(X) \mid x \notin U\},$$

is an apartness homeomorphism from X to $Pries(Prox(X))$.

Proof. By Priestley duality, ϵ_X is an order-preserving homeomorphism from X onto $\mathcal{I}_{\mathcal{P}}(\mathcal{O}^{\mathcal{T}}(X))$. We have

$$\begin{aligned} x \propto y &\Longleftrightarrow (\exists U \in \mathcal{O}^{\mathcal{T}}(X))(\exists V \in \mathcal{U}^{\mathcal{T}}(X))\ x \in U, y \in V \text{ and } U \propto V \\ &\Longleftrightarrow (\exists U \in \mathcal{O}^{\mathcal{T}}(X))(\exists V \in \mathcal{U}^{\mathcal{T}}(X))\ x \in U, y \in V \text{ and } U \prec_{\propto} X \setminus V \\ &\Longleftrightarrow \{U \in \mathcal{O}^{\mathcal{T}}(X) \mid x \notin U\} \propto_{\prec_\propto} \{U \in \mathcal{O}^{\mathcal{T}}(X) \mid y \notin U\} \\ &\Longleftrightarrow \epsilon_X(x) \propto_{\prec_\propto} \epsilon_X(y). \end{aligned}$$

The first equivalence is true by Lemma 2.2. □

Remark 4.5 Since we know that the trivial apartness and proximity relations get translated into each other, it is clear that our representation theorem is a proper extension of that of Priestley.

4.2 Morphism I: apartness maps and proximity homomorphisms

Lemma 4.6 *Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively, and $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices.*

(i) *For $f : L_1 \longrightarrow L_2$ a proximity homomorphism the map*
$$\mathit{Pries}(f) : \mathcal{I}_{\mathcal{P}}(L_2) \longrightarrow \mathcal{I}_{\mathcal{P}}(L_1); I \longmapsto f^{-1}(I),$$
is an apartness map from $\mathit{Pries}(L_2)$ to $\mathit{Pries}(L_1)$.

(ii) *For $\varphi : X_1 \longrightarrow X_2$ an apartness map the function*
$$\mathit{Prox}(\varphi) : \mathcal{O}^{\mathcal{T}}(X_2) \longrightarrow \mathcal{O}^{\mathcal{T}}(X_1); U \longmapsto \varphi^{-1}(U),$$
is a proximity homomorphism from $\mathit{Prox}(X_2)$ to $\mathit{Prox}(X_1)$.

Proof.

(i) $\mathit{Pries}(f)$ is a well defined continuous order preserving map by [3, Theorem 11.31]. For every $I, J \in \mathcal{I}_{\mathcal{P}}(L_2)$,
$$\begin{aligned}\mathit{Pries}(f)(I) \propto_1 \mathit{Pries}(f)(J) &\Longrightarrow (\exists\, a \notin \mathit{Pries}(f)(I))\\ &\quad\;\; (\exists\, b \in \mathit{Pries}(f)(J))\ a \prec_1 b\\ &\Longrightarrow f(a) \prec_2 f(b)\\ &\Longrightarrow I \propto_2 J,\ \text{because } f(a) \notin I\\ &\qquad \text{and } f(b) \in J.\end{aligned}$$

(ii) $\mathit{Prox}(\varphi)$ is a lattice homomorphism by [3, Theorem 11.31]. We prove that it preserves the proximity relation. For every $U_1, U_2 \in \mathcal{O}^{\mathcal{T}}(X_2)$,
$$\begin{aligned}U_1 \prec_{\propto_2} U_2 &\Longrightarrow U_1 \propto_2 X_2 \setminus U_2\\ &\Longrightarrow \varphi^{-1}(U_1) \propto_1 \varphi^{-1}(X_2 \setminus U_2)\\ &\Longrightarrow \varphi^{-1}(U_1) \propto_1 X_1 \setminus \varphi^{-1}(U_2)\\ &\Longrightarrow \varphi^{-1}(U_1) \prec_{\propto_1} \varphi^{-1}(U_2)\\ &\Longrightarrow \mathit{Prox}(\varphi)(U_1) \prec_{\propto_1} \mathit{Prox}(\varphi)(U_2).\end{aligned}$$

