SEMANTIC SPACES
IN PRIESTLEY FORM

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To my family.

Table of Contents

Abstract

The connection between topology and computer science is based on two fundamental insights: the first, which can be traced back to the beginning of recursion theory, and even intuitionism, is that computable functions are necessarily continuous when input and output domains are equipped with their natural topologies. The second, due to M. B. Smyth in 1981, is that the observable properties of computational domains are contained in the collection of open sets. The first insight underlies Dana Scott's categories of semantics domains, which are certain topological spaces with continuous functions. The second insight was made fruitful for computer science by Samson Abramsky, who showed in his "Domain Theory in Logical Form" that instead of working with Scott's domains one can equivalently work with lattices of observable properties. Thus he established a precise link between denotational semantics and program logic.

Mathematically, the framework for Abramsky's approach is that of Stone duality, which in general terms studies the relationship between topological spaces and their lattices of opens sets. While for his purposes, Abramsky could rely on existing duality results established by Stone in 1937, it soon became clear that in order to capture continuous domains, the duality had to be extended. Continuous domains are of interest to semantics because of the need to model the probabilistic behaviour and computation over real numbers. The extension of the Stone duality was achieved by Jung and Sünderhauf in 1996; the main outcome of this investigation is the realisation that the observable properties of a continuous space form a strong proximity lattice.

The present thesis examines strong proximity lattices with the tools of Priestley duality, which was introduced in 1970 as an alternative to Stone's duality for distributive lattices.

The advantage of Priestley duality is that it yields compact Hausdorff spaces and thus stays within classical topological ideas.

The thesis shows that Priestley duality can indeed be extended to cover strong proximity lattices, and identifies the additional structure on Priestley spaces that corresponds to the proximity relation. At least three different types of morphism have been defined between strong proximity lattices, and the thesis shows that each of them can be used in Priestley duality. The resulting maps between Priestley spaces are characterised and given a computational interpretation.

This being an alternative to the Jung–Sünderhauf duality, it is examined how the two dualities are related on the side of topological spaces.

Finally, strong proximity lattices can be seen as algebras of the logic MLS, introduced by Jung, Kegelmann, and Moshier. The thesis examines how the central notions of MLS are transformed by Priestley duality.

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Birmingham, England — Mohamed El-Zawawy
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Chapter 1

Introduction

1.1 Background

The semantics of programming languages is about developing techniques for designing and describing programming languages. Such techniques are quite important because, in most cases, if one relies only on ones intuition to design a programming language, then one will run into counter-intuitive situations, and will need a rigorous technique to guide one through the designing process. One of those situations where semantics techniques are needed in designing languages is demonstrated in [114, chapter 7] on a language that is like ALGOL 60 [64].

Among the semantics approaches [76, 78, 84] are the axiomatic, operational and denotational ones. The axiomatic approach (also termed "the program logic"), that originated in [25, 33] and proved a success in [98, 31, 38, 74], is about establishing a set of axioms to help in developing and verifying programs. An example of a program logic is *Hoare logic* [33] whose axioms have the form $\{P\}\ C\ \{Q\}$. This is to be read as follows; if the pre-condition P is satisfied before running the program C and if C terminates then the post-condition Q will be satisfied afterwards. The operational approach [88] of semantics emphasises the way programs are executed on an abstract machine such as the *Java Abstract Machine*. Early versions and properties of operational techniques are discussed in [63, 69] and [91, 87], respectively. In the denotational approach [112, 30, 97, 71], the emphasis is on the mathematical meaning of language constructs. These three techniques of the semantics of programming languages are related and, in some way, complement each other [75, 21, 68].

The idea in denotational semantics, which can be traced back to [26, 17, 113], is to use a category [70] to interpret programming language constructs; data types and programs are represented by objects and morphisms, respectively. The properties of data types and programs restrict the choice of an appropriate category to carry the denotational semantics. In other words, a category that can successfully develop the denotational semantics is subject to conditions. Two of these requirements are raised by the use of recursion in defining procedures and data structures (data-types) and are stated for a concrete category as follows.

1. A map which assigns to every endomorphism f on an object M a point $m \in M$ such that $f(m) = m$ (a fix-point for f).

2. With every suitably well-behaved functor $G : A^{op} \times A \longrightarrow A$, there should exist an object M such that $G(M, M) \cong M$.

Classical categories in mathematics such as sets, topological spaces, vectors spaces, and groups fail to meet the requirements for a suitable universe for denotational semantics.

Domain theory, which was invented in 1969 by Dana Scott [99, 100, 101, 102, 104], has proved so far to be a very convenient mathematical environment for denotational semantics. In this theory, data types are represented by domains, which are ordered sets satisfying certain conditions, and programs are represented by functions between domains. Scott equipped domains with a certain topology named after him as the Scott-topology. Hence computability of programs can be checked by testing the topological continuity of their representing functions with respect to Scott-topologies on domains.

The Scott-topology helps establishing axiomatic semantics for programs as follows. According to the *geometric logic (logic of observable properties)* [118] the Scott-open sets of a domain are interpreted as properties, where a point in the domain satisfies a property if and only if it belongs to the open set. Suppose C is a continuous map (computable function) from a domain D_1 to a domain D_2. If P_2 is a property (a Scott-open subset) of D_2 then $P_1 := C^{-1}(P_2)$ is a property of D_1, by continuity of C. Moreover, it is certain that if an input x to the program C satisfies P_1 then the output $C(x)$ will satisfy property P_2. This idea was first proposed by Smyth in [106].

In 1936, Stone presented his famous representation theorem [109] which proved that totally disconnected compact Hausdorff spaces represent Boolean algebras. This was the starting point of a whole area of research known as *Stone duality* in which mathematicians are interested in establishing "Stone-type" dualities between classes of topological spaces and algebras. Stone duality was introduced to computer science by Samson Abramsky in his famous paper, *Domain Theory in Logical Form* [2]. In this paper Abramsky showed that Stone duality is the appropriate mathematical framework for studying the relationship between denotational and axiomatic semantics.

In [2], Abramsky presented a logical representation for a particular Cartesian-closed category of domains, namely the bifinite domains. In this framework, bifinite domains are represented by propositional theories, and functions between bifinite domains are represented by a program logic axiomatising the properties of domains. Moreover, Abramsky proved that the domain interpretation via bifinite domains and his logical interpretation are Stone duals to each other and specify each other up to isomorphism.

Later in [54, 55, 50, 52, 56] Abramsky's work was extended by Achim Jung, Philipp Sünderhauf, Mathias Kegelmann, and Andrew Moshier to a class of topological spaces, *stably compact spaces* defined as follows.

Definition 1.1.1. A *stably compact space* is a topological space which is sober, compact, locally compact, and for which the collection of compact saturated subsets is closed under finite intersections, where a saturated set is an intersection of open sets.

These spaces are the appropriate general topological setting for compact coherent domains in their Scott topologies. Coherent domains include bifinite domains and other interesting Cartesian-closed categories of domains such as FS domains.

Bifinite domains are included in algebraic domains and in the latter every element can be approximated by finite (compact) elements below it. This is the main reason that one can go from the domain side to the logic side of Abramsky [2] by constructing the lattice of open-compact (with respect to the Scott-topology) sets. But for coherent domains this is not true anymore; finite elements do not necessarily approximate elements of domains and hence we can not just consider the lattice of open-compact sets. We have to work harder to understand the more general situation of coherent domains. Achim Jung and Philipp Sünderhauf in their papers [54, 55] managed to find a way to appropriately construct the lattices of observable properties of compact coherent domains. According to their work if $\langle X, \mathcal{T} \rangle$ is a stably compact space then its lattice $\mathcal{B}_X$ of observable properties is defined as follows:

$$\mathcal{B}_X = \{\langle O, K \rangle \mid O \in \mathcal{T}, K \in \mathcal{K}_X \text{ and } O \subseteq K\},$$

where $\mathcal{K}_X$ is the set of compact saturated subsets of X. The computational interpretation is as follows. For a point $x \in X$ and a property $\langle O, K \rangle \in \mathcal{B}_X$:

- $x \in O \iff x$ *satisfies* the property $\langle O, K \rangle$,
- $x \in X \setminus K \iff x$ does not satisfy the property $\langle O, K \rangle$, and
- $x \in K \backslash O \iff$ the property $\langle O, K \rangle$ is *unobservable for* x.

We note that the condition $O \subseteq K$ in the definition of $\mathcal{B}_X$ is necessary for avoiding any contradictions; without this condition a point may be considered having and *not* having a property at the same time.

This idea of constructing the lattice $\mathcal{B}_X$ and interpreting it as the lattice of observable properties is very convenient for dealing with continuous phenomena. For example, if we want to classify people according to their heights, one can argue that:

- people of height in $[160 \text{ cm} - 180 \text{ cm}]$ have about average height,
- people of height less than 150 cm or more than 190 cm do not have about average height, and
- other people may or may not be considered having about average height and therefore the property is unobservable for them.

As another example if we try to specify an interval on the thermometer inside which the weather is cool then we will face a situation similar to the one explained in the example above.

On the lattice $\mathcal{B}_X$ of observable properties a binary relation (*strong proximity relation*) was defined by Jung and Sünderhauf in [54] as:

$$\langle O, K \rangle \prec \langle O', K' \rangle \overset{\text{def}}{\iff} K \subseteq O'.$$

The computational interpretation of the strong proximity relation $\prec$ can be stated as follows:

$$\langle O, K \rangle \prec \langle O', K' \rangle \iff (\forall x \in X) \text{ either } \langle O', K' \rangle \text{ is observably satisfied for } x \text{ or } \langle O, K \rangle \text{ is (observably) not satisfied for } x.$$

Thus we can say that $\prec$ behaves like a classical implication.

A different way to understand the relation $\prec$ is the following. For a property $\langle O, K \rangle$ we interpret O as the set of positive information about the property and $X \backslash K$ as the set of negative information about the property. It makes sense to consider a property $\langle O, K \rangle$ to be more definite than a property $\langle O', K' \rangle$ if the positive and negative information of the first property include the positive and negative information of the second property, respectively, i.e. if $O' \subseteq O$ and $X \backslash K' \subseteq X \backslash K$. Now, it is easy to see that if a property $\langle O_1, K_1 \rangle$ is related to a property $\langle O_2, K_2 \rangle$ under $\prec$ then any property that is more definite than $\langle O_1, K_1 \rangle$ must be related to any property that is less definite than $\langle O_2, K_2 \rangle$.

The following algebraic structures (introduced in [54]) represent stably compact spaces and abstractly capture their lattices of observable properties (of the form of $\mathcal{B}_X$).

Definition 1.1.2. A binary relation $\prec$ on a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1 \rangle$ is called a *strong proximity* if, for every $a, x, y \in L$ and $M \subseteq_{fin} L$,

$$\begin{aligned}
&(\prec\prec) && \prec \circ \prec \, = \, \prec, \\
&(\vee - \prec) && M \prec a \Longleftrightarrow \bigvee M \prec a, \\
&(\prec - \wedge) && a \prec M \Longleftrightarrow a \prec \bigwedge M, \\
&(\prec - \vee) && a \prec x \vee y \Longrightarrow (\exists\, x', y' \in L)\; x' \prec x,\; y' \prec y \text{ and } a \prec x' \vee y', \\
&(\wedge - \prec) && x \wedge y \prec a \Longrightarrow (\exists\, x', y' \in L)\; x \prec x',\; y \prec y' \text{ and } x' \wedge y' \prec a.
\end{aligned}$$

$M \prec a$ and $a \prec M$, respectively, stand for $(\forall m \in M)\; m \prec a$ and $(\forall m \in M)\; a \prec m$. A *strong proximity lattice* is a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1 \rangle$ together with a strong proximity relation $\prec$ on L.[1]

The notion of strong proximity lattice subsumes that of bounded distributive lattice as the lattice order $\leq$ is always a strong proximity.

Now we want to define a suitable notion of morphism between strong proximity lattices in order to capture (represent) continuous maps (computable programs) between stably compact spaces. While a map that preserves order and strong proximity relations (a proximity homomorphism) between strong proximity lattices is the obvious choice for such a notion, it is computationally too specialised; if we consider its manifestation on semantic

[1] The qualifier "strong" distinguishes the concept from its precursor in [108], where $(\wedge - \prec)$ was not a requirement.

domains (compact coherent domains as stably compact spaces) then we realise it as a (an order-preserving) Lawson continuous function [28]. This does not cover the computable maps which typically are only Scott-continuous. Therefore Jung and Sünderhauf arrived at a situation similar to domain theory where also more than one kind of map is studied on a fixed class of spaces such as embedding-projection pairs, Scott-continuous function, strict Scott-continuous function, stable function, etc.

Two more general notions of morphism between strong proximity lattices were considered in order to capture all computable (Scott-continuous) functions between compact coherent domains (in their topological setting of stably compact spaces). On the side of strong proximity lattices, homomorphisms were replaced with *approximable relations*; a notion that dates back to Scott's morphisms for *information systems* in [103]. The definition is as follows:

Definition 1.1.3. Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be a binary relation from L_1 to L_2. The relation $\vdash$ is called *approximable* if for every $a \in L_1, b \in L_2$, $M_1 \subseteq_{fin} L_1$ and $M_2 \subseteq_{fin} L_2$,

$$
\begin{array}{rl}
(\vdash - \prec_2) & \vdash \circ \prec_2 = \vdash, \\
(\prec_1 - \vdash) & \prec_1 \circ \vdash = \vdash, \\
(\vee - \vdash) & M_1 \vdash b \Longleftrightarrow \bigvee M_1 \vdash b, \\
(\vdash - \wedge) & a \vdash M_2 \Longleftrightarrow a \vdash \bigwedge M_2, \\
(\vdash - \vee) & a \vdash \bigvee M_2 \Longrightarrow (\exists\, N \subseteq_{fin} L_1)\ a \prec_1 \bigvee N \text{ and } (\forall n \in N) \\
& (\exists\, m \in M_2)\ n \vdash m.
\end{array}
$$

The relation $\vdash$ is called *weakly approximable* if it satisfies all of the above conditions but not necessarily $(\vdash - \vee)$.

The relationship between proximity homomorphisms and approximable relations is analogous to that between Dijkstra's *weakest preconditions*: A homomorphism h from

L_2 to L_1 specifies the weakest precondition $h(\phi)$ needed to be satisfied by the input so that the corresponding output satisfies the condition ϕ at the end of the computation.

Now let us look at the computational interpretation of approximable relations. Suppose $f : X_1 \longrightarrow X_2$ is a continuous map between stably compact spaces $\langle X_1, \mathcal{T}_1 \rangle$ and $\langle X_2, \mathcal{T}_2 \rangle$. Then the binary relation $\vdash_f \subseteq \mathcal{B}_{X_1} \times \mathcal{B}_{X_2}$ defined as:

$$\langle O, K \rangle \vdash_f \langle O', K' \rangle \overset{\text{def}}{\iff} f(K) \subseteq O',$$

is an approximable relation [52]. Therefore the relation $\vdash_f$ relates all propositions (properties) $\langle O, K \rangle, \langle O', K' \rangle$ where the satisfaction or un-observability of $\langle O, K \rangle$ by the input entails the satisfaction of $\langle O', K' \rangle$ by the corresponding output. Moreover, the notion of approximable relation was basically introduced to abstractly capture the set of all relations established in this way, i.e. the set:

$$\{\vdash_f \mid f \text{ is a continuous map between two stably compact spaces}\}.$$

Dealing with relations, rather than functions, puts no constraints on turning around the direction. Thus Jung and Sünderhauf arrived at an equivalence of categories rather than a duality; the category of stably compact spaces and continuous maps between them and the category of strong proximity lattices and approximable relations between them.

The equivalence established in [54] played later a pivotal role in developing a logical description (expounded in [50, 52]) for stably compact spaces similar to Samson Abramsky's *domain theory in logical form*. The logical description is via the category **MLS** (Multi Lingual Sequents) of logical systems. This category is the primary object of interest in Mathias Kegelmann's PhD thesis [56]. Figure 1.1 illustrates the situation.

Continuous domains have been the primary objects of interest of much research carried out by both computer theorists and mathematicians. In the following, we briefly review some of the recent research activity concerning these domains. Tix, Keimel, and Plotkin, in [115], introduced domain theoretical tools for combining probability with nondeterminism

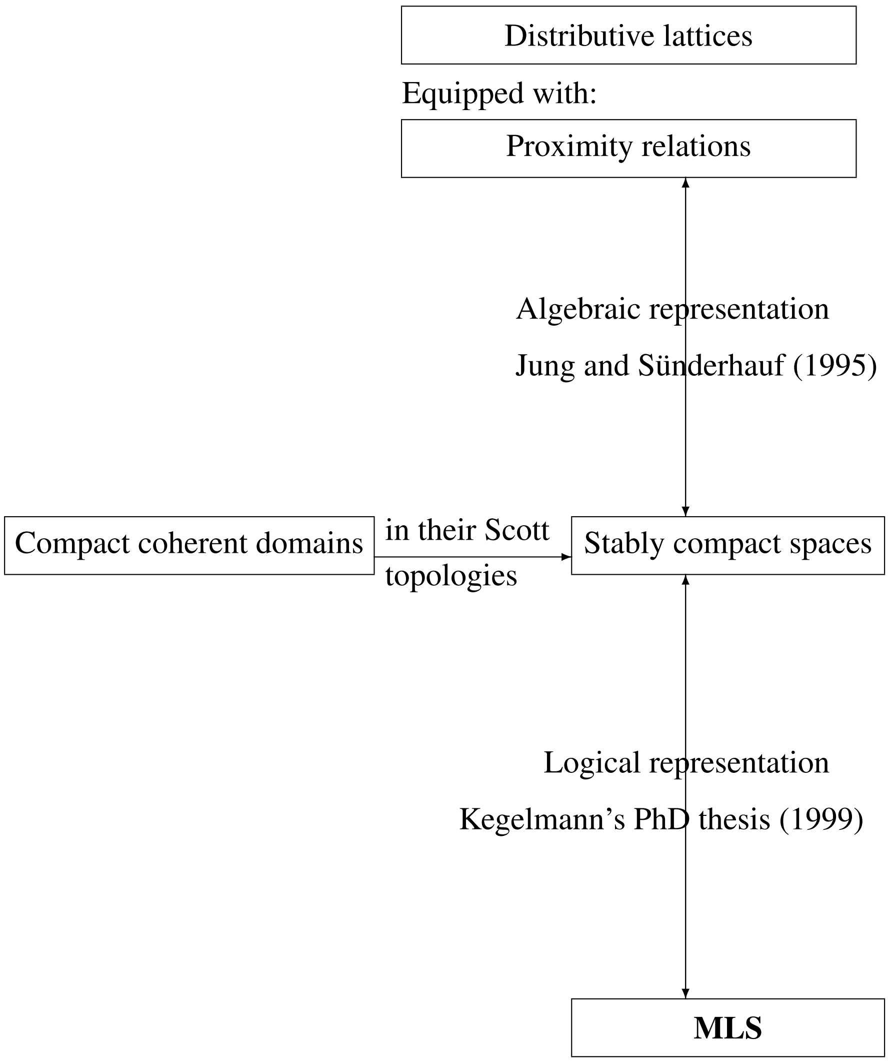


Figure 1.1: Stably compact spaces

for continuous domains. later, in [58], Keimel showed that these tools can be extended to cover larger classes of spaces including stably compact spaces. Lawson and Xu, in [67], showed that the function space of continuous locally bounded functions, from core compact spaces into posets, equipped with the Scott-topology and for which each interval is a continuous sup-semilattice, has intervals that are continuous sup-semilattices. Smyth, in [105], studied the partial metrizability of ω-continuous domains. In [10], a study of the logical content of continuous domains presented a finite information logic which was seen as a complementary logic to **MLS** [50, 52, 56].

In 1937, Marshall Stone extended his celebrated representation theorem for Boolean algebras [109, 110] to cover bounded distributive lattices [111]; the notion of Stone space was extended to that of spectral space (stably compact space with a basis of compact open sets). Typically the spaces defined by the latter notion are T_0 but not Hausdorff. In 1970, Hilary Priestley realised that Stone's duality for Boolean algebras can be differently extended to bounded distributive lattices to produce Hausdorff spaces; Priestley enriched Stone spaces with an order relation and obtained ordered topological spaces that are compact and totally order-disconnected [89, 90]. These spaces are known as Priestley spaces and defined as follows:

Definition 1.1.4. A *Priestley space* is a compact ordered space $\langle X; \mathcal{T}, \leq \rangle$ such that for every $x, y \in X$, if $x \ngeq y$ then there exists a clopen upper set U such that $y \in U$ and $x \notin U$.

Ever since they were invented, Priestley spaces have been receiving growing interest and attention by both mathematicians and computer scientists. In the following, we cite recent research related to Priestley duality. Many successful attempts for presenting Priestley duality constructively (using constructive mathematics) in a localic form exist; for example [116, 117] by C. Townsend, and [60] by M. Korostenski and C. C. A. Labuschagne. In [81, 82], A. Palmigiano presented a categorical equivalence between the category $\mathbf{K}^+$ of

K^+-spaces (ordered sets equipped with certain binary relations and associated with certain sublattices) and the category **Coalg(V)** of coalgebras of a certain endofunctor V on the category of Priestley spaces. A study about linear Heyting algebras, using Priestley duality, was presented by L. Rueda in [92]. In [72], N. Martinez and A. Petrovich used a Priestley duality for MV-algebras to specify a condition that guarantees the uniqueness of the implication for totally ordered MV-algebras.

1.2 Aim of our Research

Let $\langle X, \mathcal{T} \rangle$ be a stably compact space, $\mathcal{K}_X$ be the collection of compact saturated subsets of X, and

$$\mathcal{B}_X = \{\langle O, K \rangle \mid O \in \mathcal{T}, K \in \mathcal{K}_X \text{ and } O \subseteq K\}$$

be the lattice of observable properties (introduced in the previous section) of $\langle X, \mathcal{T} \rangle$. This is a strong proximity lattice. In logic, *theories* (or *models*) of $\mathcal{B}_X$ are represented by prime filters, which are the points of the Priestley dual space of $\mathcal{B}_X$ as a bounded distributive lattice. Therefore if we can extend the Priestley duality of bounded distributive lattices to cover strong proximity lattices then the resulting notion (on the topological side of the duality) will be an abstract description of the models (theories) of properties of stably compact spaces including compact coherent domains. This is indeed the first and the basic objective of this thesis.

Therefore we can say that **the primary aim of this thesis is to introduce Priestley spaces to the world of semantics of programming languages.**

Hence the research programme of the thesis is as follows:

1.2.1 The First Aim

The first goal of the thesis is to answer the following questions:

1. *How can Priestley duality for bounded distributive lattices be extended to strong proximity lattices?*

2. *What is the computational interpretation of the answer of the previous question?*

1.2.2 The Second Aim

Having question 1 answered naturally motivates a curiosity to understand the direct links between stably compact spaces and the answer of question 1. Therefore it is the second aim of the research to answer the following question:

3. *What is the direct relationship between the category resulting from answering question* 1 *and the category* **SCS** *of stably compact spaces (e.g. compact coherent domains in their Scott topologies seen as data types) and continuous maps (seen as computable programs) between them?*

1.2.3 The Third Aim

The third goal of the research is to answer the following questions:

4. *What are the direct links between the category* **MLS** *(a logical representation for the category* **SCS***) and the category resulting from answering question* 1*?*

5. *How can the category resulting from answering question* 1 *help establishing new semantics, in Priestley form, for essential concepts and facts of the category* **MLS***?*

6. *How can domain constructions be done in Priestley form (the category resulting from answering question* 1*)?*

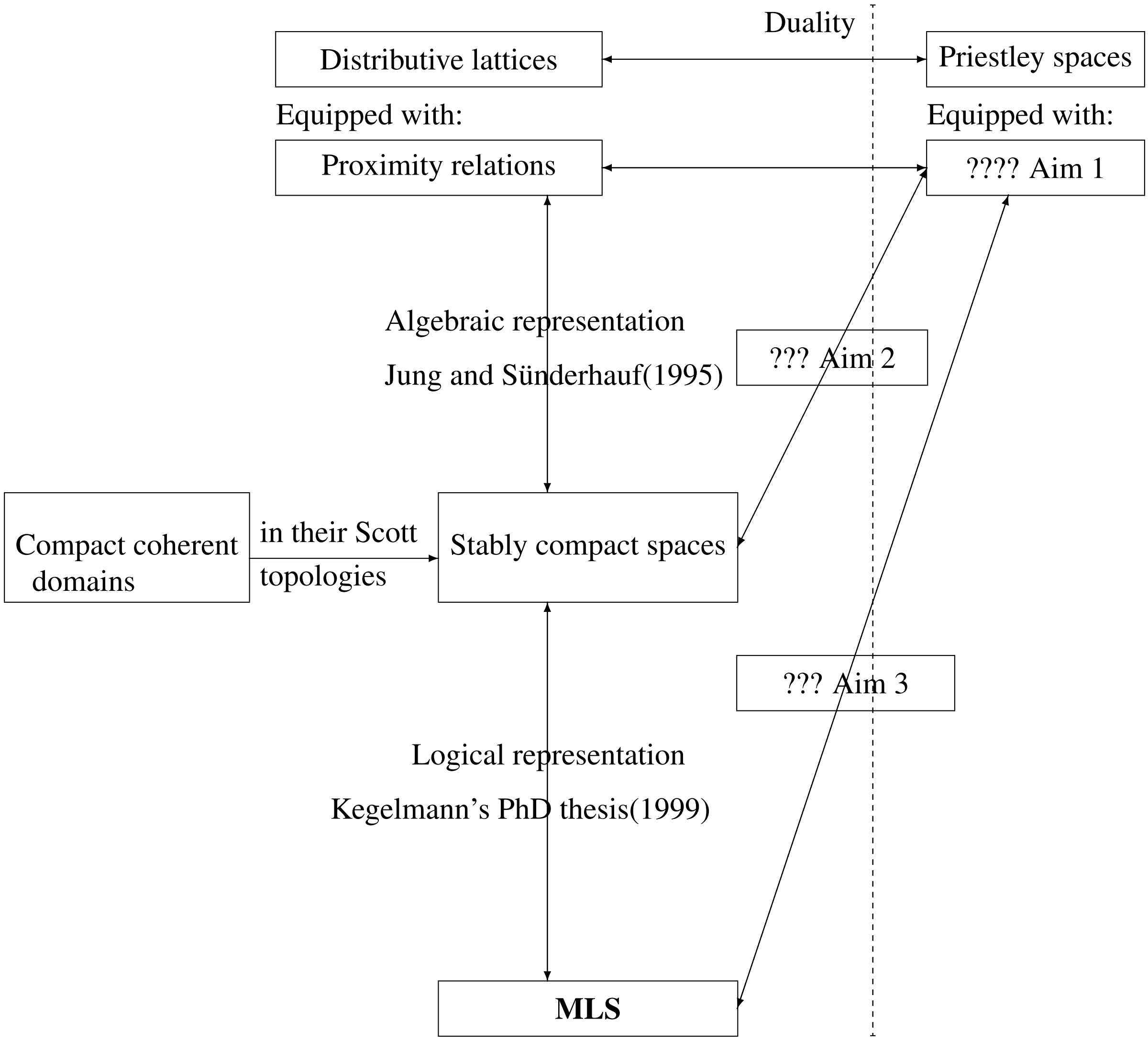


Figure 1.2: Aim of the research

We are also aiming for

7. *developing a study of the elements of the category resulting from answering question* 1*. This will be done by different means including directly linking the elements of this category to existing structures in mathematics.*

1.3 Achievements of the Research

1.3.1 Concerning the First Aim

To answer question 1, we introduce the notion of apartness relation [23, 24] on Priestley spaces as follows:

Definition 1.3.1. A binary relation $\propto$ on a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ is called an *apartness* if, for every $a, c, d, e \in X$,

$$\begin{aligned}
&(\propto\mathcal{T}) && \propto \text{ is open in } \langle X;\mathcal{T}\rangle \times \langle X;\mathcal{T}\rangle\\
&(\uparrow\propto\downarrow) && a \geq c \propto d \geq e \Longrightarrow a \propto e,\\
&(\propto\forall) && a \propto c \Longleftrightarrow (\forall b \in X)\; a \propto b \text{ or } b \propto c,\\
&(\propto\downarrow\downarrow) && a \propto ({\downarrow}c \cap {\downarrow}d) \Longrightarrow (\forall b \in X)\; a \propto b,\; b \propto c \text{ or } b \propto d,\\
&(\uparrow\uparrow\propto) && ({\uparrow}c \cap {\uparrow}d) \propto a \Longrightarrow (\forall b \in X)\; d \propto b,\; c \propto b \text{ or } b \propto a.
\end{aligned}$$

where $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

Remark 1.3.2.

1. For any Priestley space $\langle X; \leq, \mathcal{T}\rangle$, $\not\leq$ is an apartness because the order is closed in the product topology for Priestley spaces.
2. $\propto$ is an apartness on $\langle X; \leq, \mathcal{T}\rangle$ if and only if the dual of $\propto$ is an apartness on $\langle X; \geq, \mathcal{T}\rangle$.
3. It aids the intuition to assume that an element can not be apart from itself but as a matter of fact our results do not rely on this assumption.

4. If we were to axiomatise *indistinguishability* instead of apartness, then $(\propto\forall)$ would express the transitivity of this relation. Axiom $(\propto\downarrow\downarrow)$, however, would not have a simple formulation.

5. On the real line, axioms $(\propto\downarrow\downarrow)$ and $(\uparrow\uparrow\propto)$ are the same as $(\propto\forall)$.

Question 1 above is answered as follows:

The dual of a strong proximity lattice L is the corresponding Priestley space of prime filters, equipped with the apartness,

$$F \propto_{\prec} G \overset{\text{def}}{\Longleftrightarrow} (\exists\, x \in F)(\exists\, y \notin G)\; x \prec y.$$

Vice versa, the dual of a Priestley space X with apartness $\propto$ is the lattice of clopen upper sets equipped with the strong proximity,

$$A \prec_{\propto} B \overset{\text{def}}{\Longleftrightarrow} A \propto (X \backslash B).$$

Up to isomorphism, the correspondence is one-to-one.

From a representation point of view the answer to this question can be classified as follows:

	T_0 spaces	Hausdorff spaces (T_2)
strong proximity lattices	Jung & Sünderhauf, [54]	the answer to question 1
distributive lattices	Stone, [111]	Priestley, [89, 90]
Boolean algebras	Stone, [109, 110]	

According to the classification above the strong proximity lattices were represented via a class of T_0 spaces (stably compact spaces) in [54]. This chapter introduces a Hausdorff representation for strong proximity lattices. Also distributive lattices were represented via a class of T_0 spaces (spectral spaces) by Stone in [111]. Distributive lattices were also later given a Hausdorff representation (via Priestley spaces) by Priestley in [89, 90]. Boolean algebras were given a Hausdorff representation (Stone spaces) by Stone in [109, 110].

From a semantics point of view, in [52] the argument was made that the strong proximity $\prec$ relates two logical propositions (pairs of the form (O, K)) ϕ and ψ if the satisfaction or un-observability (actually true) of ϕ always implies the satisfaction (observably true) of ψ. Consequently, the logical system does not necessarily satisfy the identity axiom $\phi \vdash \phi$, and while the paper [52] demonstrates that a satisfactory and even elegant logical apparatus can still be built, the lack of this basic law of logic may feel strange. In this thesis, the view is that the strong proximity is *additional structure*, over and above the lattice operations, and that for the latter the usual axioms of logic are still valid. Consequently, a *model* of the logic is given by a prime filter as it is usually. The additional structure on the logic then gives rise to an *additional structure* on the space of all models (the Priestley space), which we read as *apartness* information.[2] The intuition is that two states of affair (i.e., models) can be observably separated if and only if they are "sufficiently apart." To give an example, consider the real numbers represented in their usual decimal representation. Mathematically, we deem $a = 1.000\ldots$ and $b = 0.999\ldots$ equal; constructively, the concrete presentation of a number is important, and in our example one would find that a and b can not be told apart in finite time but their equality can also not be established in finite time (if our only access to the numbers is by successively reading digits).

Definition 3.1.1 attempts to capture the intuitive notion of apartness on a Priestley space.

We will show that the action of Priestley duality on morphisms can also be adapted to the current setting.

> *Continuous order-preserving maps that reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms that preserve the strong proximity relation.*

[2]Indeed, there is a rather conventional way to fill in the right upper position in the table above. For this one equips the collection of *round* prime filters of L with the topology generated by all $U_x := \{F \mid x \in F\}$, and all $O_x := \{F \mid \exists y \notin F. x \prec y\}$, $x \in L$. This yields the *patch topology* of a stably compact space which is already obtainable from the Jung-Sünderhauf dual.

Next, on the side of Priestley spaces, we introduce a notion for morphism which corresponds to the notion of approximable (weakly approximable) relations. Whereas in [54] morphisms corresponding to approximable relations were proved to be functions that are continuous with respect to the upper topology, now we can not expect a similar situation at all. The reason is that the Priestley dual contains more points than the *spectrum* considered in [54] and there is no reason to assume that the process acts functionally on the additional elements. In keeping with the spirit of Definition 3.1.1, we instead consider *relations* between Priestley spaces which relate those pairs of elements that are "observably unrelated" by the computational process. Here is the definition:

Definition 1.3.3. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces with apartness relations $\propto_1$ and $\propto_2$, respectively, and let $\ltimes$ be a binary relation from X_1 to X_2. The relation $\ltimes$ is called *separating* (or a *separator*) if it is open in $\mathcal{T}_1 \times \mathcal{T}_2$ and if, for every $a, b \in X_1, d, e \in X_2$ and $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$,

$$(\uparrow_1\ltimes\downarrow_2) \qquad a \geq_1 b \ltimes d \geq_2 e \Longrightarrow a \ltimes e,$$

$$(\forall\ltimes) \qquad b \ltimes d \Longleftrightarrow (\forall c \in X_1)\, b \propto_1 c \text{ or } c \ltimes d,$$

$$(\ltimes\forall) \qquad b \ltimes d \Longleftrightarrow (\forall c \in X_2)\, b \ltimes c \text{ or } c \propto_2 d,$$

$$(\ltimes n\downarrow) \qquad b \ltimes \bigcap \downarrow d_i \Longrightarrow (\forall c \in X_1)\, b \propto_1 c \text{ or } (\exists\, i)\; c \ltimes d_i.$$

The relation $\ltimes$ is called *weakly separating (or weak separator)* if it satisfies all of the above conditions, but not necessarily $(\ltimes n\downarrow)$.

Some effort is required to show that we get a category whose objects are Priestley spaces with apartness and whose morphisms are separators (weak separators) (see Section 3.3.3), but the expected equivalence does hold:

> *Let X_1 and X_2 be Priestley spaces with apartness relation. Then (weakly) separating relations from X_1 to X_2 are in one-to-one correspondence with (weakly) approximable relations from the dual of X_1 to the dual of X_2.*

1.3.2 Concerning the Second Aim

We show that the answer to question 3 will take the following form:

For a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ with apartness $\propto$ and $A, B \subseteq X$ we define:

1. $\propto[A] = \{x \in X \mid x \propto A\}$ and $[A]\propto = \{x \in X \mid A \propto x\}$, where, as before, $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

2. $core(X) = \{x \in X \mid [x]\propto = X \backslash {\uparrow} x\}$.

3. $\mathcal{T}' = \{O \cap core(X) \mid O$ is an open upper subset of $X\}$.

One of our primary results, then, is the following:

Theorem 1.3.4. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle core(X), \mathcal{T}'\rangle$ is a stably compact space. Moreover, every stably compact space can be obtained in this way.*

We also present a study about the relationship between frame homomorphisms, continuous maps, and separators. This leads to the result that the categories **SCS** and **PSs** (of Priestley spaces with apartness and separators) are dual equivalent. Moreover, using Priestley spaces with apartness, we prove some facts about the co-compact topology of a stably compact space. We also show that a notion (of isolated set) that is crucial for proving the theorem above is equivalent to the notion of round filter (ideal) of strong proximity lattices.

Let f be a continuous map from a stably compact space Y_1 to another one Y_2. A computational reading of the separator $\ltimes_f$ corresponding to f could be the following. As explained earlier, a property $\langle O, K\rangle$ is related to a property $\langle O', K'\rangle$ in the approximable relation $\vdash_f$ if and only if the satisfaction or un-observability of the former property by an input to the function f implies the satisfaction of the latter property by the corresponding output. Let us fix this as a definition:

Definition 1.3.5. Let $f : Y_1 \longrightarrow Y_2$ be a continuous function between stably compact spaces Y_1 and Y_2. Let $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$, as defined above, be their lattices of observable properties. Then we say that a property $\langle O, K\rangle \in \mathcal{B}_{Y_1}$ *implies* a property $\langle O', K'\rangle \in \mathcal{B}_{Y_2}$ under f if $f(K) \subseteq O'$.

The separator $\ltimes_f$ relates primes filters (models or theories of properties) of the strong proximity lattice $\mathcal{B}_{Y_1}$ to prime filters of $\mathcal{B}_{Y_2}$. Under $\ltimes_f$ two filters are related if and only if the first filter contains a property that implies a property in the complement of the second filter. Therefore if $\ltimes'_f$ is the complement of $\ltimes_f$ then a pair $\langle F, G\rangle$ belongs to $\ltimes'_f$ if and only if the set of all properties implied by a property in F is contained in G.

Moreover, in this thesis we answer the following question:

> *How can the Jung-Sünderhauf duality between the category* **PLa**, *of strong proximity lattices and approximable relations between them, and the category* **SCS** *be extended to cover stably locally compact spaces (the same notion of stably compact space without requiring the topological space to be compact)?*

Hence the objective here is to remove the compactness condition from the topological side of the Jung-Sünderhauf duality. Answering the last question is very interesting because the resulting algebraic structures will be computationally interpreted as abstract descriptors for the lattices of observable properties of stably locally compact spaces which include all coherent domains in their Scott topologies.

Concerning this question, we show that removing the compactness requirement from the notion of stably compact space is equivalent to removing the condition of having a top element from the notion of strong proximity lattice. Removing the latter condition must be associated by removing any use of the empty meet which is the top element. The resulting definition is the following.

Definition 1.3.6. A binary relation $\prec$ on a distributive lattice $\langle L; \vee, \wedge\rangle$ with 0 is called a *strong proximity* if, for every $a, x, y \in L$ and $M \subseteq_{fin} L$,

$$
\begin{array}{ll}
(\prec\prec) & \prec \circ \prec = \prec, \\
(\vee\!-\prec) & M \prec a \Longleftrightarrow \bigvee M \prec a, \\
(\prec -\wedge)' & a \prec x \text{ and } a \prec y \Longleftrightarrow a \prec x \wedge y, \\
(\prec -\vee) & a \prec x \vee y \Longrightarrow (\exists\, x', y' \in L)\; x' \prec x,\; y' \prec y \text{ and } a \prec x' \vee y', \\
(\wedge\!-\prec) & x \wedge y \prec a \Longrightarrow (\exists\, x', y' \in L)\; x \prec x',\; y \prec y' \text{ and } x' \wedge y' \prec a.
\end{array}
$$

$M \prec a$ stands for $(\forall m \in M)\; m \prec a.$ A *pointed-strong proximity lattice* is a distributive lattice $\langle L; \vee, \wedge \rangle$ with 0 together with a strong proximity relation $\prec$ on L.

Similar changes are made to morphisms to obtain pointed-approximable and weakly pointed-approximable relations. Therefore we show that the category of stably locally compact spaces and continuous maps is equivalent to the category of pointed-strong proximity lattices and pointed-approximable relations.

1.3.3 Concerning the Third Aim

Concerning question 4, we present direct functors between the category **MLS** and the category **PSws** of Priestley spaces with apartness and weak separators. These functors are then used to prove the equivalence of these categories.

Concerning question 5, we introduce Priestley semantics (in **PSws**) for **MLS**'s concepts and facts such as compatibility, Gentzen's cut rule, round ideals and filters, and consistency.

Concerning question 6, we show how some domain constructions such as lifting, sum, product, and Smyth power domain can be done in the Priestley form.

Concerning 7:

1. We introduce a full and faithful functor from the category **PSws** to the category **SL** of directed-complete meet semilattices and Scott-continuous semilattice homomorphisms. This proves that the image of this functor is equivalent to **PSws**. Hence

bearing in mind that **PSws** is self-dual, the full subcategory consisting of the image of the functor is also self dual. The self-duality of this full sub-category was first noticed and proved in [65].

2. We prove an equivalence of categories between **PSws** and the Kleisli category $\mathbf{SCS}_{\mathcal{K}}$ of the Smyth power monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$ where $\mathcal{K}$ [52, section 6] is an endofunctor $\mathcal{K}$ on **SCS**. For an object $X \in \mathbf{SCS}$, $\mathcal{K}(X)$ is the set of compact saturated subsets of X equipped with the Scott-topology and for a morphism $f : X \longrightarrow Y$ in **SCS**, $\mathcal{K}(f)$ assigns to each compact saturated subset A of X the saturation of $f(A)$.

3. We prove an equivalence of categories between **PSws** and the category **SCSc** of stably compact spaces as objects and upper relations of the form $R \subseteq X \times Y_c$ as morphisms where Y_c is the co-compact topology on Y.

Finally we describe the way **MLS** can be extended to obtain an extension that provides logical descriptions for stably locally compact spaces.

1.4 Structure of the Thesis

The thesis consists of five chapters and three appendixes. The first chapter is an introduction to the research area and the research programme behind the thesis. Chapter 2 reviews the literature related to the work presented in this thesis. Chapters 3, 4, and 5 achieve the first, the second, and the third aims of the research programme, respectively. Appendixes A, B, and C present basic concepts from order theory, topology, and category theory, respectively.

Chapter 2

Related Work

This chapter reviews work related to that presented in this thesis. In order to make the thesis as self-contained as possible, concepts from order theory, topology and category theory are introduced in Appendixes A, B, and C, respectively.

2.0.1 Organisation

The chapter is organised as follows.

1. Section 2.1 introduces Stone's representation theorem for Boolean algebras and its extension for bounded distributive lattices. A computational reading of Stone dualities is presented as well.

2. Priestley's representation theorem for bounded distributive lattices, as another way of extending Stone's representation of Boolean algebras, is reviewed in Section 2.2. The Birkhoff representation theorem, which is a special case of Priestley duality for finite distributive lattices is reviewed as well. Also, the relationship between representations of bounded distributive lattices by Stone and Priestley is discussed.

3. Domain theory as a foundation for denotational semantics of programming languages

is introduced briefly in Section 2.3. We discuss why domains, as defined in this section, provide a convenient mathematical universe for the purposes of denotational semantics. We also introduce in this section classes of domains which are of special importance to us.

4. Locale theory, its relationship to domains theory and Stone dualities, and its role in computer science are discussed briefly in Section 2.4. Various dualities for classes of domains are discussed as well.

5. Samson Abramsky's localic and logical representations of bifinite domains are presented in Section 2.5. The way domain constructions can be carried out in these representations is briefly discussed.

6. Stably compact spaces as the primary objects of interest in this thesis are introduced in Section 2.6. We present a literature review of the work related to stably compact spaces. This includes the relationship between stably continuous frames and stably compact spaces, upper relations between stably compact spaces, and their probabilistic power domains.

7. Section 2.7 reviews the Jung-Sünderhauf representation theorem for stably compact spaces by a class of bounded distributive lattices known as strong proximity lattices. An extension to this theory is presented in this thesis.

8. Finally, Section 2.8 reviews the category **MLS** which provides logical descriptions for stably compact spaces and which is an extension to Samson Abramsky's domain theory in logical form for bifinite domains.

2.1 Stone Duality

This section has two goals. The first goal is purely mathematical and the second one is purely computational. The mathematical goal is to review Stone's representation theorems for Boolean algebras and bounded distributive lattices. The computational goal is to present a computer-science interpretation of Stone duality. This section is based on [20, 41, 18, 2].