□

We let **PS** be the category whose objects are Priestley spaces equipped with apartness relations, and whose morphisms are apartness maps. **PL** is the category of strong proximity lattices and proximity homomorphisms. Theorems 4.2 and 4.4 and Lemma 4.6 show that the classical Priestley duality between distributive lattices and Priestley spaces can be restricted to the (non-full) sub-categories **PL** and **PS**. To summarise:

Theorem 4.7 *The functors Pries and Prox establish a dual equivalence between the categories* **PS** *and* **PL***.*

4.3 Morphism II: approximable relations and separators

As explained in the introduction, the intended application of our results in semantics suggests to consider wider classes of morphisms for spaces and lattices. To this end we equip proximity lattices with (weakly) approximable relations as presented in Definition 1.5, and obtain categories **PLa** and **PLwa** (in which composition is given by relational product).

Their counterparts are the categories **PSs** and **PSws** of Priestley spaces with apartness as objects, and (weakly) separating relations, Definition 1.6, as morphisms. To make this meaningful, we must present identities and a definition of composition, and then show that the laws for a category are satisfied.

Proposition 4.8 *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\propto$ is a separating relation from X to X.*

Proof. $(\downarrow_1\ltimes\uparrow_2)$, $(\forall\ltimes)$ and $(\ltimes\forall)$ are clearly satisfied. $(\ltimes n\uparrow)$ is proved by induction on n as follows. The cases where $n = 0, 1$ or 2 are clear. For the induction hypothesis, suppose $(\ltimes n\uparrow)$ is true for $n = m$ and $m \geq 2$. We now prove that $(\ltimes n\uparrow)$ is true for $n = m + 1$. Suppose that for some $b \in X$, $b \propto \bigcap_{1\leq i\leq m+1} \uparrow_i d_i$. We note the following

$$\begin{aligned}\bigcap_{1\leq i\leq m+1} \uparrow d_i &= \bigcap_{1\leq i\leq m} \uparrow d_i \cap \uparrow d_{m+1} = \bigcup\{\uparrow t \mid t \in \bigcap_{1\leq i\leq m} \uparrow d_i\} \cap \uparrow d_{m+1}\\ &= \bigcup\{\uparrow t \cap \uparrow d_{m+1} \mid t \in \bigcap_{1\leq i\leq m} \uparrow d_i\}.\end{aligned}$$

Therefore

$$b \propto \bigcap_{1\leq i\leq m+1} \uparrow_i d_i \iff (\forall t \in \bigcap_{1\leq i\leq m} \uparrow d_i)\; b \propto (\uparrow t \cap \uparrow d_{m+1}).$$

Now let $c \in X$ be such that $b \not\propto c$ and $c \not\propto d_{m+1}$. Then by $(\propto\uparrow\uparrow)$, $(\forall t \in \bigcap_{1\leq i\leq m} \uparrow d_i)$ $c \propto t$ and by the induction hypothesis there exists $1 \leq j \leq m$ such that $c \propto d_j$ which completes the proof. □

Definition 4.9 Let X_1, X_2, and X_3 be Priestley spaces equipped with apartness relations $\propto_1, \propto_2$, and $\propto_3$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ and $\ltimes' \subseteq X_2 \times X_3$ be separating relations. The composition $\ltimes \circ \ltimes' \subseteq X_1 \times X_3$ is defined as follows:

$$(\forall - comp)\; a \ltimes \circ \ltimes' c \overset{\text{def}}{\iff} (\forall b \in X_2)\; a \ltimes b \text{ or } b \ltimes' c.$$

The following technical lemma is needed to show that the composition of two separators satisfies ($\ltimes n\uparrow$):