2.1.1 Stone Duality, mathematically

In 1936, Marshall Stone showed that the category **BoolAlg** of Boolean algebras and Boolean algebra homomorphisms is dual to the category **StoSpc** of Stone spaces (compact totally disconnected Hausdorff spaces) and continuous functions [109]. Stone's duality maps a Stone space X to its algebra $\mathcal{P}^{\mathcal{T}}(X)$ of clopen subsets of X, and a Boolean algebra B to its space $\mathit{filt}_p(B)$ of prime filters of B, equipped with the topology $\mathcal{T}$ generated by the following basis set:

$$\mathcal{C} = \{U_a \mid a \in B\}, \text{ where } U_a = \{F \in \mathit{filt}_p(B) \mid a \in F\}.$$

The topological space $\langle \mathit{filt}_p(B); \mathcal{T} \rangle$ is called the *Stone dual space* of B.

Remark 2.1.1. Let $\langle B; \vee, \wedge, 0, 1, ' \rangle$ be a Boolean algebra. For every $F \in \mathit{filt}_p(B)$ and $a \in B$, a belongs to F if and only if a' does not belong to F. Therefore $\mathit{filt}_p(B) \setminus U_a = U_{a'}$ meaning that U_a is also closed in the Stone dual space and hence is clopen.

Proofs of results presented in this section can be found in [20, Chapter 11] and [41, II.3.4]. This duality is justified as follows.

The lemma below proves that the Stone dual space of a Boolean algebra is actually a Stone space.

Lemma 2.1.2. *Let $\langle B; \vee, \wedge, 0, 1, ' \rangle$ be a Boolean algebra. Then the Stone dual space $\langle \mathit{filt}_p(B); \mathcal{T} \rangle$ is compact, Hausdorff, and totally disconnected. Moreover, the set $\mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(B))$ of clopen subsets of $\mathit{filt}_p(B)$ equals the set $\{U_a \mid a \in B\}$.*

Proof. To prove the compactness of the Stone dual space , it is enough to show that every open cover consisting of basic open sets has a finite subcover. Suppose $\mathcal{O} \subseteq \mathcal{C}$ is an open cover to $\mathit{filt}_p(B)$. Hence $\mathcal{O} = \{U_a \mid a \in A\}$ for some $A \subseteq B$. Now let J be the ideal generated by A, i.e.

$$J = \bigcup \{\downarrow(a_1 \vee \ldots \vee a_n) \mid a_1, \ldots, a_n \in A\}.$$

By Lemma A.17, if J is a proper ideal then J is contained in a prime ideal $I \in idl_p(B)$. Therefore $A \subseteq I$. Set $F = B \setminus I$. By Lemma A.15, $F \in \mathit{filt}_p(B)$. Hence $F \notin U_a$ for every $a \in A$. This contradicts the fact that $\mathcal{O}$ is an open cover for $\mathit{filt}_p(B)$. Therefore I is not proper, i.e. $I = B$. This implies $1 \in I$ implying $1 = a_1 \vee \ldots \vee a_n$, for some $a_1, \ldots, a_n \in A$. Hence

$$\mathit{filt}_p(B) = U_1 = U_{a_1 \vee \ldots \vee a_n} = U_{a_1} \cup \ldots \cup U_{a_n},$$

which completes the proof of compactness.

Let $F_1, F_2 \in \mathit{filt}_p(B)$ be distinct elements of the Stone dual space . Then, without loss of generality, there exists $a \in F_1 \setminus F_2$. Hence $F_1 \in U_a$ and $F_2 \in \mathit{filt}_p(B) \setminus U_a$. This completes the proof that the Stone dual space is Hausdorff and totally disconnected as U_a is clopen.

We now prove that $\mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(B)) = \{U_a \mid a \in B\}$. One inclusion is given by Remark 2.1.1. For the other inclusion, let $V \in \mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(B))$. Then $V = \bigcup\{U_a \mid a \in A\}$, for some $A \subseteq B$, because V is open. V is closed in $\mathit{filt}_p(B)$ and hence it is compact because the space is compact Hausdorff. Therefore there exists a finite subset $A' \subseteq A$ such that $V = \bigcup\{U_a \mid a \in A'\} = U_{\bigvee A'}$. □

Theorem 2.1.3. *If B is a Boolean algebra then B and the Boolean algebra of clopen subsets $\mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(B))$ of the Stone dual space $\langle \mathit{filt}_p(B); \mathcal{T}\rangle$ are isomorphic via the map*

$$\eta : B \longrightarrow \mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(B)); a \longmapsto U_a.$$

Proof. By Lemma 2.1.2 η is well defined and onto. If a, b are distinct elements of B then there exists a prime filter $F \in \mathit{filt}_p(B)$ such that a belongs to F and b does not belong to F (by Lemma A.16 where $J = \uparrow a$ and $G = \downarrow b$ provided that $b \not\geq a$). Therefore η is one-to-one. It is easy to check that η is a lattice isomorphism. Since $\eta(0) = \emptyset$ and $\eta(1) = X$, η is also a Boolean algebra isomorphism. □

Theorem 2.1.4. *If X is a Stone space then X is homeomorphic to the Stone dual space of the Boolean algebra $\mathcal{P}^{\mathcal{T}}(X)$ of clopen subsets of X.*

Proof. Clearly,

$$\varepsilon : X \longrightarrow \mathit{filt}_p(\mathcal{P}^{\mathcal{T}}(X)); x \longmapsto \{V \in \mathcal{P}^{\mathcal{T}}(X) \mid x \in V\},$$

is a well-defined map. ε is one-to-one because X is totally disconnected. To show that ε is continuous, it suffices to show that the pre-image of a clopen set is clopen (the set of clopen subsets of a Stone space is a basis for the topology). But this is true because for a clopen set V of X we have

$$\varepsilon^{-1}(U_V) = \{x \in X \mid \varepsilon(x) \in U_V\} = \{x \in X \mid V \in \varepsilon(x)\} = V.$$

Now we show that ε is onto. Note that $\varepsilon(X)$ is closed (being the image of a compact Hausdorff space under a continuous map – Lemma B.6). Hence for $F \in \mathit{filt}_p(\mathcal{P}^{\mathcal{T}}(X)) \backslash \varepsilon(X)$, there exists $V \in \mathcal{P}^{\mathcal{T}}(X)$ such that $\varepsilon(X) \cap U_V = \emptyset$ and $F \in U_V$. This implies $\emptyset = \varepsilon^{-1}(U_V) = V$ contradicting the fact that $F \in U_V$. It follows that ε is a homeomorphism because both X and $\mathit{filt}_p(\mathcal{P}^{\mathcal{T}}(X))$ are compact Hausdorff spaces. □

On the morphism level, the duality sends a Boolean algebra homomorphism $f : B_1 \longrightarrow B_2$ to the function $f^{-1} : \mathit{filt}_p(B_2) \longrightarrow \mathit{filt}_p(B_1)$ and a continuous map $g : X_1 \longrightarrow X_2$, where X_1 and X_2 are Stone spaces, to the map $g^{-1} : \mathcal{P}^{\mathcal{T}}(X_2) \longrightarrow \mathcal{P}^{\mathcal{T}}(X_1)$.

In 1937, Stone extended his representation theorem for Boolean algebras to cover bounded distributive lattices [110]. The extension proves that the category **DLat** of bounded distributive lattices and lattice homomorphisms is dual to the category **SpecSpc**, of spectral spaces and perfect maps between them. The notions of object and morphism of the latter category are defined as follows.

Definition 2.1.5. A topological space $\langle X, \mathcal{T} \rangle$ is *spectral* if it is T_0, well-filtered, and the set of compact-open subsets of X is closed under finite intersections and is a basis for $\mathcal{T}$. A topological space is *well-filtered* if for every filter base $\{A_i \mid i \in I\}$ of compact saturated sets and an open set O:

$$\bigcap_i A_i \subseteq O \Longrightarrow (\exists j \in I)\ A_j \subseteq O.$$

Definition 2.1.6. A continuous map between spectral spaces is *perfect* if the pre-image of each compact-open subset is compact.

It is easy to check that every Stone space is spectral.

The new duality still works in the same way on objects and morphisms except that it sends a spectral space X to the algebra $K\Omega(X)$ of compact-open subsets of X rather than to the algebra $\mathcal{P}^{\mathcal{T}}(X)$ of clopen subsets.

Remark 2.1.7. A Stone space X is compact Hausdorff, therefore the set $K\Omega(X)$ equals the set $\mathcal{P}^{\mathcal{T}}(X)$.

2.1.2 Stone Duality, computationally

In this section, we describe a computational interpretation of Stone duality. Suppose we are trying to develop a program P. Logically speaking, developing the program P will include three steps. The first step is to specify the properties $\{\phi_i \mid i \in I\}$ that the program P has to satisfy (expressed as $(\forall i \in I)\ P \vDash \phi_i$). The second step is to actually design the program P. The third and the final step is to prove that every property ϕ_i specified in the first step is actually satisfied by the program P; for every property ϕ_i specified in the first step, $P \vDash \phi_i$. It must now be apparent that the logical relationship $P \vDash \phi_i$ is central to program design and development.

Stone duality can be interpreted as a tool providing two equivalent perspectives for the logical relationship $P \vDash \phi_i$. One perspective is provided by the topological side of the duality as follows.

1. Data types of a programming language correspond to topological spaces.

2. Programs correspond to points in the topological spaces.

3. Properties of programs correspond to open sets.

4. The logical relationship $P \vDash \phi_i$ is interpreted as follows. A program (point) x satisfies a property (an open set) O if and only if $x \in O$.

The other equivalent perspective is provided by the logical (localic) side of Stone duality.

1. Data types of a programming language correspond to spaces of prime filters of lattices.

2. Programs correspond to prime filters of lattices.

3. Properties of programs correspond to points of lattices.

4. The logical relationship $P \vDash \phi_i$ is interpreted as follows. A program (prime filter) F satisfies a property (point) x if and only if $x \in F$.

The localic side of Stone duality can be described using axioms and inference rules. This fact motivates using Stone duality, and other similar dualities, in establishing logical descriptions (in the form of logical systems) for program development. Examples of such logical systems are Abramsky's domain theory in logical form for Bifinite Domains [2, 4] and **MLS** [50, 51, 52, 77].

2.2 Priestley Duality

In 1970, H. Priestley observed another way of extending Stone duality for Boolean algebras to bounded distributive lattices [89, 90]. She showed that the category **DLat** of bounded distributive lattices and lattice homomorphisms is dual to the category **PSpc** of Priestley spaces (also known as *ordered stone spaces*) and continuous order-preserving functions. This section is based on [20, 41].

Definition 2.2.1. A *Priestley space* is a compact ordered space (Definition B.4) $\langle X; \mathcal{T}, \leq \rangle$ that is totally order-disconnected, i.e. for every $x, y \in X$, if $x \ngeq y$ then there exists a clopen upper set V such that $y \in V$ and $x \notin V$.

There are two alternative ways for presenting Priestley duality; using prime ideals or prime filters. Mathematically, they are entirely equivalent. Computationally however, it is preferable to work with prime filters. This is so because if we consider prime ideals then every point of a Priestley space will correspond to the prime ideal of clopen lower sets that do *not* contain the point. In the logic of observable properties, this ideal is the set of properties that are *not* satisfied by the point. Whereas if we consider prime filters then every point of a Priestley space will correspond to the prime filter of clopen upper sets containing the point. This filter is the set of observable properties satisfied by the point. Therefore, we present and work with Priestley duality using prime filters rather than prime ideals.

Priestley duality sends a Priestley space X to its algebra $\mathcal{U}^{\mathcal{T}}(X)$ of clopen upper subsets of X, and a bounded distributive lattice L to its space $\mathit{filt}_p(L)$ of prime filters of L ordered by inclusion and equipped with the topology $\mathcal{T}$ generated by the following basis set:

$$\mathcal{C} = \{U_a \setminus U_b \mid a, b \in L\}, \text{ where } U_a = \{F \in \mathit{filt}_p(L) \mid a \in F\}.$$

For every $a \in L$ we let O_a denote the set $\mathit{filt}_p(L) \backslash U_a$. Therefore the topology $\mathcal{T}$ has a subbasis consisting of the sets U_a and their complements, the sets O_a. The topological space $\langle \mathit{filt}_p(B); \subseteq, \mathcal{T}\rangle$ is known as the *Priestly dual space* of L.

In the case of finite distributive lattices, Priestley duality cuts down to Birkhoff's representation theorem, which establishes a correspondence between finite distributive lattices and finite ordered sets (on the topological side). In this case, the set of prime filters $\mathit{filt}_p(L)$ corresponds to the set of join-irreducible elements $\mathcal{J}(L)$ as follows:

$$F \in \mathit{filt}_p(L) \iff (\exists x \in \mathcal{J}(L))\ F = {\uparrow} x.$$

This is easy to prove because the meet of a prime filter is join-irreducible by primeness of the filter. Moreover, the upper set of a join-irreducible element is a prime filter because every join irreducible element is join-prime (Lemma A.8). We note however that the inclusion order on prime filters gets reversed when we consider their corresponding join-irreducible elements. This is so because

$$(\forall x, y \in \mathcal{J}(L))\ x \leq y \iff {\uparrow} y \subseteq {\uparrow} x.$$

Under Birkhoff duality, a finite ordered set P is sent to its lattice of lower sets $\mathcal{O}(P)$ ordered by inclusion, and a finite distributive lattice $\langle L; \leq\rangle$ is sent to its set of join-irreducible elements $\mathcal{J}(L)$ with the order induced from L.

Therefore, bounded distributive lattices (finite distributive lattices) are precisely the lattices of clopen upper sets (of lower sets) of Priestley spaces (of finite ordered sets). There is a trade-off between having a simple dual for a finite distributive lattice and the way this dual is ordered. On the one hand the dual in the finite case is a subset (join-irreducible elements) of the lattice rather than a set of subsets (prime filters) in the general case, but on the other hand the order of the finite case is the reverse of that of general case. This can be optimised by ordering join-irreducible elements by the dual of the order induced from the lattice.

Consequently we will have a version of Birkhoff duality under which the image of a finite ordered set P is its lattice of upper sets $\mathcal{U}(P)$ ordered by inclusion and the image of a finite distributive lattice $\langle L; \leq \rangle$ is $\langle \mathcal{J}(L), \leq^{\partial} \rangle$, where

$$a \leq^{\partial} b \Longleftrightarrow a \geq b.$$

We first introduce this version of Birkhoff's representation theorem and then extend it to Priestley duality.

2.2.1 The Finite Case: Birkhoff's Representation Theorem

We would like to confirm that all principal upper and lower sets, sets of the form $\uparrow x$ and $\downarrow x$, below are calculated with respect to the order $\leq$ not $\leq^{\partial}$. Proofs of results presented in this section can be found in [20, Chapter 5].

We start with reviewing the isomorphism between join-irreducible and meet-irreducible elements of finite distributive lattices.

Lemma 2.2.2. *Let L be a finite distributive lattice. Then the set of join-irreducible elements $\mathcal{J}(L)$ ordered by the lattice order and the set of meet-irreducible elements $\mathcal{M}(L)$ ordered by the lattice order are isomorphic via the following map:*

$$g : \mathcal{J}(L) \longrightarrow \mathcal{M}(L); x \longmapsto \bigvee (L \backslash \uparrow x).$$

Proof. As we have discussed above, $\langle \mathcal{J}(L), \leq \rangle$ and $\langle \mathit{filt}_p(L), \supseteq \rangle$ are isomorphic via the map

$$f : \mathcal{J}(L) \longrightarrow \mathit{filt}_p(L); x \longmapsto \uparrow x$$

whose inverse is

$$f' : \mathit{filt}_p(L) \longrightarrow \mathcal{J}(L); F \longmapsto \bigwedge F.$$

Dually, $\langle idl_p(L), \subseteq \rangle$ and $\langle \mathcal{M}(L), \leq \rangle$ are isomorphic via the map

$$h : \mathcal{M}(L) \longrightarrow idl_p(L); x \longmapsto \downarrow x$$

whose inverse is

$$h' : idl_p(L) \longrightarrow \mathcal{M}(L); I \longmapsto \bigvee I.$$

Also by Lemma A.15, $\langle idl_p(L), \subseteq\rangle$ and $\langle \mathit{filt}_p(L), \supseteq\rangle$ are isomorphic via the map

$$m : idl_p(L) \longrightarrow \mathit{filt}_p(L); I \longmapsto L\backslash I.$$

whose inverse is

$$m' : \mathit{filt}_p(L) \longrightarrow idl_p(L); F \longmapsto L\backslash F.$$

The map g is order-isomorphism because it equals $h \circ m \circ f$. □

Birkhoff duality is justified by the following facts:

1. Every ordered set $\langle P, \leq\rangle$ is isomorphic to $\langle \mathcal{J}(\mathcal{U}(P)), \supseteq\rangle$ via the following map:

$$\epsilon_P : P \longrightarrow \mathcal{J}(\mathcal{U}(P)); x \longmapsto \uparrow x.$$

2. Every finite distributive lattice $\langle L, \leq\rangle$ is isomorphic to $\langle \mathcal{U}(\langle \mathcal{J}(L), \leq^\partial\rangle), \subseteq\rangle$ via the following map

$$\eta_L : L \longrightarrow \mathcal{U}(\langle \mathcal{J}(L), \leq^\partial\rangle); a \longmapsto \{x \in \mathcal{J}(L) \mid x \leq a\}.$$

Proof. For every a in L, $\eta_L(a)$ is a lower subset of $\langle \mathcal{J}(L), \leq\rangle$ and therefore an upper subset of $\langle \mathcal{J}(L), \leq^\partial\rangle$. Hence η_L is well defined. We show that η_L is an order-isomorphism. Clearly $a \leq b$ implies $\eta_L(a) \subseteq \eta_L(b)$. For the other direction, suppose $\eta_L(a) \subseteq \eta_L(b)$. Then by Lemmas A.6 and A.18

$$a = \bigvee \eta_L(a) \leq \bigvee \eta_L(b) = b.$$

It remains to show that η_L is onto. Obviously $\emptyset = \eta_L(0)$. Let

$$\emptyset \neq \{a_1, \ldots, a_k\} \in \mathcal{O}(\langle \mathcal{J}(L), \leq\rangle) = \mathcal{U}(\langle \mathcal{J}(L), \leq^\partial\rangle)$$

and set $a = a_1 \vee \ldots \vee a_k$. We show that $\eta_L(a) = \{a_1, \ldots, a_k\}$. Every a_i is join-irreducible and below a in the lattice order. Therefore every a_i belongs to $\eta_L(a)$. For the other inclusion, let $x \in \eta_L(a)$. Then $x \leq a_1 \vee \ldots \vee a_k$ which implies $x \leq a_i$ for some i, by Lemma A.8. Hence $x \in \{a_1 \ldots a_k\}$ because $\{a_1 \ldots a_k\}$ is a lower subset of the lattice. $\square$

Let $\langle L_1; \leq_1\rangle$ and $\langle L_2; \leq_2\rangle$ be finite distributive lattices and P_1 and P_2 be finite ordered sets. Then Birkhoff duality sends a lattice homomorphism $f : L_1 \longrightarrow L_2$ to the order preserving map

$$\varphi_f : \langle \mathcal{J}(L_2), \leq_2^{\partial}\rangle \longrightarrow \langle \mathcal{J}(L_1), \leq_1^{\partial}\rangle; y \longmapsto \min_{L_1}\{x \in \mathcal{J}(L_1) \mid x \in f^{-1}(\uparrow y)\},$$

and an order preserving map $\varphi : P_1 \longrightarrow P_2$ to the lattice homomorphism:

$$f_{\varphi} : \mathcal{U}(P_2) \longrightarrow \mathcal{U}(P_1); U \longmapsto \varphi^{-1}(U).$$

Remark 2.2.3. The function φ_f above is well-defined because by Lemma A.18 the set of prime filters of a finite distributive lattice L is

$$\mathit{filt}_p(L) = \{\uparrow x \mid x \in \mathcal{J}(L)\}.$$

Moreover, the inverse image of a prime filter under a lattice homomorphism is a prime filter.

It is not hard to show that the following diagrams commute.

$$\begin{array}{ccc} L_1 & \xrightarrow{\ f\ } & L_2 \\ \downarrow{\scriptstyle \eta_{L_1}} & & \downarrow{\scriptstyle \eta_{L_2}} \\ \mathcal{U}(\langle \mathcal{J}(L_1), \leq_1^{\partial}\rangle) & \xrightarrow{f_{\varphi_f}} & \mathcal{U}(\langle \mathcal{J}(L_2), \leq_2^{\partial}\rangle) \end{array} \qquad \begin{array}{ccc} P_1 & \xrightarrow{\ \varphi\ } & P_2 \\ \downarrow{\scriptstyle \epsilon_{P_1}} & & \downarrow{\scriptstyle \epsilon_{P_2}} \\ \langle \mathcal{J}(\mathcal{U}(P_1)), \supseteq\rangle & \xrightarrow{\varphi_{f_{\varphi}}} & \langle \mathcal{J}(\mathcal{U}(P_2)), \supseteq\rangle \end{array}$$

Proof. 1. Let $a \in L_1$. Then

$$\begin{aligned}
\eta_{L_2}(f(a)) &= \{x \in \mathcal{J}(L_2) \mid x \leq f(a)\} \\
&= \{x \in \mathcal{J}(L_2) \mid \varphi_f(x) \leq a\} \\
&= \varphi_f^{-1}(\{y \in \mathcal{J}(L_1) \mid y \leq a\}) \\
&= f_{\varphi_f}(\eta_{L_1}(a)).
\end{aligned}$$

The second equality is true as follows

$$\varphi_f(x) \leq a \Longrightarrow x \leq f(\varphi_f(x)) \leq f(a) \Longrightarrow x \leq f(a).$$

For the other direction,

$$x \leq f(a) = f(\bigvee\{y \in \mathcal{J}(L_1) \mid y \leq a\}) = \bigvee\{f(y) \mid y \in \mathcal{J}(L_1) \text{ and } y \leq a\}.$$

Therefore by Lemma A.8 there exists $y \in \mathcal{J}(L_1)$ such that $x \leq f(y)$ and $y \leq a$. This implies $y \in f^{-1}(\uparrow x)$. Therefore $\varphi_f(x) \leq y$ implying $\varphi_f(x) \leq a$.

2. Let $x \in P_1$. Then

$$\begin{aligned}
\varphi_{f_\varphi}(\uparrow x) &= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid U \in f_\varphi^{-1}(\uparrow(\uparrow x))\} \\
&= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid f_\varphi(U) \in (\uparrow(\uparrow x))\} \\
&= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid \uparrow x \subseteq f_\varphi(U)\} \\
&= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid x \in f_\varphi(U)\} \\
&= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid x \in \varphi^{-1}(U)\} \\
&= \min_{\mathcal{U}(P_2)}\{U \in \mathcal{U}(P_2) \mid \varphi(x) \in U\} \\
&= \uparrow\varphi(x).
\end{aligned}$$

□

Furthermore for a lattice homomorphism f between finite distributive lattices L_1 and L_2, f is one-to-one if and only if φ_f is onto. And f is onto if and only if φ_f is an order-embedding (Definition A.13).

2.2.2 The General Case

We now justify Priestley duality in its general case. Proofs of results presented in this section can be found in [20, Chapter 11] and [41, II.4].

Lemma 2.2.4. *Let L be a bounded distributive lattice. Then the Priestley dual space $\langle \mathit{filt}_p(L); \subseteq, \mathcal{T}\rangle$ is compact and totally order-disconnected. Moreover, the set $\mathcal{U}^{\mathcal{T}}(\mathit{filt}_p(L))$ of clopen upper subsets of $\mathit{filt}_p(L)$ equals the set $\{U_a \mid a \in L\}$ and the set $\mathcal{P}^{\mathcal{T}}(\mathit{filt}_p(L))$ of clopen subsets of $\mathit{filt}_p(L)$ equals the set $\{U_a \setminus U_b \mid a, b \in L\}$.*

Proof. By Alexander's subbasis lemma (Lemma B.7), it is sufficient to show that every open cover to $\mathit{filt}_p(L)$ by members of

$$\{U_a \mid a \in L\} \cup \{O_b \mid b \in L\}$$

has a finite sub-cover. Suppose

$$\mathcal{O} = \{U_a \mid a \in A\} \cup \{O_b \mid b \in B\}, A, B \subseteq L$$

is an open cover to $\mathit{filt}_p(L)$. Let I be the ideal generated by A and F be the filter generated by B. By Lemma A.16, if $I \cap F = \emptyset$ then there exists a prime filter F' such that $F \subseteq F'$ and $I \cap F' = \emptyset$. Hence $F' \notin U_a$ for every $a \in A$ and $F' \notin O_b$ for every $b \in B$. This contradicts the assumption that $\mathcal{O}$ is an open cover of $\mathit{filt}_p(L)$. Hence $I \cap F \neq \emptyset$. Fix $a \in I \cap F$ and suppose that A and B are nonempty (the empty cases cut down to Lemma 2.1.2). Then there exist $\{a_1, \ldots, a_n\} \subseteq A$ and $\{b_1, \ldots, b_m\} \subseteq B$ such that

$$b_1 \wedge \ldots \wedge b_m \leq a \leq a_1 \vee \ldots \vee a_n.$$

For every $G \in \mathit{filt}_p(L)$ either $a \in G$ or $a \notin G$. If $a \in G$ then $a_1 \vee \ldots \vee a_n \in G$ implying $a_j \in G$ for some j therefore $G \in U_{b_j}$. If $a \notin G$ then $b_1 \wedge \ldots \wedge b_m \notin G$ implying $a_j \notin G$ for some j therefore $G \in O_{a_j}$. Hence

$$\mathit{filt}_p(L) = U_1 = U_{a_1} \cup \ldots \cup U_{a_n} \cup O_{b_1} \cup \ldots \cup O_{b_m}.$$

It is fairly easy to prove that the Priestley dual space is totally order-disconnected. The proof of the rest of the lemma is similar to Lemma 2.1.2. $\square$

Theorem 2.2.5. *Let L be a bounded distributive lattice. Then L is isomorphic to the lattice $\mathcal{U}^{\mathcal{T}}(\mathit{filt}_p(L))$ of clopen upper subsets of the Priestley dual space $\langle \mathit{filt}_p(L), \subseteq, \mathcal{T}\rangle$ of L via the following map:*

$$\eta_L : L \longrightarrow \mathcal{U}^{\mathcal{T}}(\mathit{filt}_p(L)); a \longmapsto U_a.$$

Theorem 2.2.6. *Let X be a Priestley space. Then X is order-homeomorphic to the Priestley dual space of the lattice $\mathcal{U}^{\mathcal{T}}(X)$ of clopen upper subsets of X via the following map:*

$$\epsilon_X : X \longrightarrow \mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X)); x \longmapsto \{V \in \mathcal{U}^{\mathcal{T}}(X) \mid x \in V\}.$$

Proof. Clearly ϵ_X is well-defined. Moreover, ϵ_X is an order-embedding because

$$x \leq y \text{ in } X \iff (\forall V \in \mathcal{U}^{\mathcal{T}}(X))(x \in V \implies y \in V) \iff \epsilon_X(x) \subseteq \epsilon_X(y).$$

Now, we show that ϵ_X is continuous by showing that the pre-images of subbasis members are open i.e. by showing that $\epsilon_X^{-1}(U_V)$ and $\epsilon_X^{-1}(\mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X)) \setminus U_V)$ are open for every $V \in \mathcal{U}^{\mathcal{T}}(X)$. This is proved as follows:

$$\epsilon_X^{-1}(\mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X)) \setminus U_V) = \{x \in X \mid \epsilon_X(x) \notin U_V\} = X \setminus \epsilon_X^{-1}(U_V),$$

and

$$\epsilon_X^{-1}(U_V) = \{x \in X \mid \epsilon_X(x) \in U_V\} = \{x \in X \mid V \in \epsilon_X(x)\} = V.$$

Finally, we prove that ϵ_X is onto. Note that $\epsilon_X(X)$ is a closed (by Lemma B.6 because X is compact Hausdorff and ϵ_X is continuous). If $F \in \mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X)) \setminus \epsilon_X(X)$ then there exists a clopen subset V of the Priestley dual space $\mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X))$ such that $\epsilon_X(X) \cap V = \emptyset$ and $F \in V$ (by totally ordered-disconnectedness). By Lemma 2.2.4 we can assume that $V = U_{C_1} \setminus U_{C_2}$, for some $C_1, C_2 \in \mathcal{U}^{\mathcal{T}}(X)$. Hence $\emptyset = \epsilon_X^{-1}(V) = C_1 \setminus C_2$ implying $C_1 \subseteq C_2$. This is a contradiction because $F \in U_{C_1} \cap (\mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X)) \setminus U_{C_2}) = U_{C_1} \cap O_{C_2}$. $\square$

Priestley duality acts on morphisms in the same way as Stone duality.

Under Priestley duality, there is a one-to-one correspondence between open upper subsets of Priestley spaces and ideals of bounded distributive lattices. If $\langle X; \mathcal{T}, \leq \rangle$ is a Priestley space represents the lattice $\mathcal{U}^{\mathcal{T}}(X)$ then under this correspondence an ideal I in $\mathcal{U}^{\mathcal{T}}(X)$ is sent to its union and an upper open subset S of X is sent to the set of all clopen upper sets contained in S. Dually, there is a one-to-one correspondence between open lower subsets of Priestley spaces and filters of bounded distributive lattices. Proofs of these correspondences are in [20, 11.29].

Now it is not surprising that the category **SpecSpc** of spectral spaces and perfect maps between them is equivalent to the category **PSpc** of Priestley spaces and order-preserving continuous maps between them. The following definition is needed.

Definition 2.2.7. Let $\langle X, \mathcal{T} \rangle$ be a topological space.

1. A subset $A \subseteq X$ is *saturated* if it is an intersection of open sets.

2. The *patch topology* of $\langle X, \mathcal{T} \rangle$ is the topology on X generated by the set of open sets and complements of elements in $\mathcal{K}_X$, the set of compact saturated subsets of X.

The details of the equivalence mentioned above are as follows. For a Priestley space $\langle X; \mathcal{T}, \leq \rangle$,

- the set $\mathcal{T}^{\uparrow}$ of open upper subsets of X is a spectral topology on X,

- the specialisation order of $\langle X, \mathcal{T}^{\uparrow} \rangle$ is $\leq$, and

- for each subset $O \subseteq X$, O belongs to $\mathcal{T}$ if and only if O is open in the patch topology of $\langle X, \mathcal{T}^{\uparrow} \rangle$.

For a spectral space $\langle Y, \mathcal{T} \rangle$, if $\mathcal{T}'$ is its patch topology and $\leq_{\mathcal{T}}$ is its specialisation order then $\langle Y, \mathcal{T}', \leq_{\mathcal{T}} \rangle$ is a Priestley space. Moreover, for each subset $O \subseteq Y$, O belongs to $\mathcal{T}$ if and only if O belongs to $\mathcal{T}'$ and is upper with respect to $\leq_{\mathcal{T}}$.

The part of the equivalence pertaining to the morphisms relies on the following fact. For spectral spaces X and Y, a map $f : X \longrightarrow Y$ is perfect if and only if it is continuous with respect to the patch topologies of X and Y and order-preserving with respect to the specialisation orders of X and Y.

2.3 Domain Theory

This section reviews basic definitions and results from domain theory with a focus on those that are related to the work presented in this thesis. The section is based on [28, 4].

Definition 2.3.1. A poset P is a *dcpo (directed-complete partial order)* if it is closed under suprema of directed subsets of P. P is *pointed* if it has a least element.

Definition 2.3.2. Let P be a dcpo and $A \subseteq P$. The set A is open in *the Scott-topology* on P if it is upper and for every directed subset $S \subseteq P$, $\bigvee^{\uparrow} S \in A$ implies $S \cap A \neq \emptyset$. The Scott-topology on P is denoted by σ_P. The *Lawson topology* on P (denoted by λ_P) is the topology generated by the following subbasis:

$$\{A \subseteq P \mid A \in \sigma_p \text{ or } A = P \setminus \uparrow x\}.$$

Definition 2.3.3. Let P and Q be dcpo's and f a map from P to Q.

1. The map f is *Scott-continuous* if for every directed subset $S \subseteq P$, $f(\bigvee^{\uparrow} S) = \bigvee^{\uparrow}\{f(s) \mid s \in S\}$.

2. The map f is *strict* if it preserves the least element.

It is straightforward to prove that maps between dcpo's are Scott-continuous if and only if they are continuous (topologically) with respect to Scott topologies on dcpo's.

Now we define some constructions on dcpo's. For dcpo's P and Q:

1. The *lifting* of P denoted by $P_{\perp}$ is obtained by adding a least element to P.

2. The *Cartesian product* $P \times Q$ is the set $\{\langle a, b\rangle \mid a \in P \text{ and } b \in Q\}$ ordered component-wise.

3. The *function space* $[P \longrightarrow Q]$ is the set of all Scott-continuous maps from P to Q ordered point-wise.

4. The *strict function space* $[P \longrightarrow_{\perp} Q]$ is the set of all strict Scott-continuous maps from P to Q ordered point-wise.

5. Suppose P and Q are pointed. Then the *coalesced sum* $A \oplus B$ is the disjoint union of A and B with identifying the least elements.

Definition 2.3.4. Let $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ be continuous maps between dcpo's P and Q. If $g \circ f = id_P$ and $f \circ g \subseteq id_Q$ then $\langle f, g\rangle$ is a *projection pair* from P to Q, f is an *embedding*, and g is a *projection*.

The category **DCPO** has dcpo's and Scott-continuous maps between them as objects and morphisms, respectively. The category **DCPO**$_{\perp}$ is the sub-category of **DCPO** whose objects are pointed dcpo's and whose morphisms are strict Scott-continuous maps. The category **DCPO** is a very convenient mathematical environment to develop denotational semantics of programs. The reasons behind this include the following. First it supports the recursive definitions of programs (represented by elements of domains) and data types (represented by domains). This was Scott's original discovery in 1969 [4]. We quote two theorems:

Theorem 2.3.5. *Let P be a pointed dcpo and f be a Scott-continuous map on P.*

1. *The map f has fix-points and the least of them is $\bigvee^{\uparrow}_{n\in\mathbb{N}} f^n(\perp)$.*

2. *The map $fpoint : [P \longrightarrow P] \longrightarrow P; f \longmapsto \bigvee^{\uparrow}_{n\in\mathbb{N}} f^n(\perp)$ is Scott continuous.*

The proof of the previous theorem can be found in [4, Chapter 2] and that of he following can be found in [86, Chapter 5]. The category $\mathbf{DCPO}^e$ is the sub-category of **DCPO** whose morphisms are the embeddings (Definition 2.3.4).

Theorem 2.3.6. *Let* $G : \mathbf{DCPO}^e \longrightarrow \mathbf{DCPO}^e$ *be a continuous functor. Then there exists a dcpo* P *such that*

$$P \cong G(P).$$

A precious definition of the notion of continuous functor, which is analogous to that of Scott-continuous maps between dcpo's, can be found in [86, Chapter 4].

Another convenient aspect of **DCPO** is that it admits a highly productive type-structure [86, Chapters 2,3].

In domain theory, the computability of processes is measured via testing the continuity of representing mathematical structures (functions as elements of function spaces between dcpo's). This is the reason that a convenient notion of continuity is required for dcpo's. Consequently an approximation notion over dcpo's is needed in defining continuity. This justifies the importance of the following two definitions.

Definition 2.3.7. Let P be a dcpo, $A \subseteq P$ and $x, y \in P$.

1. x *approximates* y (denoted by $x \ll y$) if for every directed subset S of P, $y \leq \bigvee^{\uparrow} S$ implies $x \leq s$, for some $s \in S$.

2. The element x is *compact (or finite)* if $x \ll x$. The set of all compact elements of P is denoted by $K(P)$.

3. $\twoheaduparrow x = \{z \in P \mid x \ll z\}$ and $\twoheaduparrow A = \bigcup\{\twoheaduparrow a \mid a \in A\}$. Dually, $\twoheaddownarrow x$ and $\twoheaddownarrow A$ are defined.

Definition 2.3.8. Let P be a dcpo.

1. P is a *continuous domain (or domain)* if for every $x \in P, x = \bigvee^{\uparrow} \twoheaddownarrow x$.

2. P is an *algebraic domain* if for every $x \in P, x = \bigvee^{\uparrow}\{z \in K(P) \mid z \ll x\}$.

3. P is an ω-*algebraic domain* if it is an algebraic domain and $K(P)$ is countable.

CONT, **ALG** and ω**ALG** are the categories whose objects are continuous, algebraic and ω-algebraic domains, respectively, and whose morphisms are Scott-continuous maps between objects in each category.

Although the categories **DCPO** and **DCPO**$_\perp$ are closed under the function-space construction (*Cartesian-closed*), the subcategories **CONT** and **ALG** are not. The closure of **DCPO** under function-space construction is very interesting for reasons including the following. Let A and B be two data structures represented by two dcpo's P and Q, respectively. Now we are pretty sure that $[P \longrightarrow Q]$, the set of all Scott-continuous functions from P to Q, represents the set of all computable programs from A to B. Therefore every computable program from A to B is represented by a point in the dcpo $[P \longrightarrow Q]$. This would not be the case if **DCPO** were not Cartesian-closed.

The importance of continuous domains and the closure under the function-space construction were the motivation for the research in [45, 47, 44, 43, 46, 48] in which Achim Jung described maximal Cartesian-closed full subcategories of **CONT** and **ALG**. In the following we present some of these subcategories.

Definition 2.3.9. Let P be a pointed continuous (algebraic) domain.

1. P is an *L-domain* (an *aL-domain*) if

$$(\forall a, b, c \in P)\ a, b \leq c \Longrightarrow a \vee b \text{ exists in } \downarrow c.$$

2. P is a *bounded-complete domain or bc-domain* (an *abc-domain*) if

$$(\forall a, b, c \in P)\ a, b \leq c \Longrightarrow a \vee b \text{ exists in } P.$$

3. P is a *continuous lattice* (an *algebraic lattice*) if

$$(\forall a, b \in P)\ a \vee b \text{ exists in } P.$$

L, **BC**, and **LAT** (**aL**,**aBC** and **aLAT**) are the full subcategories of $\mathbf{DPCO}_\perp$, whose objects are L-domains, bc-domains and continuous lattices (aL-domains, abc-domains and a-continuous lattices), respectively, and whose morphisms are the Scott-continuous.

Proofs of results presented in the rest of this section can be found in [4, Chapter 4].

Lemma 2.3.10. *The categories* $\mathbf{L}, \mathbf{BC}, \mathbf{LAT}, \mathbf{aL}, \mathbf{aBC}$ *and* $\mathbf{aLAT}$ *are Cartesian-closed.*

Definition 2.3.11. Let P be a poset and A a subset of P.

1. P has the *property m* if for every $M \subseteq_{fin} P$, the set of upper bounds of M equals the set $\uparrow mub(M)$ where $mub(M)$ stands for the set of minimal upper bounds of M.

2. The *mub-closure of* A (denoted by $mc(A)$) is the smallest set that contains A and the set $mub(M)$ for every $M \subseteq_{fin} mc(A)$.

3. P has the *finite mub property* if it has the property m and $mc(M)$ is finite for every $M \subseteq_{fin} P$.

4. P is a *Plotkin order* if it is pointed and has the finite mub property.

Definition 2.3.12. Let P be an algebraic domain. P is a *bifinite domain* if $K(P)$ is a Plotkin order. **B** is the full subcategory of $\mathbf{ALG}_\perp$, whose objects are the bifinite domains.

Remark 2.3.13. Note that the category $\mathbf{ALG}_\perp$ is the sub-category of **ALG**, whose objects are pointed algebraic domains and whose morphisms are strict Scott-continuous maps.

Definition 2.3.14. Let P be a pointed dcpo and f a Scott-continuous map on P. f is *finitely-separated* from the identity map on P if there exists $M \subseteq_{fin} P$ such that:

$$(\forall x \in P)(\exists m \in M)\ f(x) \leq m \leq x.$$

Definition 2.3.15. Let P be a pointed dcpo. P is an *FS-domain* if the identity map id_P on P is the supremum $\bigvee^{\uparrow}_{i \in I} f_i$ where $\{f_i \mid i \in I\}$ is a directed family of Scott-continuous maps on P, each of them finitely-separated from id_P.

Remark 2.3.16. It is not hard to prove that FS-domains are continuous. Therefore we define the category **FS** to be the full sub-category, of **CONT**, whose objects are FS-domains.

Lemma 2.3.17. *The categories* **B** *and* **FS** *are Cartesian-closed.*

Remark 2.3.18. It is a fact that a bifinite domain is an FS-domain and an algebraic FS-domain is a bifinite domain.

L and **FS** (**aL** and **B**) are maximal Cartesian-closed full subcategories of $\mathbf{CONT}_\perp$ ($\mathbf{ALG}_\perp$).

The following lemma which characterise compact saturated subsets of continuous domains will be needed in Chapter 5.

Lemma 2.3.19. *Let P be a continuous domain and K a compact saturated subset of P that is contained in an open set O. Then there exists a finite set $M \subseteq P$ such that*

$$K \subseteq \mathop{\uparrow\!\!\!\!\uparrow} M = \{y \in P \mid (\exists m \in M)\, m \ll y\} \subseteq \uparrow M \subseteq O.$$

Coherent Domains

Coherent domains as defined below, are the primary objects of interest in this thesis. More precisely, we are interested in studying compact (with respect to the Scott-topology) coherent domains. This will be clarified in the next section. For the moment, we just introduce the notion of coherent domain and state some results concerning it.

Definition 2.3.20. A continuous domain is *coherent* if the binary intersection of compact saturated subsets is compact.

Alternatively, on a continuous domain P coherence can be defined via the lattice of Scott-open sets σ_P, or via the Lawson topology λ_P as follows:

$$\begin{aligned} P \text{ is coherent} \quad &\iff \quad ((\forall O_1, O_2, O \in \sigma_P)\, O \ll O_1, O \ll O_2 \Longrightarrow O \ll O_1 \cap O_2) \\ &\iff \quad \text{the Lawson topology on } P \text{ is compact.} \end{aligned}$$

For an algebraic domain P, we can describe coherence via the set of compact elements $K(P)$ as follows. P is coherent if and only if

1. $K(P)$ has property m, and

2. the set of minimal upper bounds of any finite subset of $K(P)$ is finite.

Remark 2.3.21. The description of coherent algebraic domains via their sets of compact elements justifies the terminology $2/3$ *bifinite domains* for coherent algebraic domains.

Lemma 2.3.22. *FS-domains and bifinite domains are coherent.*

Some interesting facts about coherent domains are the following.

Lemma 2.3.23. *Suppose P and Q are continuous domains such that Q is pointed.*

1. *If $[P \longrightarrow Q]$ is continuous then P is coherent or Q is L-domain.*

2. *Q is an FS-domain if and only if Q and $[Q \longrightarrow Q]$ are coherent.*

2.4 Locales

This section presents basic ideas from locale theory and a computational interpretation for the theory. The section is based on [4, 2, 41]. Proofs of results presented in this section can be found in [4, Chapter 7] and [41, II.3.2-4].

Definition 2.4.1. A *frame* is a complete lattice satisfying the *infinite distributive law*:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

Frame homomorphisms are functions between frames that preserve finite infima and arbitrary suprema.

The category **Frm** is the category of frames and frame homomorphisms. The category **Loc** is the opposite of the category **Frm**.

The starting point of locale theory is the following observation. There is an adjunction $(\Omega, pt, \eta, \epsilon)$ between the category **Top** of topological spaces and continuous maps between them and the category **Loc**. This observation and some other facts from category theory and constructive mathematics led mathematicians (in particular topologists) to focus their research on the category **Loc** instead of the category **Top**. **Loc** is an abstract category; concretely one works with **Frm**.