Lemma 4.10 *For every $a \in X_1$ and $\{A_i \mid 1 \leq i \leq n\} \subseteq \mathcal{U}(X_2)$,*

$$a \ltimes \bigcap_i A_i \Longrightarrow (\forall x \in X_1)\ a \propto_1 x \text{ or } (\exists i)\ x \ltimes A_i.$$

Proof. Suppose by way of contradiction that $x \in X_1$ and $d_i \in A_i$ such that $a \not\propto_1 x$ and $(\forall i)\ x \not\ltimes d_i$. On the other hand, we note that $a \ltimes \bigcap \uparrow d_i \subseteq \bigcap A_i$. Then by ($\ltimes n\uparrow$) there exists i such that $x \ltimes d_i$, which is a contradiction. □

Lemma 4.11 *The composition of two separators $\ltimes$ and $\ltimes'$ is again a separating relation.*

Proof. Suppose $x \ltimes \circ \ltimes' y$ and set $O = \{t \in X_2 \mid t \propto_2 y\}$. Then O is an open subset of X_2 with $x \ltimes (X_2 \setminus O)$. By Lemma 4.13 below, there exist $U \in \mathcal{O}^{\mathcal{T}}(X_1)$ and $V \in \mathcal{U}^{\mathcal{T}}(X_2)$ such that $x \in U$, $X_2 \setminus O \subseteq V$ and $U \ltimes V$. Therefore $(X_2 \setminus V) \ltimes' y$. By applying Lemma 4.13 again, there exists $W \in \mathcal{U}^{\mathcal{T}}(X_3)$ such that $y \in W$ and $(X_2 \setminus V) \ltimes' W$. Hence $U \ltimes V$ and $(X_2 \setminus V) \ltimes' W$ implying $U \ltimes \circ \ltimes' W$. This proves that $\ltimes \circ \ltimes'$ is open and satisfies ($\downarrow_1 \ltimes \uparrow_2$).

($\forall \ltimes$) is proved as follows. Let $x \ltimes \circ \ltimes' y$ and $z \in X_1$ such that $x \not\propto_1 z$. We claim that $z \ltimes \circ \ltimes' y$. Let $t \in X_2$ such that $z \not\ltimes t$. Therefore, by ($\forall \ltimes$) $x \not\ltimes t$, and so $t \ltimes' y$, by definition of composition. Hence $z \ltimes \circ \ltimes' y$. ($\ltimes \forall$) is proved similarly.

($\ltimes n\uparrow$) is proved as follows. In the following let $O_i = \{t \in X_2 \mid t \ltimes' d_i\}$.

$$\begin{aligned}
b \ltimes \circ \ltimes' \bigcap \uparrow d_i &\Longrightarrow (\forall r \in X_2)\ b \ltimes r \text{ or } r \ltimes' \bigcap \uparrow d_i \\
&\Longrightarrow (\forall r, t \in X_2)\ b \ltimes r, r \propto_2 t \text{ or } (\exists i)\ t \ltimes' d_i, \text{ by } (\ltimes n\uparrow) \\
&\Longrightarrow (\forall t \in X_2)\ b \ltimes t \text{ or } (\exists i)\ t \ltimes' d_i, \text{ by } (\ltimes\forall) \\
&\Longrightarrow b \ltimes \bigcap_i (X_2 \setminus O_i) \text{ and } (\forall i)\ O_i \ltimes'\ d_i \\
&\Longrightarrow (\forall c \in X_1)\ b \propto_1 c \text{ or } (\exists i)\ c \ltimes (X_2 \setminus O_i) \\
&\qquad \text{and } O_i \ltimes' d_i, \text{ Lemma 4.10} \\
&\Longrightarrow (\forall c \in X_1)\ b \propto_1 c \text{ or } (\exists i)\ c \ltimes \circ \ltimes' d_i
\end{aligned}$$