In terms of **Frm**, the adjunction $(\Omega, pt, \eta, \epsilon)$ is detailed as follows. The functor,

$$\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$$

sends a topological space X to its open-sets lattice $\Omega(X)$ (ordered by inclusion) and a continuous map f to its inverse image f^{-1}.

The functor,

$$pt : \mathbf{Frm} \longrightarrow \mathbf{Top}$$

sends a frame A to its collection of completely prime filters $pt(A)$ equipped with the topology whose open sets have the following form:

$$U_a = \{F \in pt(A) \mid a \in F\}, a \in A.$$

The functor pt sends a frame homomorphism $f : A \longrightarrow B$ to the continuous map

$$pt(f) : pt(B) \longrightarrow pt(A); F \longmapsto f^{-1}(F).$$

The elements of $pt(A)$ are knows as the *points* of A and the topology defined on $pt(A)$ above is known as the *point topology*.

For a topological space X, the map

$$\eta_X : X \longrightarrow pt(\Omega(X)); x \longmapsto \{O \mid x \in O\},$$

is continuous, open onto its image, and commutes with continuous functions. Therefore the collection of all η_X, where X is a topological space, is a natural transformation from the identity functor on **Top** to the functor $pt \circ \Omega$.

For a locale A, the map

$$\epsilon_A : A \longrightarrow \Omega(pt(A)); a \longmapsto U_a,$$

is a frame homomorphism and commutes with frame homomorphisms. Hence the collection of all ϵ_A, where A is a frame, is a natural transformation from the identity functor on **Frm** to the functor $\Omega \circ pt$. Moreover, for a topological space X the composition

$$\Omega(X) \xrightarrow{\epsilon_{\Omega(X)}} \Omega(pt(\Omega(X))) \xrightarrow{\Omega(\eta_X)} \Omega(X),$$

equals the identity on $\Omega(X)$ and for a frame A the composition

$$pt(A) \xrightarrow{\eta_{pt(A)}} pt(\Omega(pt(A))) \xrightarrow{pt(\epsilon_A)} pt(A),$$

equals the identity on $pt(A)$.

Therefore the functors Ω and pt are dual adjoints to each other with η and ϵ as their units.

Remark 2.4.2. Let L be a bounded distributive lattice. Then the Stone dual space of L is homeomorphic to the space $pt(idl(L))$ of the locale $idl(L)$ (ideals of L) equipped with the point topology. The locales arising as the sets of ideals of bounded distributive lattices are called *coherent locales*. The points of coherent locales equipped with the point topology are precisely the spectral spaces. Moreover, the open-set lattices of spectral spaces are precisely the coherent locales [41]. This establishes a duality between the category **CohLoc** of coherent locales and perfect maps (locales homomorphisms whose corresponding frame homomorphism maps compact (finite) elements to compact elements), and the category **SpecSpc** of spectral spaces and perfect maps between them. This duality sends a perfect maps f between spectral spaces, to the locale homomorphism corresponding to the frame homomorphism f^{-1}. A locale homomorphism f is sent, under the duality, into the inverse image f^{*-1}, where f^* is the frame homomorphism corresponding to f.

To complete our survey of the relationship between coherent locales, bounded distributive lattices and spectral spaces, we should mention the following. The sublattices of compact (finite) elements (elements approximating themselves) of coherent locales are precisely the bounded distributive lattices.

Moreover, the category **CohLoc** is dual to the category **DLat**, of bounded distributive lattices and lattice homomorphisms between them. On the morphisms level, this duality sends a coherent map $f : A \longrightarrow B$ to the restriction of the corresponding frame homomorphism to compact elements $f^* : K(B) \longrightarrow K(A)$. Suppose A, B are coherent locales and $g : K(A) \longrightarrow K(B)$ is a lattice homomorphism. The locales A and B are freely generated by $K(A)$ and $K(B)$, respectively. Therefore g extends uniquely to a frame homomorphism $g' : A \longrightarrow B$. Hence the duality, being discussed in this section, sends g into the locale homomorphism corresponding to the frame homomorphism g'.

The observations reviewed in this remark are the core of the direct relationship between Stone duality and locale theory.

Definition 2.4.3.

1. A topological space is *sober* if every closed irreducible set (a set which is not the union of two of its closed proper-subsets) is the closure of a unique point in the space.
2. A *stably locally compact space* is a topological space that is sober, locally compact and such that the collection of compact saturated subsets is closed under binary intersection.
3. A *stably compact space* is a compact stably locally compact space.
4. A *spectral space* is a stably compact space which has a basis of compact open sets.
5. A complete lattice is *spatial* if its elements are separated by completely prime filters.

6. A complete lattice L is *completely distributive* if for every set $\{A_i \mid i \in I \text{ and } A_i \subseteq L\}$,

$$\bigwedge_{i\in I} \bigvee A_i = \bigvee_{f: I \xrightarrow{\odot} \cup A_i} \bigwedge_{i \in I} f(i),$$

where $\odot$ means that $f(i) \in A_i$ for every $i \in I$.

7. A *continuous lattice* is a complete lattice in which every element is the supremum of elements approximating it.

8. An *arithmetic lattice* is a distributive continuous lattice in which

$$x \ll y \text{ and } x \ll z \Longrightarrow x \ll y \wedge z.$$

9. A *stably continuous frame* is an arithmetic lattice in which $1 \ll 1$ (the top element approximates itself).

Sober spaces have interesting properties including the celebrated Hofmann-Mislove theorem [59] which states that, in a sober space X, the set $\mathit{Sfilt}(\Omega(X))$ of Scott-open filters of the open-sets lattice $\Omega(X)$, is isomorphic to the set $\mathcal{K}_X$, of compact saturated subset of X. This isomorphism simply maps a compact saturated set A to the set of all open sets containing A. Conversely, a Scott open filter $\mathcal{F}$ of open set in X is mapped under this isomorphism to the intersection $\bigcap \mathcal{F}$.

The functors Ω and pt establish a dual equivalence between the following categories:

Remark 2.4.4. In Table 2.1, the morphisms on the topology side are always continuous functions and on the lattice side are always the frame homomorphisms.

Remark 2.4.5. From Remark 2.4.2 and the table above it should be clear that the duality between sober spaces and spatial lattices is the most general topological version of Stone duality.

the category of sober spaces	and	the category of spatial frames
the category of sober locally compact spaces	and	the category distributive continuous lattices
the category of stably locally compact spaces	and	the category of arithmetic lattices
the category of stably compact spaces	and	the category of stably continuous frames
the category of spectral spaces	and	the category of algebraic arithmetic lattices
CONT	and	the category of completely distributive lattices
ALG	and	the category of algebraic completely distributive lattices
the category of coherent domains	and	the category of arithmetic completely distributive lattices
the category of coherent algebraic domains	and	the category of algebraic arithmetic completely distributive lattices

Table 2.1: Stone dualities

In this thesis, we are interested in studying coherent domains which are compact in their Scott topologies. As we have already mentioned before, coherent domains include Cartesian-closed subcategories of continuous domains like FS-domains and bifinite domains and this points to the importance of studying the coherent domains. From Table 2.1 above, it is clear that coherent domains are stably locally compact spaces. *Stably compact spaces* are compact stably locally compact spaces. Therefore the notion of *stably compact space*, the primary notion of interest in this thesis, is a topological generalisation of compact coherent domains in their Scott topologies.

Now we have two perspectives for different concepts in topology; the classical perspective appearing on the topological side of the dualities described above and the perspective arising from the lattices (localic) side, using the point topology. In other words, classical concepts from topology correspond to lattice-theoretic concepts on the frame side of the dualities above. This is clarified in Table 2.2, where $\langle X, \mathcal{T} \rangle$ is a topological space and L is a complete lattice.

$\langle X, \mathcal{T} \rangle$	$pt(L)$ equipped with the point topology
point	completely prime filter
specialisation order	the inclusion-of-sets order
open set	completely prime filters containing an element $x \in L$
compact saturated set	completely prime filters containing a Scott open filter of L

Table 2.2: Topological concepts on duality sides

Lattice homomorphisms between distributive lattices can be replaced with certain upper relations to obtain the category **DLatj** of distributive lattices and join-approximable relations between them. This category provides a finitary representation for the category **SpecSpc** of spectral spaces and continuous maps between them. The notion of join-approximable relation is defined as follows.

Definition 2.4.6. Let L and K be two lattices. A binary relation $B \subseteq L \times K$ is *join-approximable* if for every $l, l', l_1, \ldots, l_n \in L$ and $k, k', k_1, \ldots, k_m \in K$:

1. $l' \leq l \; B \; k \leq k' \Longrightarrow l' \; B \; k'$.
2. $(\forall 1 \leq i \leq m) \; l \; B \; k_i \Longrightarrow l \; B \; (\bigwedge_{1 \leq i \leq m} k_i)$.
3. $(\forall 1 \leq i \leq n) \; l_i \; B \; k \Longrightarrow (\bigvee_{1 \leq i \leq n} l_i) \; B \; k$.
4. $l \; B \; (\bigvee_{1 \leq i \leq m} k_i) \Longrightarrow (\exists M \subseteq_{fin} L) \; l = \bigvee M$ and $(\forall m \in M)(\exists i) \; m \; B \; l_i$.

Theorem 2.4.7. *The categories* **DLatj** *and* **SpecSpc** *are equivalent. The equivalence sends a spectral space X to its lattice of compact open subsets $K\Omega(X)$ and a distributive lattice L to its spectrum $spec(L)$ of prime filters, equipped with the topology generated by the following basis:*

$$U_a = \{F \in \mathit{filt}_p(L) \mid a \in F\}, a \in L.$$

On the morphism level, a continuous map $f : X \longrightarrow Y$ between spectral spaces X and Y is sent to the binary relation $B_f \subseteq K\Omega(Y) \times K\Omega(X)$ defined as follows:

$$U \; B_f \; V \overset{\text{def}}{\Longleftrightarrow} f^{-1}(U) \subseteq V.$$

A join-approximable relation $B \subseteq L \times K$ is sent to the following continuous map:

$$f_B : spec(K) \longrightarrow spec(L); F \longmapsto \{l \in L \mid (\exists k \in F)\ l\ B\ k\}.$$

Moreover, this equivalence cuts down to an equivalence between coherent algebraic domains and lattices whose elements are joins of finite subsets of join-prime elements.

We now introduce the concept of bifinite lattice together with a result which is crucial for Abramsky's logical form for bifinite domains.

Definition 2.4.8. A bounded lattice L is *bifinite* if

1. the top element is join-prime,
2. elements of L are finite suprema of join-prime elements, and
3. for every finite subset M of join-prime elements, there is another finite subset N of join-primes such that $M \subseteq N$ and

$$(\forall A \subseteq N)(\exists B \subseteq N) \bigwedge A = \bigvee B.$$

Theorem 2.4.9. *A lattice is bifinite if and only if it is isomorphic to the lattice of compact open subsets of a bifinite domain.*

2.4.1 Computational Reading of Locale Theory

In this section, we present a computational interpretation of locale theory. The idea is that the elements of topological spaces denote programs and then the elements of the corresponding lattices act as semi-observable properties of the programs. The properties are semi-observable because the satisfaction of a property by a program can be determined given a finite amount of information about the program but the violation of a property by a program can not, in general, be determined in finite time.

Moreover, the Sierpinski space $\mathbb{O}$ (thought of as 2 equipped with the upper topology) fits into the picture as follows. Given a topological space X, we have the following

$$\text{properties of programs} = \text{open sets of topological spaces}$$
$$= \text{points of corresponding lattice.}$$

Then the properties (open sets of X ($\Omega(X)$)) of elements (programs) of X correspond to the continuous maps from X to $\mathbb{O}$ i.e.

$$\Omega(X) \cong (X \longrightarrow \mathbb{O}).$$

2.5 Abramsky's Logical Form for Bifinite Domains

In this section, we briefly review Abramsky's famous paper, *Domain Theory in Logical Form* [2]. The section is based on [2] and [4, section 7.3]. In this paper, Abramsky presented a logical representation for bifinite domains and he showed that the logical form is Stone dual to the domain form. Moreover, Abramsky showed how to get domain constructions (like the function-space construction) done in the logical formalism. He also proved that the results of the same domain construction on domains and their logical representations are Stone duals to each other.

We first introduce the category **DPL** of domains prelocales and approximable relations between them. This category is a main pillar for Abramsky's work.

Proofs of results presented in this section can be found in [2, Chapters 3,4].

Definition 2.5.1. A *domain prelocale* is a structure $\langle A, \lesssim, \approx, \vee, \wedge, 0, 1, C, T \rangle$ such that,

1. A is a countable set equipped with the preorder $\lesssim$. 0 and 1 are least and greatest elements in A, respectively. $\approx$ is the equivalence relation given by the intersection of $\lesssim$ and $\gtrsim$.

2. $\vee$ and $\wedge$ are binary operations on A producing a join and a meet, respectively.

3. C is a unary predicate on A such that

- $(\forall a \in A)\ C(a) \overset{\text{def}}{\Longleftrightarrow} a$ is $\vee$-prime.
- $C(1)$.

$C(A)$ is the set of all $\vee$-prime elements in A.

4. $(\forall a \in A)(\exists M \subseteq_{fin} C(A))\ a \approx \bigvee M$.

5. $(\forall U \subseteq_{fin} C(A))(\exists V \subseteq_{fin} C(A))\ U \subseteq V$ and $(\forall W \subseteq U)(\exists Z \subseteq V)\ \bigwedge W \approx \bigvee Z$.

6. T is a unary predicate such that for every $M \subseteq_{fin} A$,
   - $(\forall a, b \in A)\ T(a), b \lesssim a \Longrightarrow T(b)$.
   - $((\forall m \in M)\ T(m)) \Longrightarrow T(\bigvee M)$.
   - $(\forall a \in A)\ T(a) \Longleftrightarrow a \not\approx 1$.

Definition 2.5.2. Let A and B be domain prelocales. A binary relation $R \subseteq A \times B$ is *approximable* if for every $a, a' \in A$, $b, b' \in B$, $\{a_1, \ldots, a_n\} \subseteq A$ and $\{b_1, \ldots, b_m\} \subseteq B$,

1. $((\forall 1 \leq i \leq m)\ a\,R\,b_i) \Longrightarrow a\ R\ \bigwedge_{1 \leq i \leq m} b_i$.

2. $((\forall 1 \leq i \leq n)\ a_i\,R\,b) \Longrightarrow \bigvee_{1 \leq i \leq n} a_i\ R\,b$.

3. $a \lesssim a'\ R\ b' \lesssim b \Longrightarrow a\,R\,b$.

4. $C_A(a), a\ R\ \bigvee_{1 \leq i \leq m} b_i \Longrightarrow (\exists j)\ a\,R\,b_j$.

One of the main results, proved by Abramsky in [2], is that the category **B** of bifinite domains and strict Scott-continuous maps between them is equivalent to the category **DPL**. The details of this equivalence are given by the following functors. The functor $F : \mathbf{B} \longrightarrow \mathbf{DPL}$ sends a bifinite domain P to the domain prelocale

$$\langle K\Omega(P); \subseteq, =, \cup, \cap, \emptyset, P, \{\uparrow a \mid a \in K(P)\}, K\Omega(P) \setminus \{\uparrow 0_P\}\rangle,$$

and a strict Scott-continuous function $f : P \longrightarrow Q$ between bifinite domains to the approximable relation $R_f \subseteq K\Omega(P) \times K\Omega(Q)$ defined by:

$$a \, R_f \, b \overset{\text{def}}{\Longleftrightarrow} a \subseteq f^{-1}(b).$$

The functor G : **DPL** $\longrightarrow$ **B** sends a domain prelocale A to the bifinite domain $spec(A)$ of prime filters $\mathit{filt}_p(A)$ ordered by inclusion and an approximable relation $R \subseteq A \times B$ to the strict Scott-continuous map f_R defined by:

$$f_R(I) = \{y \mid (\exists x \in I) \, x \, R \, y\}.$$

Definition 2.5.3. *Pre-isomorphisms* are onto, order-preserving, and order-reflecting maps between domain pre-locales. A domain prelocale A is a *localic representation* of a bifinite domain P if A is pre-isomorphic to $K\Omega(P)$.

Although the morphisms in **DPL** are relations, the isomorphisms in this category can be described via pre-isomorphisms as follows. Let $\varphi : A \longrightarrow B$ be a pre-isomorphism. Then the relation

$$a \, R_\phi \, b \overset{\text{def}}{\Longleftrightarrow} \varphi(a) \leq_B b$$

is an isomorphism in **DPL**.

Moreover, Stone's isomorphisms in the case of distributive lattices (Section 2.1), work as natural transformations between the functors F and G. These natural transformations are defined as follows:

$$\eta : I_{\mathbf{DPL}} \longrightarrow F \circ G,$$

where $\eta_A = R_{\varphi_A}$ and $\varphi_A : A \longrightarrow K\Omega(spec(A))$ is the pre-isomorphism defined as

$$\varphi_A(a) = \{I \in spec(A) \mid a \in I\}$$

and

$$\varepsilon : I_{\mathbf{B}} \longrightarrow G \circ F,$$

where $\varepsilon_x = \{U \in K\Omega(P) \mid x \in U\}$. This is, more or less, the reason why Stone duality was considered the appropriate mathematical framework for studying the relationship between denotational and axiomatic semantics.

Lemma 2.5.4. *Let A be a domain prelocale, P a bifinite domain, and $\varphi : A \longrightarrow K\Omega(P)$ a pre-isomorphism. Then the map*

$$\tau = \varepsilon_P^{-1} \circ G(R_\varphi)$$

is a domain isomorphism from $spec(A)$ to P. Moreover, a pre-isomorphism from A to $K\Omega(P)$ exists if and only if $spec(A)$ and P are isomorphic.

Remark 2.5.5. In Lemma 2.5.4, $G(\varphi) : spec(K\Omega(P)) \longrightarrow spec(A)$ sends a prime filter F to its inverse image $\varphi^{-1}(F)$.

Definition 2.5.6. Let A and B be domain prelocales. Then A is *sub-prelocale* from B (denoted by $A \trianglelefteq B$) if

1. A is a subalgebra of B with respect to $\vee, \wedge, 0$ and 1.
2. $\lesssim_A = A \cap \lesssim_B$ and $C(A) = A \cap C(B)$.

Lemma 2.5.7. *Let A and B be domain prelocales and $A \trianglelefteq B$. Then the following mappings are embeddings:*

$$e : spec(A) \longrightarrow spec(B); F \longmapsto \uparrow_B F,$$

and

$$p : spec(B) \longrightarrow spec(A); F \longmapsto F \cap A.$$

One of Abramsky's basic goals in [2], was to have domains constructions done on the localic side. This means the following: suppose we are given two bifinite domains P_1 and P_2 which have localic representations A_1 and A_2 via pre-isomorphisms ϕ_1 and ϕ_2, respectively. Further suppose that F is a binary domain-construction; hence $F(P_1, P_2)$ denotes the domain resulting from applying the construction F to domains P_1 and P_2. The goal now is to construct, using only A_1 and A_2, a domain prelocale $T(A_1, A_2)$ and a pre-isomorphism $\phi : T(A_1, A_2) \longrightarrow K\Omega(F(P_1, P_2))$. Abramsky outlined a schema for establishing various domain constructions in the localic form. This schema guarantees that the construction $T(A_1, A_2)$ satisfies natural and desired properties like the following. If B_1 and B_2 are sub-prelocales of A_1 and A_2, respectively, then $T(B_1, B_2)$ is a sub-prelocale of $T(A_1, A_2)$. Moreover, the following diagram commutes where I, e and e' are the embedding mappings resulting from Lemma 2.5.7.

$$\begin{array}{ccc} spec(T(B_1, B_2)) & \xrightarrow{\ I\ } & spec(T(A_1, A_2)) \\ \downarrow{\scriptstyle \tau_B} & & \downarrow{\scriptstyle \tau_A} \\ F(Q_1, Q_2) & \xrightarrow{\ F(e, e')\ } & F(P_1, P_2) \end{array}$$

As an example, we explain the function-space construction. Suppose P and Q are bifinite domains which have localic representations A and B via pre-isomorphisms ϕ_1 and ϕ_2, respectively. The function space (the pre-locale) $A \longrightarrow B$ is constructed as follows. The carrier set of the prelocale $A \longrightarrow B$ is generated by the following set:

$$G = \{(a \longrightarrow b) \mid a \in A \text{ and } b \in B\}.$$

The relations $\lesssim_{A \longrightarrow B}$ and $\approx_{A \longrightarrow B}$ and the predicates $C_{A \longrightarrow B}$ and $T_{A \longrightarrow B}$ are defined inductively as follows, where $a, a_1, \ldots, a_n \in A$, $b, b_1, \ldots, b_n \in B$:

Axioms:

1. $(a \longrightarrow \bigwedge_{1 \leq i \leq n} b_i) \approx \bigwedge_{1 \leq i \leq n} (a \longrightarrow b_i)$.
2. $(\bigvee_{1 \leq i \leq n} a_i \longrightarrow b) \approx \bigwedge_{1 \leq i \leq n} (a_i \longrightarrow b)$.
3. The prelocale $A \longrightarrow B$ is distributive.

Rules:

1. If $C(a)$ then $(a \longrightarrow \bigvee_{1\leq i\leq n} b_i) \approx \bigvee_{1\leq i\leq n}(a \longrightarrow b_i)$.

2. If $a' \lesssim a$ and $b \lesssim b'$ then $(a \longrightarrow b) \lesssim (a' \longrightarrow b')$.

3. If $(\forall i \in I)\ C(a_i), C(b_i)$, and $(\forall K \subseteq \{1,\ldots,n\})(\exists L \subseteq \{1,\ldots,n\})$ such that $\bigwedge_{k\in K} a_k \approx \bigvee_{l\in L} a_l$ and $(\forall k \in K, l \in L)\ b_k \lesssim b_l$ then $C(\bigwedge_{i\in I}(a_i \longrightarrow b_i))$.

4. If $C(a'), a' \lesssim a$ and $T(b)$ then $T(a \longrightarrow b)$.

The required pre-isomorphism from $(A \longrightarrow B)$ to $K\Omega(P \longrightarrow Q)$ is defined as follows:

$$(a \longrightarrow b) \longmapsto \{f : P \longrightarrow Q \mid f \text{ is Scott-continuous and } f(\phi_1(a)) \subseteq \phi_2(b)\}.$$

Bifinite domains were then given by Abramsky in [2, Chapter 4] a meta-language and a logical interpretation which was established via the localic description illustrated above. The logical interpretations were proved to be Stone duals of the to corresponding bifinite domains.

The meta-language has the following grammar:

$$\sigma \quad ::= \quad 1 \mid X \mid (\sigma{\rightarrow}\sigma) \mid (\sigma{\times}\sigma) \mid (\sigma{\oplus}\sigma) \mid (\sigma)_\perp \mid \mathrm{P}^p(\sigma) \mid \mathrm{rec}X.\sigma,$$

where X belongs to a collection TV of type variables.

Suppose each variable in TV is assigned a bifinite domain via the map (environment) $\rho_P : TV \longrightarrow \mathbf{B}$. Then the semantic clauses of the grammar above are as follows:

$$\begin{aligned}
\mathfrak{I}_P(1;\rho_P) &= \text{the one element dcpo;}\\
\mathfrak{I}_P(X;\rho_P) &= \rho_P(X);\\
\mathfrak{I}_P((\sigma \longrightarrow \tau);\rho_P) &= (\mathfrak{I}_P(\sigma;\rho_P) \longrightarrow \mathfrak{I}_P(\tau;\rho_P));\\
&\vdots\\
\mathfrak{I}_P(\mathrm{rec}X.\sigma;\rho_P) &= \mathrm{FIX}(F_T),\ \text{where } F_T(Q) = \mathfrak{I}_P(\sigma;\rho_P[X \longmapsto Q]).
\end{aligned}$$

In the last clause, FIX(F_T) is the solution to the domain equation FIX$(F_T) = F_T$ and $\rho_P[X \longmapsto Q]$ is the function (environment) from TV to **B** that maps X to Q and any other variable $Y \in TV$ to $\rho_P(Y)$.

Every type expression σ is given a language $\mathfrak{L}(\sigma)$ of (computational or observational) *properties* via the following inductive definition:

$$
\begin{array}{rcl}
 & \Longrightarrow & \text{true}, \text{false} \in \mathfrak{L}(\sigma); \\
\phi, \psi \in \mathfrak{L}(\sigma) & \Longrightarrow & \phi \wedge \psi, \phi \vee \psi \in \mathfrak{L}(\sigma); \\
\phi \in \mathfrak{L}(\sigma), \psi \in \mathfrak{L}(\tau) & \Longrightarrow & (\phi \longrightarrow \psi) \in \mathfrak{L}(\sigma \longrightarrow \tau), \\
\phi \in \mathfrak{L}(\sigma), \psi \in \mathfrak{L}(\tau) & \Longrightarrow & (\phi \times \psi) \in \mathfrak{L}(\sigma \times \tau); \\
\phi \in \mathfrak{L}(\sigma) & \Longrightarrow & (\phi \oplus \text{false}) \in \mathfrak{L}(\sigma \oplus \tau); \\
\psi \in \mathfrak{L}(\tau) & \Longrightarrow & (\text{false} \oplus \psi) \in \mathfrak{L}(\sigma \oplus \tau); \\
\phi \in \mathfrak{L}(\sigma) & \Longrightarrow & (\phi)_\perp \in \mathfrak{L}((\sigma)_\perp); \\
\phi \in \mathfrak{L}(\sigma) & \Longrightarrow & \Box\phi, \Diamond\phi \in \mathfrak{L}(\mathrm{P}^p(\sigma)); \\
\phi \in \mathfrak{L}(\sigma[\text{rec}X.\sigma/X]) & \Longrightarrow & \phi \in \mathfrak{L}(\sigma).
\end{array}
$$

There are many applications to Abramsky's domain theory in logical form; for example [40, 39, 80, 32, 14, 3, 1].

2.6 Stably Compact Spaces

Stably compact spaces are the primary objects of interest in this thesis and they have been researched extensively [35, 34, 54, 62, 108, 49, 7]. One reason why stably compact spaces are interesting for us is that they simply cover compact coherent domains in their Scott topologies. Topologically, stably compact spaces are seen as a generalisation of compact Hausdorff spaces in the T_0 setting. This is a reason why mathematicians are interested in studying stably compact spaces. As a reminder, stably compact spaces are defined as follows:

Definition 2.6.1. A *stably compact space* is a topological space that is sober, compact, locally compact and such that the collection of compact saturated subsets is closed under binary intersection.

Stably compact spaces can be defined alternatively as follows.

Definition 2.6.2. A stably compact space is a topological space which is T_0, compact, locally compact, well-filtered, and the collection of compact saturated subsets is closed under binary intersections. A topological space is *well-filtered* if for every filter base $\{A_i \mid i \in I\}$ of compact saturated sets and an open set O:

$$\bigcap_i A_i \subseteq O \Longrightarrow (\exists j \in I)\ A_j \subseteq O.$$

The mathematical reason behind the equivalence of these apparently different definitions of stably compact spaces is the following theorem proved in [28, Theorem II-1.21].

Theorem 2.6.3. *For a locally compact T_0 space X, the following statements are equivalent:*

1. *X is sober.*
2. *X is well-filtered.*
3. *For a Scott-open filter of open sets $\mathcal{F}$ and open set $\mathcal{O}$,*

$$\mathcal{O} \in \mathcal{F} \Longleftrightarrow \bigcap \mathcal{F} \subseteq \mathcal{O}.$$

We have seen in Section 2.4 that the Stone duals of stably compact spaces are the stably continuous frames.

In this section, we review some properties and results concerning stably compact spaces. This section is based on [28, 49, 7, 53]. Let us start with the famous Hofmann-Mislove theorem [59, 29].

Theorem 2.6.4. *For every sober space, there is a one-to-one correspondence between the set of Scott-open filters of open sets and the set of compact saturated sets.*

This one-to-one correspondence works as follows. The intersection of the elements of any Scott-open filter of open sets is compact saturated. And the set of all open neighbourhoods of a compact saturated set is a Scott-open filter.

As a corollary of the Hofmann-Mislove theorem, we get the following result.

Corollary 2.6.5. *Let $\langle X, \mathcal{T} \rangle$ be a sober space and $\{K_i \mid i \in I\}$ be a filtered family of nonempty compact saturated subsets of X. Then $\bigcap_{i \in I} K_i$ is nonempty compact saturated and for every $O \in \mathcal{T}$,*

$$\bigcap_{i \in I} K_i \subseteq O \iff (\exists i \in I)\ K_i \subseteq O.$$

This corollary makes it clear that compact saturated subsets of stably compact spaces (which are T_0 spaces) play a parallel role to that played by compact subsets of Hausdorff (T_2) spaces. Indeed, a stably compact space $\langle X, \mathcal{T} \rangle$ is Hausdorff if and only if $\mathcal{T} = \mathcal{T}_c$ where $\mathcal{T}_c$ is the topology (known as the *co-compact topology*) on X which has as a basis the set of all complements of compact saturated subsets of X.

Proofs of results presented in the rest of this section can be found in [53].

For a stably compact space $\langle X, \mathcal{T} \rangle$, we already know that $\mathcal{T}$ is an arithmetic lattice whose order is the inclusion of sets. What is new is that the set of compact saturated subsets of X denoted by $\mathcal{K}_X$ and ordered by the reversed inclusion of sets is also an arithmetic lattice. The order of approximation $\ll$ can be characterised in $\mathcal{T}$ and $\mathcal{K}_X$ as follows:

Theorem 2.6.6. *For a stably compact space X,*

- *$\mathcal{T}$ and $\mathcal{K}_X$ are arithmetic lattices.*
- $(\forall O_1, O_2 \in \mathcal{T})\ O_1 \ll O_2 \iff (\exists K \in \mathcal{K}_X)\ O_1 \subseteq K \subseteq O_2.$
- $(\forall K_1, K_2 \in \mathcal{K}_X)\ K_1 \ll K_2 \iff (\exists O \in \mathcal{T})\ K_2 \subseteq O \subseteq K_1.$

Now we consider the relationship between stably compact spaces and their co-compact topologies. This results is crucial for the next subsection.

Lemma 2.6.7. *For a stably compact space* $\langle X, \mathcal{T} \rangle$*,*

- $\mathcal{T}_c$ *is the collection of complements of compact saturated subsets of* $\langle X, \mathcal{T} \rangle$*.*
- $\mathcal{T}$ *is the collection of complements of compact saturated subsets of* $\langle X, \mathcal{T}_c \rangle$*.*
- $\langle X, \mathcal{T}_c \rangle$ *is stably compact and its co-compact topology is* $\langle X, \mathcal{T} \rangle$*.*
- *The specialisation order of* $\langle X, \mathcal{T}_c \rangle$ *is the dual of the specialisation order of* $\langle X, \mathcal{T} \rangle$*.*
- *The specialisation order of* X *is closed in* $\langle X, T \rangle \times \langle X, \mathcal{T}_c \rangle$*.*

2.6.1 Compact Ordered Spaces

There is a one-to-one correspondence between stably compact spaces and compact ordered spaces [79]. The correspondence sends a compact ordered space $\langle X, \mathcal{T}, \leq \rangle$ to the space X equipped with the *upwards topology* $\langle X, \mathcal{T}^{\uparrow} \rangle$ where

$$\mathcal{T}^{\uparrow} = \{ \mathcal{O} \in \mathcal{T} \mid \uparrow\mathcal{O} = \mathcal{O} \},$$

and it sends a stably compact space $\langle Y, \mathcal{G} \rangle$ to $\langle Y, \mathcal{G}_p, \leq_{\mathcal{G}} \rangle$ where $\mathcal{G}_p$ is the *patch topology* on Y generated by the following subbasis:

$$\mathcal{G} \bigcup \{ Y \setminus K \mid K \text{ is compact saturated subset of } Y \},$$

and $\leq_{\mathcal{G}}$ is the specialisation order of the topology $\mathcal{G}$.

Proofs of results presented in this section can be found in [7].

Definition 2.6.8. A map f from a stably compact space Y to a stably compact space Y' is *perfect* if it is continuous and the compact saturated subsets of Y' are preserved under the pre-image f^{-1}.

On the morphism level, there is a one-to-one correspondence between continuous order-preserving maps between compact ordered spaces and perfect maps, between stably compact spaces. This correspondence is detailed as follows.

Lemma 2.6.9. *A map f from a stably compact space Y to a stably compact space Y' is perfect if and only if*

1. *the map f is continuous with respect to patch topologies on Y and Y', and*
2. *the map f is order-preserving with respect to specialisation orders of Y and Y'.*

Interestingly, stably compact spaces are closed under arbitrary products and the product topology can be obtained as the upwards topology of the product of corresponding compact ordered spaces.

2.6.2 Upper Relations

Definition 2.6.10. Let $\langle Y_1, \mathcal{T}_1\rangle$ and $\langle Y_2, \mathcal{T}_2\rangle$ be topological spaces. Then a binary relation $R \subseteq Y_1 \times Y_2$ is said to be *upper* if it is closed in the product topology $\langle Y_1 \times Y_2, \mathcal{T}_1 \times \mathcal{T}_2\rangle$.

In [53], upper relations between stably compact spaces were studied as morphisms for the category of stably compact spaces. They were seen as an extension to continuous maps and were shown to be in one-to-one correspondence with *pre-frame* maps (maps that preserve finite meets and directed joins) [8] on the localic side of Stone duality. [1]

The motivation for studying these relations is that experiments show that in order to generalise Abramsky's logical form for bifinite domains [2] to continuous domains one needs to consider a purer logic than Abramsky's. This turns out to be translated into considering upper relations between stably compact spaces rather than continuous maps. This need for relations appeared already in denotational semantics like in [12, 16].

Upper relations between stably compact spaces can be treated as functions as follows.

[1] The notion of upper relation is the same as that of closed relation in [53].

Proofs of results presented in this section can be found in [53].

Proposition 2.6.11. *Let R be an upper relation from a stably compact space X to another one Y. Suppose that $\mathcal{K}_Y$ is equipped with the Scoot topology. Then*

$$f_R : X \longrightarrow \mathcal{K}_Y; x \longmapsto \{y \in Y \mid x\ R\ y\},$$

is a continuous map. If $f : X \longrightarrow \mathcal{K}_Y$ is a continuous map then

$$R_f = \{\langle x, y\rangle \in X \times Y \mid y \in f(x)\}$$

is an upper relation from X to Y. Moreover, the maps $f \longmapsto R_f$ and $R \longmapsto f_R$ are inverse to each other.

However, the upper relations that correspond to functions can be characterised as follows.

Proposition 2.6.12. *Let R be an upper relation from a stably compact space X to another one Y such that for every $x \in X$, $f_R(x)$ has a least element denoted by $r(x)$. Then*

$$r : X \longrightarrow Y; x \longmapsto r(x),$$

is a continuous map. If $f : X \longrightarrow Y$ is a continuous map then

$$\{\langle x, y\rangle \in X \times Y \mid f(x) \leq y\}$$

is an upper relation from X to Y where the order on Y is the specialisation one. Moreover, the two maps presented above are inverses to each other.

Lemma 2.6.13. *Let X and Y be stably compact spaces and Y_c be the set underlying Y equipped with the co-compact topology. If $R \subseteq X \times Y_c$ is an upper relation then the map*

$$f_R : X \longrightarrow \mathcal{K}_Y; x \longmapsto \{y \in Y \mid x\ R\ y\}$$

is continuous with respect to the Scott-topology on $\mathcal{K}_Y$.

The details of the relationship between upper relations between stably compact spaces, and pre-frame maps between stably continuous frames are as follows.

Lemma 2.6.14. *Let* $R \subseteq X \times Y$ *be an upper relation between stably compact spaces* $\langle X, \mathcal{T}_X \rangle$ *and* $\langle Y, \mathcal{T}_Y \rangle$*. Then*

$$\Omega_R : \mathcal{T}_Y \longrightarrow \mathcal{T}_X; O \longmapsto \{x \in X \mid (\forall y \in Y)\; x\, R\, y \Longrightarrow y \in O\},$$

is a pre-frame morphism.

Lemma 2.6.15. *Let* $f : L_1 \longrightarrow L_2$ *be a pre-frame map between stably continuous frames* L_1 *and* L_2*. Then* $pt(f) \subseteq pt(L_2) \times pt(L_1)$ *where for completely prime filters* $F_1 \in pt(L_1)$ *and* $F_2 \in pt(L_2)$*,*

$$F_2\; pt(f)\; F_1 \overset{\text{def}}{\Longleftrightarrow} f^{-1}(F_2) \subseteq F_1,$$

is an upper relation.

Therefore we have the following equivalence of categories:

Theorem 2.6.16. *The category* **SCSc***, of stably compact spaces and upper relations between them, is dually equivalent to the category* **ALatp***, of stably continuous frames and pre-frame maps between them.*

2.6.3 Probabilistic Power Domain

Suppose that P is a program that is modelled by a continuous map $f : P_1 \longrightarrow P_2$ from a continuous domain P_1 to another one P_2 and that makes some decisions randomly. Then a function (called a (probability) valuation) from the Scott-topology on P_2 to $[0, \infty]$ is defined and used in modelling the behaviour of the program. The idea is that every Scott-open set O of P_2 is assigned a weight by the valuation function, which is supposed to measure the probability that the output of the program will belong to the open set O. This idea originated in [93, 61].

Proofs of results presented in this section can be found in [7].

The notion of (probability) valuation, appearing in mathematics in [83, 37, 13] and introduced to computer science in [103, 42], is defined as follows.

Definition 2.6.17. Let $\langle X, \mathcal{T}\rangle$ be a topological space. A function $\mu : \mathcal{T} \longrightarrow [0, \infty]$ is a *valuation* if

1. $\mu(\emptyset) = 0$,
2. $(\forall O_1, O_2 \in \mathcal{T})\ O_1 \subseteq O_2 \Longrightarrow \mu(O_1) \leq \mu(O_2)$, and
3. $(\forall O_1, O_2 \in \mathcal{T})\ \mu(O_1) + \mu(O_2) = \mu(O_1 \cup O_2) + \mu(O_1 \cap O_2)$.

μ is a *probability valuation* if additionally $\mu(X) = 1$. A valuation μ is *continuous* if for every directed family $\{O_i \mid O_i \in \mathcal{T} \text{ and } i \in I\}$,

$$\mu(\bigcup_{i\in I} O_i) = sup_{i\in I}\mu(O_i).$$

For many classes of topological spaces, continuous valuations can be extended to Borel measures, as in [66, 6, 5], which are defined as follows.

Definition 2.6.18. For a topological space $\langle X, \mathcal{T}\rangle$, the σ-algebra generated by $\mathcal{T}$ is the smallest subset of $\wp(X)$, the power set of X, that contains $\mathcal{T}$ and is closed under complementation and countable unions of its elements.

Definition 2.6.19. Let $\langle X, \mathcal{T}\rangle$ be a Hausdorff space and $\mathcal{B}(X)$ be the σ-algebra generated by elements of $\mathcal{T}$. A function $m : \mathcal{B}(X) \longrightarrow [0, \infty]$ is a *measure (or Borel measure)* if

1. $m(\emptyset) = 0$,
2. $(\forall A, B \in \mathcal{B}(X))\ A \cap B = \emptyset \Longrightarrow m(A) + m(B) = m(A \cup B)$, and
3. for every increasing sequence $\{A_n \mid n \in \mathbb{N}\} \subseteq \mathcal{B}(X)$,

$$m(\bigcup_{n\in\mathbb{N}} A_n) = sup_{n\in\mathbb{N}} m(A_n).$$

A measure m is a *Radon measure* if

1. it is *inner regular*; i.e for every $A \in \mathcal{B}(X)$

$$m(A) = sup\{m(K) \mid K \text{ is a compact subset of } A\}, and$$

2. for every compact subset of X, $m(K) \leq \infty$.

In [57, 49, 7], the focus is on studying valuations and measures on stably compact spaces; integration is used to study the effect of these valuations and measures, that are defined on stably compact spaces, on continuous maps.

In [7], the Riesz representation theorem was used to provide a direct proof for the fact (proved in [66]) that a valuation on a stably compact space X can be uniquely extended to a Radon measure on the σ-algebra generated by the open sets of the compact ordered space corresponding to X.

Let X be a compact Hausdorff space and $C(X)$ be the space of all continuous maps from X to the real line. A *linear functional* on $C(X)$ is a linear map from $C(X)$ to the real line. A linear functional ϕ on $C(X)$ is *positive* if $\phi(f) > 0$ for every positive function $f \in C(X)$. The Riesz representation theorem states that every positive linear functional ϕ on $C(X)$ there is a unique Radon measure m such that

$$(\forall f \in C(X))\ \phi(f) = \int_0^\infty m(\{x \in X \mid f(x) > r\})dr.$$

The other interesting fact that is proved in [7] is that a natural topology can be defined on the set of continuous valuations on a stably compact space to get another stably compact space. This can be achieved as follows.

Theorem 2.6.20. *Let $\langle X, \mathcal{T}\rangle$ be a stably compact space and let $\mathcal{B}_{\leq 1}(X)$ be the set of all continuous valuations μ, on X, in which $\mu(X) \leq 1$. Let $\mathcal{S}$ be the the weakest topology on $\mathcal{B}_{\leq 1}(X)$ such that for every bounded continuous (with respect to $\mathcal{T}$ the topology generated* *by open upper sets of* $[0,\infty]$*)* $g : X \longrightarrow [0,\infty]$*, the mapping*

$$\mu \longrightarrow \int_0^\infty \mu(\{x \in X \mid g(x) > r\})dr,$$

is continuous. Then $\langle \mathcal{B}_{\leq 1}(X), \mathcal{S}\rangle$ *is a stably compact space.*

The above theorem is still true if we replace $\mathcal{B}_{\leq 1}(X)$ with the set $\mathcal{B}_1(X)$ of all continuous valuations μ, on X, in which $\mu(X) = 1$.

2.7 Jung-Sünderhauf Representation Theorem

In this section, we review Jung-Sünderhauf representation theorem [54] which associates bounded distributive lattices with certain binary relations (strong proximity relations) to provide finitary representations for stably compact spaces. The resulting structure is defined as follows:

Definition 2.7.1. A binary relation $\prec$ on a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ is called a *strong proximity* if, for every $a, x, y \in L$ and $M \subseteq_{fin} L$,

$$
\begin{array}{rl}
(\prec\prec) & \prec \circ \prec = \prec, \\
(\vee\!-\prec) & M \prec a \Longleftrightarrow \bigvee M \prec a, \\
(\prec -\wedge) & a \prec M \Longleftrightarrow a \prec \bigwedge M, \\
(\prec -\vee) & a \prec x \vee y \Longrightarrow (\exists\, x', y' \in L)\; x' \prec x,\; y' \prec y \text{ and } a \prec x' \vee y', \\
(\wedge\!-\prec) & x \wedge y \prec a \Longrightarrow (\exists\, x', y' \in L)\; x \prec x',\; y \prec y' \text{ and } x' \wedge y' \prec a.
\end{array}
$$

$M \prec a$ and $a \prec M$, respectively, stand for $(\forall m \in M)\; m \prec a$ and $(\forall m \in M)\; a \prec m$. A *strong proximity lattice* is a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ together with a strong proximity relation $\prec$ on L.

The concept of strong proximity lattice has precursors in the literature in [108, 29, 107].

Semantically, for a stably compact space $\langle Y, \mathcal{T}\rangle$, we explained in Chapter 1 that its lattice of properties

$$\{\langle O, K\rangle \mid O \in \mathcal{T}, K \text{ is a compact saturated subset of } Y \text{ and } O \subseteq K\},$$

has special importance. The concept of strong proximity lattice is meant to capture these lattices in a purely algebraic form.

The continuous maps between stably compact spaces were given finite descriptions via approximable relations which are defined as follows.