□

Lemma 4.12 **PSs** *and* **PSws** *are categories.*

Proof. For associativity of composition we compute:

$$\begin{aligned}
x\ (\ltimes \circ \ltimes') \circ \ltimes''\ y &\Longleftrightarrow (\forall r \in X_3)\ x \ltimes \circ \ltimes' r \text{ or } r \ltimes'' y \\
&\Longleftrightarrow (\forall r \in X_3)(\forall s \in X_2)\ x \ltimes s,\ s \ltimes' x \text{ or } t \ltimes'' y
\end{aligned}$$

$$\Longleftrightarrow (\forall s \in X_2)\ x \ltimes s \text{ or } s \ltimes' \circ \ltimes'' y$$
$$\Longleftrightarrow x \ltimes \circ (\ltimes' \circ \ltimes'')\ y.$$

Identities are given by the internal apartness relations which is a valid choice by Lemma 4.8. They satisfy $\propto_1 \circ \ltimes = \ltimes$ and $\ltimes \circ \propto_2 = \ltimes$ by definition. □

Let us now turn to the translation between separating and approximable relations. For the rest of this section we make the following conventions:

(i) $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ are Priestley spaces with apartness relations $\propto_1$ and $\propto_2$, respectively. $\ltimes$ is a separating relation from X_1 to X_2.

(ii) $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ are strong proximity lattices. $\vdash$ is an approximable relation from L_1 to L_2.

(iii) For $A \in \mathit{Prox}(X_1) = \mathcal{O}^{\mathcal{T}}(X_1)$, $B \in \mathit{Prox}(X_2) = \mathcal{O}^{\mathcal{T}}(X_2)$, $A \vdash_{\ltimes} B \overset{\text{def}}{\Longleftrightarrow} A \ltimes (X_2 \setminus B)$.

(iv) For $I \in \mathit{Pries}(L_1) = \mathcal{I}_{\mathcal{P}}(L_1)$, $J \in \mathit{Pries}(L_2) = \mathcal{I}_{\mathcal{P}}(L_2)$, $I \ltimes_{\vdash} J \overset{\text{def}}{\Longleftrightarrow} (\exists\, x \notin I)(\exists\, y \in J)\ x \vdash y$.

The following facts are proved similarly to their counterparts earlier in the paper.

Lemma 4.13 *For closed subsets $A \subseteq X_1$ and $B \subseteq X_2$, if $A \ltimes B$ then there exist $U \in \mathcal{O}^{\mathcal{T}}(X_1)$ and $V \in \mathcal{U}^{\mathcal{T}}(X_2)$ such that $A \subseteq U$, $B \subseteq V$ and $U \ltimes V$.*

Lemma 4.14 $(\forall a \in L_1)(\forall b \in L_2)\ a \vdash b \Longleftrightarrow O_a \ltimes_{\vdash} U_b$.

Theorem 4.15 *The relation $\vdash_{\ltimes}$ satisfies $(\vdash - \prec_2)$, $(\prec_1 - \vdash)$, $(\vee - \vdash)$, $(\vdash - \wedge)$ and $(\vdash - \vee)$. So it is an approximable relation between $\mathit{Prox}(X_1)$ and $\mathit{Prox}(X_2)$.*

Theorem 4.16 *The relation $\ltimes_{\vdash}$ satisfies $(\downarrow_1\ltimes\uparrow_2)$, $(\forall\ltimes)$, $(\ltimes\forall)$ and $(\ltimes n\uparrow)$. Therefore it is a separating relation between $\mathit{Pries}(L_1)$ and $\mathit{Pries}(L_2)$.*

Lemma 4.17 • $x \ltimes y \Longleftrightarrow \epsilon_{X_1}(x) \ltimes_{\vdash_{\ltimes}} \epsilon_{X_2}(y)$.

• $a \vdash b \Longleftrightarrow \eta_{L_1}(a) \vdash_{\ltimes_{\vdash}} \eta_{L_2}(b)$.

Theorem 4.18 *The categories* **PSs** *(***PSws***) and* **PLa** *(***PLwa***) are equivalent to each other.*

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