Definition 2.7.2. Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be a binary relation from L_1 to L_2. The relation $\vdash$ is called *approximable* if for every $a \in L_1, b \in L_2$, $M_1 \subseteq_{fin} L_1$ and $M_2 \subseteq_{fin} L_2$,

$$\begin{aligned}
(\vdash - \prec_2) &\quad \vdash \circ \prec_2 = \vdash,\\
(\prec_1 - \vdash) &\quad \prec_1 \circ \vdash = \vdash,\\
(\vee - \vdash) &\quad M_1 \vdash b \Longleftrightarrow \bigvee M_1 \vdash b,\\
(\vdash - \wedge) &\quad a \vdash M_2 \Longleftrightarrow a \vdash \bigwedge M_2,\\
(\vdash - \vee) &\quad a \vdash \bigvee M_2 \Longrightarrow (\exists\, N \subseteq_{fin} L_1)\ a \prec_1 \bigvee N \text{ and } (\forall n \in N)\\
&\quad (\exists\, m \in M_2)\ n \vdash m.
\end{aligned}$$

The relation $\vdash$ is called *weakly approximable* if it satisfies all of the above conditions but not necessarily $(\vdash - \vee)$.

The semantics interpretation of the elements (pairs $\langle O, K\rangle$) of property-lattices is that each pair represents a property satisfied by members of O, unsatisfied by elements in the complement of K and undecidable for elements in $K \setminus O$. Given a continuous map (computable program) $f : Y_1 \longrightarrow Y_2$ between stably compact spaces, a property $\langle O, K\rangle$ of Y_1 implies a property $\langle O', K'\rangle$ of Y_2 if and only if the output under f, of any input that satisfies or is undecided for the property $\langle O, K\rangle$, satisfies the property $\langle O', K'\rangle$ or equivalently if and only if $f(K) \subseteq O'$. The approximable relations are meant to be a mathematically abstracted form of binary relations between property-lattices where two properties are related in these relations if and only if the first property implies the second one.

The basic output of [54] is the equivalence between the category **PLa** of strong proximity lattices and approximable relations, and the category **SCS** of stably compact spaces and continuous maps between them. This equivalence is proved by making use of the already established duality, reviewed in Section 2.4 and in [28], between the latter category and the category of stably continuous frames and frame homomorphisms between them.

2.7.1 From Strong Proximity Lattices to Stably Compact Spaces

Definition 2.7.3. Let $\langle L, \vee, \wedge, 0, 1, \prec \rangle$ be a strong proximity lattice and I be a non-empty subset of L. The set I is a *round ideal* if it is closed under binary suprema and lower with respect to the strong proximity relation i.e.

$$(\forall x \in L)\; x \in I \iff (\exists y \in I)\; x \prec y.$$

Round filters of L are defined dually. The set of all (prime) round ideals of L is denoted by $(ridl_p(L))$ $ridl(L)$. Dually $(rfilt_p(L))$ $rfilt(L)$ denotes the set of all (prime) round filters of L.

The following theorem presents some interesting results about round ideals of strong proximity lattices.

Theorem 2.7.4. *For a strong proximity lattice* $\langle L, \vee, \wedge, 0, 1, \prec \rangle$*, the ordered set* $\langle ridl(L), \subseteq \rangle$ *is a stably continuous frame in which*

- *the meet of a subset* $\{I_j \mid j \in J\}$ *is* $\{x \in L \mid (\exists y \in \bigcap_j I_j)\; x \prec y\}$ *and finite meets are the intersections,*
- *the join of a subset* $\{I_j \mid j \in J\}$ *is* $\{x \in L \mid (\exists M \subseteq_{fin} \bigcup_j I_j)\; x \prec \bigvee M\}$ *and directed joins are the unions, and*

- $I \ll J \iff (\exists x \in L)\ I \subseteq \{y \in L \mid y \prec x\} \subseteq J$.

The following corollary follows from the duality between stably continuous frames and stably compact space.

Corollary 2.7.5. *For a strong proximity lattice* $\langle L, \vee, \wedge, 0, 1, \prec\rangle$*, the point topology on* $pt(ridl(L))$ *is a stably compact space whose set of open sets is isomorphic to* $ridl(L)$*.*

The corollary explains how we get stably compact spaces from strong proximity lattices. However, this way of obtaining stably compact spaces can be smoothed as follows.

We start off with a strong proximity lattice $\langle L, \vee, \wedge, 0, 1, \prec\rangle$ and then equip the space $rfilt_p(L)$ (known as *spectrum* of L and also denoted by $spec(L)$) with the topology (known as *canonical topology*) generated by the sets of the following form:

$$\mathcal{O}_a = \{F \in spec(L) \mid a \in F\},\ a \in L.$$

Then this topological space is homeomorphic to $pt(ridl(L))$ equipped with the point topology. This is proved in [54, Theorem 21]. This justification makes it clear that the lattice $ridl(L)$ is isomorphic to the lattice of open subsets of the canonical topology on $spec(L)$.

Next we present an interesting correspondence between round filters of strong proximity lattices and ordered sets of compact saturated subsets of stably compact spaces ordered by superset inclusion.

Lemma 2.7.6. *For a strong proximity lattice* $\langle L; \vee, \wedge, 0, 1, \prec\rangle$*, the ordered set of of Scott-open filters of* $ridl(L)$ *(denoted by* $Sfilt(L)$*) is isomorphic to the lattice* $rfilt(L)$ *via the following mappings:*

$$\phi : Sfilt(L) \longrightarrow rfilt(L); \mathcal{F} \longmapsto \{x \in L \mid \{y \in L \mid y \prec x\} \in \mathcal{F}\},$$

and

$$\psi : rfilt(L) \longrightarrow Sfilt(L); F \longmapsto \{I \in ridl(L) \mid I \cap F \neq \emptyset\}.$$

The following corollary follows from the one-to-one correspondence between the lattice of compact saturated subsets of $pt(ridl(L))$, on the stably compact spaces side, and the lattice of Scott-open filters of $ridl(L)$, on the stably continuous frames side. This correspondence is an entry in Table 2.2 of Section 2.4. This particular one-to-one correspondence follows from Hofmann-Mislove Theorem [4, 59, 36] which states that, in a sober space X, the set of Scott-open filters of the open-sets lattice $\Omega(X)$ is isomorphic to the set of compact saturated subset of X.

Corollary 2.7.7. *For a strong proximity lattice $\langle L; \vee, \wedge, 0, 1, \prec\rangle$, the ordered set of compact saturated subsets $\langle \mathcal{K}_{pt(ridl(L))}, \supseteq\rangle$ is isomorphic to the ordered set $\langle rfilt(L), \subseteq\rangle$.*

2.7.2 From Stably Compact Spaces to Strong Proximity Lattices

Definition 2.7.8. For a stably compact space $\langle Y, \mathcal{T}\rangle$, let $\mathcal{K}_Y$ denote the set of compact saturated subsets of Y.

Theorem 2.7.9. *For a stably compact space $\langle Y, \mathcal{T}\rangle$, the algebra $\langle \mathcal{B}_Y; \vee, \wedge, 0, 1; \prec\rangle$, where*

1. $\mathcal{B}_Y = \{\langle O, K\rangle \in \mathcal{T} \times \mathcal{K}_Y \mid O \subseteq K\}$.

2. $\langle O, K\rangle \vee \langle O', K'\rangle = \langle O \cup O', K \cup K'\rangle$.

3. $\langle O, K\rangle \wedge \langle O', K'\rangle = \langle O \cap O', K \cap K'\rangle$.

4. $0 = \langle \emptyset, \emptyset\rangle$ *and* $1 = \langle Y, Y\rangle$.

5. $\langle O, K\rangle \prec \langle O', K'\rangle \overset{\text{def}}{\Longleftrightarrow} K \subseteq O'$.

is a strong proximity lattice. Moreover, the topological space $\langle Y, \mathcal{T}\rangle$ is homeomorphic to the canonical topology on the space $spec(\mathcal{B})$. Therefore the spectra of strong proximity lattices are exactly the stably compact space.

2.7.3 Morphisms

Lemma 2.7.10. • *Given an approximable relation $R \subseteq L_1 \times L_2$,*

$$f_R : ridl(L_2) \longrightarrow ridl(L_1); I \longmapsto \{x \in L_1 \mid (\exists y \in I)\; x\; R\; y\},$$

is a frame homomorphism.

- *For a frame homomorphism $f : ridl(L_2) \longrightarrow ridl(L_1)$, where L_1 and L_2 are strong proximity lattice, the binary relation $R_f \subseteq L_1 \times L_2$ defined as :*

$$a\; R_f b \overset{\text{def}}{\Longleftrightarrow} a \in f(\{x \in L \mid x \prec b\}),$$

is an approximable relation.

Moreover, the maps $f \longmapsto R_f$ and $R \longmapsto f_R$ are inverses of each other.

Bearing in mind the one-to-one correspondence between frame homomorphisms and continuous maps between stably compact spaces, the one-to-one correspondence between the latter and approximable relations follows from the lemma above.

2.8 Multi Lingual Sequent Calculus

In this section, we review the category **MLS** (Multi Lingual Sequent) [50, 51, 52, 77] which was presented to provide logical descriptions for stably compact spaces in a way that generalises Abramsky's domain theory in a logical form for bifinite domains.

The elements of this category are defined as follows.

Definition 2.8.1. 1. Let $\langle A; \vee, \wedge, \top, \bot\rangle$ and $\langle B; \vee, \wedge, \top, \bot\rangle$ be two algebras of type $\langle 2, 2, 0, 0\rangle$. A binary relation $\vdash$ from finite subsets of A to those of B is a *consequence relation* if for every $\phi, \psi \in A, \Gamma, \Gamma' \subseteq_{fin} A, \phi', \psi' \in B$ and $\Delta, \Delta' \subseteq_{fin} B$,

$(L\bot)$ $(\forall\, \Theta \subseteq_{fin} B)\; \bot \vdash \Theta.$

$(L\top)$ $\Gamma \vdash \Delta \Longleftrightarrow \top, \Gamma \vdash \Delta.$

$(L\wedge)$ $\phi, \psi, \Gamma \vdash \Delta \Longleftrightarrow \phi \wedge \psi, \Gamma \vdash \Delta.$

$(L\vee)$ $\phi, \Gamma \vdash \Delta$ and $\psi, \Gamma \vdash \Delta \Longleftrightarrow \phi \vee \psi, \Gamma \vdash \Delta.$

$(R\bot)$ $\Gamma \vdash \Delta \Longleftrightarrow \Gamma \vdash \Delta, \bot.$

$(R\top)$ $(\forall\, \Theta \subseteq_{fin} A)\; \Theta \vdash \top.$

$(R\wedge)$ $\Gamma \vdash \Delta, \phi'$ and $\Gamma \vdash \Delta, \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \wedge \psi'.$

$(R\vee)$ $\Gamma \vdash \Delta, \phi', \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \vee \psi'.$

(W) $\Gamma \vdash \Delta \Longrightarrow \Gamma', \Gamma \vdash \Delta, \Delta'.$

2. A consequence relation $\Vdash$ on an algebra $\langle A; \vee, \wedge, \bot, \top\rangle$ is *closed under* (Cut) if

$$(Cut) \qquad \Gamma \Vdash \Delta, \phi \text{ and } \phi, \Theta \Vdash \Lambda \Longrightarrow \Delta, \Theta \Vdash \Delta, \Lambda.$$

3. A consequence relation $\Vdash$ on an algebra $\langle A; \vee, \wedge, \bot, \top\rangle$ *has interpolants* if

$$(L-Int) \qquad \phi, \Gamma \Vdash \Delta \Longrightarrow (\exists \phi' \in A)\; \phi \Vdash \phi' \text{ and } \phi', \Gamma \Vdash \Delta.$$

$$(R-Int) \qquad \Gamma \Vdash \Delta, \phi \Longrightarrow (\exists \phi' \in A)\; \Gamma \Vdash \Delta, \phi' \text{ and } \phi' \Vdash \phi.$$

4. A *coherent sequent calculus* is an algebra $\langle A; \vee, \wedge, \top, \bot\rangle$ together with a consequence relation $\Vdash$ on A such that $\Vdash$ is closed under (Cut) and has interpolants.

5. A consequence relation $\vdash$ from a coherent sequent calculus $\langle A; \vee, \wedge, \top, \bot, \Vdash_A\rangle$ to a coherent sequent calculus $\langle B; \vee, \wedge, \top, \bot, \Vdash_B\rangle$ is *compatible* if

$$\Vdash_A \dagger \vdash \,=\, \vdash \,=\, \vdash \dagger \Vdash_B,$$

where $\dagger$ is the composition in **MLS** which is defined as follows,

$$\frac{\begin{array}{cc} \Gamma \vdash \Delta_1 & \Theta_1 \vdash' \Lambda \\ \vdots & \vdots \\ \Gamma \vdash \Delta_n & \Theta_m \vdash' \Lambda \end{array}}{\Gamma(\vdash \dagger \vdash')\Lambda}\;(Cut-Comp)$$

where for every choice functions $f \in \prod_i \Delta_i$ there exists $j \in \{1, \ldots, m\}$ such that $\Theta_j \subseteq \{f_1 \ldots, f_n\}$.

Coherent sequent calculi and compatible consequence relations are, respectively, the objects and morphisms of the category **MLS**. The composition in **MLS** is $(Cut - Comp)$.

Compatible consequence relations can be characterised as follows.

Lemma 2.8.2. *Let $\vdash$ be a consequence relation from a coherent sequent calculus A to another one B. Then $\vdash$ is compatible if and only if*

$$
\begin{array}{ll}
(L - Int') & \phi, \Gamma \vdash \Delta \Longrightarrow (\exists \phi' \in A)\ \phi \Vdash_A \phi' \textit{ and } \phi', \Gamma \vdash \Delta; \\
(R - Int') & \Gamma \vdash \Delta, \phi \Longrightarrow (\exists \phi' \in B)\ \Gamma \vdash \Delta, \phi' \textit{ and } \phi' \Vdash_B \phi; \\
(L - Cut) & \Gamma \Vdash_A \phi \textit{ and } \phi, \Theta \vdash \Lambda \Longrightarrow \Gamma, \Theta \vdash \Lambda; and \\
(R - Cut) & \Gamma \vdash \Delta, \phi \textit{ and } \phi \Vdash_B \Lambda \Longrightarrow \Gamma \vdash \Delta, \Lambda.
\end{array}
$$

The composition rules (Cut) and $(Cut - Comp)$ are related as follows.

Lemma 2.8.3. *Let $\langle A; \vee, \wedge, \top, \bot, \Vdash \rangle$ be a coherent sequent calculus such that $\Vdash$ has interpolants. Then*

$$\Vdash \dagger \Vdash = \Vdash \Longleftrightarrow \textit{ the relation } \Vdash \textit{ is closed under } (Cut).$$

Definition 2.8.4. Let A and B be coherent sequent calculi, $\vdash$ a compatible consequence relation from A to B, $X \subseteq A$ and $Y \subseteq B$. Then

1. $X[\vdash] = \{\phi \in B \mid (\exists \Gamma \subseteq_{fin} X)\ \Gamma \vdash \phi\}$.

2. $[\vdash]Y = \{\phi \in A \mid (\exists \Delta \subseteq_{fin} Y)\ \phi \vdash \Delta\}$.

Definition 2.8.5. Let A be a coherent sequent calculus, $\Vdash$ a compatible consequence relation on A and $I \subseteq A$. The set I is a *round ideal* of A if $I = [\Vdash]I$. *Round filters* are defined dually. The set of all round ideals (filters) of A is denoted by $ridl(A)$ ($rfilt(A)$). The set of all prime round ideals (filters) of A is denoted by $ridl_p(A)$ ($rfilt_p(A)$).

Remark 2.8.6. Let $\langle A; \vee, \wedge, \top, \bot, \Vdash \rangle$ be a coherent sequent calculus. For every $X \subseteq A$, $X[\vdash]$ is a round filter and $[\vdash]X$ is a round ideal.

Logically, round filters represent theories. The consistency of a theory in logic means that it is a proper subset of the whole language. Therefore a theory is consistent if and only if its representing round filter is contained in a prime round filter (model).

Suppose A and B are coherent sequent calculi and $\vdash$ is a compatible consequence relation from A to B. Suppose X is a set of formulas that are satisfied in A and Y is a set of formulas that are not satisfied in B. If $\vdash$ respects the latter information about satisfaction and dissatisfaction of formulas i.e it does not link any finite subset of X to any finite subset of Y then the pair X and Y is consistent with respect to $\vdash$ (or $\vdash-$*consistent*). Formally the definition is as follows.

Definition 2.8.7. Let $\langle A; \vee, \wedge, \top, \bot, \Vdash_A \rangle$ and $\langle B; \vee, \wedge, \top, \bot, \Vdash_B \rangle$ be coherent sequent calculi and $\vdash$ a compatible consequent relation from A to B. Let $X \subseteq A$ and $Y \subseteq B$. Then the pair $\langle X, Y \rangle$ is $\vdash-$*consistent* if for every $\Gamma \subseteq_{fin} X$ and $\Delta \subseteq_{fin} Y$, $\langle \Gamma, \Delta \rangle \not\in \vdash$.

The following lemma links consistency to round ideals and filters.

Lemma 2.8.8. *Let $\langle A; \vee, \wedge, \top, \bot, \Vdash_A \rangle$ and $\langle B; \vee, \wedge, \top, \bot, \Vdash_B \rangle$ be coherent sequent calculi and $\vdash$ a compatible consequent relation from A to B. Let $X \subseteq A$ and $Y \subseteq B$. Then the following statements are equivalent:*

1. *$\langle X, Y \rangle$ is $\vdash-$consistent.*
2. *$\langle X, [\Vdash_B]Y \rangle$ is $\vdash-$consistent.*
3. *$\langle X[\Vdash_A], Y \rangle$ is $\vdash-$consistent.*
4. *$\langle X[\vdash], Y \rangle$ is $\Vdash_B-$consistent.*
5. *$\langle X, [\vdash]Y \rangle$ is $\Vdash_A-$consistent.*

6. $X[\vdash] \cap [\Vdash_B]Y = \emptyset$.

7. $X[\Vdash_A] \cap [\vdash]Y = \emptyset$.

8. $(\exists\, I \in ridl_p(B))\ [\Vdash_B]Y \subseteq I$ *and* $\langle X, I\rangle$ *is* $\vdash$*−consistent.*

9. $(\exists\, I \in \mathit{filt}_p(A))\ X[\Vdash_A] \subseteq F$ *and* $\langle F, Y\rangle$ *is* $\vdash$*−consistent.*

2.8.1 MLS and Strong Proximity Lattices

Strong proximity lattices were the basis for establishing the category **MLS**. Therefore this category and the category **PLwa** of strong proximity lattices and weakly approximable relations between them are equivalent. The equivalent sends a strong proximity lattice $\langle L; \vee, \wedge, 0, 1, \prec\rangle$ to the coherent sequent calculus $\langle L; \vee, \wedge, 0, 1, \vdash_{\prec}\rangle$ where

$$\Gamma \vdash_{\prec} \Delta \stackrel{\text{def}}{\Longleftrightarrow} \bigwedge \Gamma \prec \bigvee \Delta,$$

and a weakly approximable relation relation G from a strong proximity lattice L to a strong proximity lattice K is sent to the compatible consequence relation $\vdash_G$ defined as:

$$\Gamma \vdash_G \Delta \stackrel{\text{def}}{\Longleftrightarrow} \bigwedge \Gamma\, G \bigvee \Delta.$$

The equivalence mentioned above sends a coherent sequent calculus $\langle A; \vee, \wedge, \top, \bot, \Vdash\rangle$ to $L/\equiv$ where $\equiv$ is the least congruence such that $L/\equiv$ is a bounded distributive lattice. The lattice $L/\equiv$ is equipped with the following strong proximity relation:

$$[\delta] \prec_{\Vdash} [\gamma] \stackrel{\text{def}}{\Longleftrightarrow} \delta \Vdash \gamma.$$

It also sends a compatible consequence relation $\vdash$ to the weakly approximable relation $G_{\vdash}$ defined by:

$$[\delta]\ G_{\vdash}\ [\gamma] \stackrel{\text{def}}{\Longleftrightarrow} \delta \vdash \gamma.$$

The relation $\equiv$ has the following interesting property: if $\gamma_1 \equiv \gamma'_1, \ldots, \gamma_n \equiv \gamma'_n$ and $\delta_1 \equiv \delta'_1, \ldots, \delta_m \equiv \delta'_m$ then

$$\gamma_1, \ldots, \gamma_n \Vdash \delta_1 \ldots \delta_m \Longleftrightarrow \gamma'_1, \ldots, \gamma'_n \Vdash \delta'_1 \ldots \delta'_m.$$

2.8.2 MLS and Stably Compact Spaces

The stably compact spaces and upper relations[2] of the form $R \subseteq X \times Y_c$, where X and Y are stably compact spaces and Y_c is the space Y topologised with the topology generated by the complements of compact saturated subsets of Y, is a category whose composition is the usual relation compositions. This category is equivalent to the category **MLS**. This equivalence is detailed as follows. If A is a coherent sequent calculus then the set $\mathit{spec}(A)$ of all prime round filters on A topologised with the topology generated by the sets of the form

$$\mathcal{O}_a = \{F \in \mathit{spec}(A) \mid \phi \in F\}, \phi \in A,$$

is a stably compact space.

Suppose $\vdash$ is a compatible consequence relation from A to B. Then the relation $R_\vdash \subseteq \mathit{spec}(A) \times \mathit{spec}(B)_c$ defined as

$$F \; R_\vdash \; F' \stackrel{\text{def}}{\Longleftrightarrow} F[\vdash] \subseteq F'$$

is upper.

If X is a stably compact space then the algebra $\langle A_X; \vee, \wedge, \bot, \top; \Vdash\rangle$, where

1. $A_X = \{\langle O, K\rangle \in \mathcal{T}_X \times \mathcal{K}_X \mid O \subseteq K\}$;

2. $\langle O, K\rangle \vee \langle O', K'\rangle = \langle O \cup O', K \cup K'\rangle$;

3. $\langle O, K\rangle \wedge \langle O', K'\rangle = \langle O \cap O', K \cap K'\rangle$;

[2]The notion of upper relation is the same as that of closed relation in [53].

4. $\bot = \langle \emptyset, \emptyset \rangle$ and $\top = \langle X, X \rangle$;

5. $\Gamma \Vdash_X \Delta \overset{\text{def}}{\iff} \bigcap\{K \mid (\exists O)(O,k) \in \Gamma\} \subseteq \bigcup\{O \mid (\exists K)(O,k) \in \Delta\}$,

is a coherent sequent calculus.

Let A and B be two coherent sequent calculi and $R \subseteq \mathit{spec}(A) \times \mathit{spec}(B)_c$ be an upper relation. Then $\Vdash_A \dagger \vdash_R \dagger \Vdash_B$, where $\vdash_R$ is the consequence relation from A to B defined as

$$\Gamma \vdash_R \Delta \overset{\text{def}}{\iff} ((\forall F \in \mathit{spec}(A)) \bigwedge \Gamma \in F \Longrightarrow \bigvee \Delta \in \bigcap\{G \in \mathit{spec}(B) \mid F \; R \; G\}),$$

is a compatible consequence relation from A to B.

Chapter 3

Strong Proximity Lattices in Priestley Form

3.1 Overview

This chapter extends the celebrated Priestley's representation theorem for bounded distributive lattices to represent the wider class of strong proximity lattices introduced in Section 2.7. We begin by studying the problem in the finite case, extending Birkhoff's representation theorem for finite distributive lattices to represent finite strong proximity lattices.

3.1.1 The Finite Case

Recall from section 2.2.1 that Birkhoff's representation theorem associates with a finite distributive lattice L the set $\mathcal{J}(L)$ of join-irreducible elements, ordered by the reverse $\leq^\partial$ of the inherited lattice order $\leq$. If the finite lattice is also equipped with a strong proximity (Definition 2.7.1), then our proposal is to equip $\mathcal{J}(L)$ with an apartness relation defined as follows:

Definition 3.1.1. A binary relation $\propto$ on an ordered set $\langle P, \leq \rangle$ is an *apartness* if, for every $a, c, d, e \in P$,

$$(\uparrow\propto\downarrow) \qquad a \geq c \propto d \geq e \Longrightarrow a \propto e,$$

$$(\propto\forall) \qquad a \propto c \Longleftrightarrow (\forall b \in P)\ a \propto b \text{ or } b \propto c,$$

$$(\propto\downarrow\downarrow) \qquad a \propto ({\downarrow}c \cap {\downarrow}d) \Longrightarrow (\forall b \in P)\ a \propto b,\ b \propto c \text{ or } b \propto d,$$

$$(\uparrow\uparrow\propto) \qquad ({\uparrow}c \cap {\uparrow}d) \propto a \Longrightarrow (\forall b \in P)\ d \propto b,\ c \propto b \text{ or } b \propto a.$$

where $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

Remark 3.1.2. 1. For any ordered set $\langle P, \leq \rangle$, $\not\leq$ is an apartness.

2. $\propto$ is an apartness on $\langle P; \leq \rangle$ if and only if $\propto^{-1}$ is an apartness on $\langle P; \geq \rangle$.

In Section 3.2, we will show that finite ordered sets with apartness represent finite strong proximity lattices. The translations between the two structures are as follows. The apartness on $\mathcal{J}(L)$ is defined by

$$x \propto_{\prec} y \overset{\text{def}}{\Longleftrightarrow} x \prec g(y),$$

where $g(y)$ is the meet-irreducible element corresponding to y (see Lemma 2.2.2 for detail of this correspondence). Vice versa, the dual of a finite ordered set $\langle P, \leq \rangle$ equipped with an apartness $\propto$ is its lattice of upper sets ordered by inclusion $\subseteq$ and equipped with the strong proximity,

$$A \prec_{\propto} B \overset{\text{def}}{\Longleftrightarrow} A \propto (X \backslash B).$$

We will further show that the action of Birkhoff duality on morphisms can also be adapted to the current setting as follows.

> *Order-preserving maps between finite ordered sets which reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms between finite strong proximity lattices that preserve the strong proximity relation.*

3.1.2 The General Case

In the general case we need to link the apartness relation with the topology of the Priestley space:

Definition 3.1.3. A binary relation $\propto$ on a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ is an *apartness* if

1. $\propto$ is open in $\langle X; \mathcal{T}\rangle \times \langle X; \mathcal{T}\rangle$.
2. $\propto$ is an apartness on the ordered set $\langle X, \leq\rangle$.

Remark 3.1.4.

1. For any Priestley space $\langle X; \leq, \mathcal{T}\rangle$, $\not\leq$ is an apartness (because the order is required to be closed for ordered spaces).
2. $\propto$ is an apartness on $\langle X; \leq, \mathcal{T}\rangle$ if and only if $\propto^{-1}$ is an apartness on $\langle X; \geq, \mathcal{T}\rangle$.
3. Intuitively, it is helpful to assume that an element can not be apart from itself but actually we mathematically do not need this assumption.
4. In case we axiomatise *indistinguishability* rather than apartness, that is $X^2 \backslash \propto$ rather than $\propto$, then $(\propto\forall)$ will express the transitivity of this relation. Axiom $(\propto\downarrow\downarrow)$, however, will not have a simple formulation. This is discussed in greater detail in Section 3.4.
5. On the real line, axioms $(\propto\downarrow\downarrow)$ and $(\uparrow\uparrow\propto)$ are the same as $(\propto\forall)$.

We will prove that:

The dual of a strong proximity lattice L is the corresponding Priestley space of prime filters ordered by inclusion and equipped with the apartness,

$$F \propto_{\prec} G \overset{\text{def}}{\iff} (\exists\, x \in F)(\exists\, y \notin G)\; x \prec y.$$

Vice versa, the dual of a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ equipped with apartness $\propto$ is the lattice of clopen upper sets equipped with the strong proximity,

$$A \prec_{\propto} B \overset{\text{def}}{\iff} A \propto (X \backslash B).$$

Up to isomorphism, the correspondence is one-to-one.

Also, the action of Priestley duality on morphisms can be adapted to the new setting as follows.

Continuous order-preserving maps that reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms that preserve the strong proximity relation.

Mathematically, Priestley maps are a good choice for establishing the duality, but computationally this is not necessarily true. Priestley maps are, in a way, not general enough; their manifestation on semantic domains are (order-preserving) Lawson continuous functions [41]. This does not cover the computable maps which typically are only Scott-continuous. This fact creates a situation similar to one that happens very often in domain theory where more than one kind of mapping is defined on a fixed class of spaces, for example, embedding-projection pairs, Scott-continuous function, strict Scott-continuous function, stable function, etc.

In order to capture more computable functions, two more general notions of morphism were introduced by Jung and Sünderhauf in [54, 52] on the side of strong proximity lattices; *approximable relations* and *weakly approximable relations* (Definition 2.7.2). This technique dates back to Scott's morphisms for *information systems* [103]. In this chapter, we study the transformation of these morphisms under our duality.

First note that we can not expect to obtain functions on the side of Priestley spaces with apartness. This is so because the Priestley dual contains more points than the *spectrum* defined in [54] and there is no guarantee that the process acts functionally on the additional elements. In keeping with the spirit of approximable relations, we alternatively consider *relations*, rather than functions. These relations between Priestley spaces are meant to relate those pairs of elements (models or theories of properties) that are "observably unrelated" by the computational process. We will discuss the computational intuition in Section 3.1.3.

Here is the definition:

Definition 3.1.5. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces with apartness relations $\propto_1$ and $\propto_2$, respectively, and let $\ltimes$ be a binary relation from X_1 to X_2. The relation $\ltimes$ is called *separating* (or a *separator*) if it is open in $\mathcal{T}_1 \times \mathcal{T}_2$ and if, for every $a, b \in X_1, d, e \in X_2$ and $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$,

$$(\uparrow_1\ltimes\downarrow_2) \qquad a \geq_1 b \ltimes d \geq_2 e \Longrightarrow a \ltimes e,$$

$$(\forall\ltimes) \qquad b \ltimes d \Longleftrightarrow (\forall c \in X_1)\, b \propto_1 c \text{ or } c \ltimes d,$$

$$(\ltimes\forall) \qquad b \ltimes d \Longleftrightarrow (\forall c \in X_2)\, b \ltimes c \text{ or } c \propto_2 d,$$

$$(\ltimes n\downarrow) \qquad b \ltimes \bigcap \downarrow d_i \Longrightarrow (\forall c \in X_1)\, b \propto_1 c \text{ or } (\exists\, i)\; c \ltimes d_i.$$

The relation $\ltimes$ is called *weakly separating* (or a *weak separator*) if it satisfies all of the above conditions, but not necessarily $(\ltimes n\downarrow)$.

Some effort will be required to show Priestley spaces equipped with apartness relations and (weak) separators between them are indeed the objects and morphisms of a category, (see Section 3.3.3), but once this is established, it can be shown that the desired equivalence does hold:

> *Let X_1 and X_2 be Priestley spaces equipped with apartness relations. Then (weakly) separating relations from X_1 to X_2 are in one-to-one correspondence with (weakly) approximable relations from the dual of X_1 to the dual of X_2.*

Remark 3.1.6. From a representation point of view the various dualities can be classified as follows:

	T_0	T_2
strong proximity lattices	Jung & Sünderhauf, [54]	this chapter
distributive lattices	Stone, [111]	Priestley, [89, 90]
Boolean algebras	Stone, [109, 110]	

According to the classification above the strong proximity lattices were represented via a class of T_0 spaces (stably compact spaces) in [54]. This chapter introduces a Hausdorff representation for strong proximity lattices. Also distributive lattices were represented via a class of T_0 spaces (spectral spaces) by Stone in [111]. Distributive lattices were also later given a Hausdorff representation (via Priestley spaces) by Priestley in [89, 90]. Boolean algebras were given a Hausdorff representation (Stone spaces) by Stone in [109, 110].

It should be mentioned that there is a rather conventional way to fill in the right upper position in the table above. For this one equips the collection of *round* prime filters of L with the topology generated by all $U_x := \{F \mid x \in F\}$, and all $O_x := \{F \mid \exists y \notin F.x \prec y\}$, $x \in L$. This yields the *patch topology* of a stably compact space which is already obtainable from the Jung-Sünderhauf dual.

3.1.3 Computational Motivation

Dealing with relations, rather than functions, puts no constraints on the direction of morphisms; so we can turn around the direction and we will get an equivalence of categories rather than a duality. The relationship between proximity homomorphisms and approximable relations is analogous to that between Dijkstra's *weakest preconditions* [85] and *Hoare logic*: A homomorphism h from L_2 to L_1 specifies the weakest precondition $h(\phi)$ necessary for ϕ to hold at the end of the computation. On the other side, an approximable relation $\vdash$ from L_1 to L_2 relates propositions ϕ, ψ if the satisfaction or un-observability (actually true) of ϕ, before the computation, guarantees ψ to be satisfied (observably true) afterwards.

Before we embark on the mathematical details of our duality, let us consider a computational motivation for it.

1. Let $\langle X, \mathcal{T} \rangle$ be a stably compact space (e.g., a compact coherent domain together with

its Scott topology) and $\mathcal{K}_X$ be the set of compact saturated subsets of X. Then

$$\mathcal{B}_X = \{\langle O, K\rangle \mid O \in \mathcal{T}, K \in \mathcal{K}_X \text{ and } O \subseteq K\},$$

is the lattice of observable properties of X, where a property $\langle O, K\rangle$ is

(a) satisfied by elements in O,

(b) unsatisfied by elements in $X \setminus K$, and

(c) unobservable for elements in $K \backslash O$.

Note that $O \subseteq K$ guarantees that no contradiction can arise concerning the satisfaction of properties in $\mathcal{B}_X$ by elements of the space X. From this we can say that strong proximity lattices are abstract algebraic descriptions for lattices of observable properties of stably compact spaces.

For a stably compact space X, models (or theorems) of properties are prime filters of elements of $\mathcal{B}_X$. Hence these models are precisely the points of the Priestley dual space of $\mathcal{B}_X$ as a bounded distributive lattice. Therefore by answering the main question in this chapter (describing the Priestley dual of strong proximity lattices) we are actually establishing purely topological descriptions (Priestley spaces with apartness) for models of observable properties of stably compact spaces.

On the morphism level, we have the following computational interpretation. Suppose $f : X \longrightarrow Y$ is a continuous map between stably compact spaces X and Y. Then the approximable relation $\vdash_f$, corresponding to f, relates a property $\langle O, K\rangle$ of X to a property $\langle O', K'\rangle$ of Y if and only if $f(K) \subseteq O'$. The last condition means that if the property $\langle O, K\rangle$ is satisfied or unobservable for the input of the function (program) f then the output $f(x)$ must satisfy the property $\langle O', K'\rangle$. In other words two properties are related under the relation $\vdash_f$ if and only if the satisfaction and non-observability of the first property, by the input of the program, implies the satisfaction of the second property by the corresponding output. We say that the property $\langle O, K \rangle$, of X, *implies* the property $\langle O', K' \rangle$, of Y, if the former property is related to the latter one under $\vdash_f$.

Let $\ltimes'_f$ be the complement of the separator $\ltimes_f$ corresponding to f. Then, as we will see in detail in the next chapter, a model F_1 is related to a model F_2 in $\ltimes'_f$ if and only if F_2 contains the set of all properties implied by any property in F_1.

2. As we have explained above, in [52] the argument was made that the strong proximity $\prec$ relates two logical propositions (properties) ϕ and ψ if the *observation* of ϕ always implies that ψ is *actually true*. Consequently, the logical systems, **MLS**, (corresponding to strong proximity lattices) do not necessarily satisfy the identity axiom $\phi \vdash \phi$. The lack of this basic law of logic may feel weird. Still the paper [52] presents a satisfactory and even elegant logical apparatus for strong proximity lattices. In this thesis, the view is that the strong proximity relation is *additional structure*, over and above the lattice operations, and that for the latter the usual axioms of logic are still valid. Consequently, a *model* of the logic is given by a prime filter, as it is usually. The additional structure on the logic then gives rise to *additional structure* on the space of all models (the Priestley dual space), which we read as *apartness* information. The intuition is that two states of affair (i.e., models) can be observably separated if and only if they are "sufficiently apart." To give an example, consider the real numbers presented in their usual decimal expansion. Mathematically, we deem $a = 1.000\ldots$ and $b = 0.999\ldots$ equal; constructively, the concrete presentation of a number is important, and in our example one would find that a and b can not be told apart in finite time but their equality can also not be established in finite time (if our only access to the numbers is by successively reading digits). This indeed agrees with what we have explained in the first part of this subsection as the relation $\ltimes'_f$ is the complement of $\ltimes_f$.

3.2 An Extension of Birkhoff's Representation Theorem

3.2.1 From Finite Ordered Sets with Apartness to Finite Strong Proximity Lattices

We define the dual for a finite ordered set with apartness as follows:

Definition 3.2.1. Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then

$$\mathit{Prox}(P) = \langle \mathcal{U}(P); \cup, \cap, \emptyset, P; \prec_{\propto} \rangle,$$

where $\mathcal{U}(P)$ is the collection of upper sets of P and $\prec_{\propto}$ is the binary relation defined on $\mathcal{U}(P)$ as:

$$A \prec_{\propto} B \overset{\text{def}}{\Longleftrightarrow} A \propto (P \setminus B).$$

Remark 3.2.2. $\propto = \not\leq \implies \prec_{\propto} = \subseteq$.

Lemma 3.2.3. *Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then the relation $\prec_{\propto}$ of $\mathit{Prox}(P)$ satisfies the condition $(\prec\prec)$.*

Proof. Suppose $A, C \in \mathcal{U}(P),\ A \prec_{\propto} C$, and $D = P \setminus C$. Hence $A \propto D$. Fix $a \in A$. Set $O_a = \{x \in P \mid a \propto x\}$. By $(\uparrow\propto\downarrow)$ O_a is a lower set with $a \propto O_a$ and $(X \backslash O_a) \propto D$ by $(\propto\forall)$ and the fact that $a \propto D$. Set $B = \bigcup_{a \in A}(P \backslash O_a)$. Therefore $B \in \mathcal{U}(P)$, $A \propto (P \backslash B)$ and $B \propto D$. Hence $A \prec_{\propto} B$ and $B \prec_{\propto} C$. Therefore $\prec_{\propto} \subseteq \prec_{\propto} \circ \prec_{\propto}$.

For the other inclusion, suppose $A \prec_{\propto} B,\ B \prec_{\propto} C$ and $D = P \backslash C$. Then $A \propto (P \backslash B)$ and $B \propto D$. For any $a \in A$ and $d \in D$:

$$(\forall b \in P)\ b \notin B \text{ or } b \in B \implies (\forall b \in P)\ a \propto b \text{ or } b \propto d \implies a \propto d, \text{ by } (\propto\forall).$$

Therefore $A \propto D$ which implies $A \prec_{\propto} C$. This proves $\prec_{\propto} \circ \prec_{\propto} \subseteq \prec_{\propto}$. $\square$

Remark 3.2.4. In the proof of Lemma 3.2.3 we only need $(\uparrow\propto\downarrow)$ and $(\propto\forall)$. Furthermore if $\propto$ is a binary relation on a finite ordered set such that $\propto$ satisfies $(\uparrow\propto\downarrow)$ and $\prec_{\propto}$ satisfies $(\prec\prec)$ then it is not hard to prove condition $(\propto\forall)$ for $\propto$.

Lemma 3.2.5. *Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then the relation $\prec_\propto$ of $Prox(P)$ satisfies $(\prec -\vee)$.*

Proof. Suppose $A \prec_\propto U \cup V$, $C = P \backslash U$ and $D = P \backslash V$. Hence $A \propto (C \cap D)$ implying

$$(\forall a \in A)(\forall c \in C)(\forall d \in D)\ a \propto (\downarrow c \cap \downarrow d).$$

Fix $c \in C$ and $d \in D$. Set $O_c = \{b \in P \mid b \propto c\}$ and $O_d = \{b \in P \mid b \propto d\}$. By $(\uparrow\propto\downarrow)$, O_c and O_d are upper subsets of P. Clearly $O_c \propto c$ and $O_d \propto d$. Moreover, by $(\propto\downarrow\downarrow)$, $A \propto (P \backslash (O_c \cup O_d))$.

Set $U' = \bigcap_{c \in C} O_c$ and $V' = \bigcap_{d \in d} O_d$. Hence U' and V' are upper subsets of P satisfying:

- $U' \propto C$, hence $U' \prec_\propto U$,
- $V' \propto D$, hence $V' \prec_\propto V$, and
- $A \propto P \backslash (U' \cup V')$, hence $A \prec_\propto U' \cup V'$.

□

Remark 3.2.6. In the proof of Lemma 3.2.5 we only need $(\uparrow\propto\downarrow)$ and $(\propto\downarrow\downarrow)$. Furthermore if $\propto$ is a binary relation on a finite ordered set such that $\propto$ satisfies $(\uparrow\propto\downarrow)$ and $\prec_\propto$ satisfies $(\prec -\vee)$ then it is not hard to prove condition $(\propto\downarrow\downarrow)$ for $\propto$.

The following lemma is proved dually to Lemma 3.2.5.

Lemma 3.2.7. *Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then the relation $\prec_\propto$ of $Prox(P)$ satisfies $(\wedge- \prec)$.*

Theorem 3.2.8. *Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then $Prox(P) = \langle \mathcal{U}(P); \cup, \cap, \emptyset, P; \prec_\propto \rangle$ is a finite strong proximity lattice.*

Proof. Clearly $\langle \mathcal{U}(P); \cup, \cap, \emptyset, P\rangle$ is a finite distributive lattice. Lemma 3.2.3 proves that $\prec_\propto$ satisfies $(\prec\prec)$. $(\vee - \prec)$ and $(\prec - \wedge)$ are proved as follows. For any $A \in \mathcal{U}(P)$ and $\{A_i \mid 1 \leq i \leq n\} \subseteq \mathcal{U}(P)$,

$$\bigcup_i A_i \prec_\propto A \iff \bigcup A_i \propto P\backslash A \iff (\forall i)\ A_i \propto P\backslash A \iff (\forall i)\ A_i \prec_\propto A, \text{ and}$$

$$A \prec_\propto \bigcap_i A_i \iff A \propto P\backslash \bigcap A_i = \bigcup_i (P\backslash A_i) \iff (\forall i)\ A \propto (P\backslash A_i) \iff (\forall i)\ A \prec_\propto A_i.$$

Lemmas 3.2.5 and 3.2.7 prove $(\prec - \vee)$ and $(\wedge - \prec)$, respectively. □

3.2.2 From Finite Strong Proximity Lattices to Finite Ordered Sets with Apartness

We remind the reader of the following. For a lattice L, the sets of join-irreducible and meet-irreducible elements of L are denoted by $\mathcal{J}(L)$ and $\mathcal{M}(L)$, respectively.

Recall from Lemma 2.2.2 that the map $g : x \longmapsto \bigvee (L\backslash\uparrow x)$ is an order-isomorphism from $\mathcal{J}(L)$ to $\mathcal{M}(L)$. It is characterised by the property

$$x \nleq a \iff a \leq g(x)$$

which holds for all $x \in \mathcal{J}(L), a \in L$.

Definition 3.2.9. Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a finite strong proximity lattice. Then

$$Pries(L) = \langle \mathcal{J}(L), \leq^\partial, \propto_\prec\rangle,$$

where $\leq^\partial = \geq$, the order inherited from L, and $\propto_\prec$ is the binary relation defined on $\mathcal{J}(L)$ as:

$$x \propto_\prec y \overset{\text{def}}{\iff} x \prec g(y).$$

Remark 3.2.10.

$$x \propto_\prec y \iff x \prec g(y) \implies x \leq g(y) \iff x \nleq y \iff x \ngeq^\partial y,$$

that is, if the strong proximity relation is trivial then so is the apartness.

Lemma 3.2.11. *Let* $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ *be a finite strong proximity lattice. Then the relation* $\propto_\prec$ *of* $Pries(L)$ *satisfies* $(\uparrow\propto\downarrow)$.

Proof. Let $a, c, d, e \in \mathcal{J}(L)$ be such that $a \geq^\partial c$ and $d \geq^\partial e$, and $c \propto_\prec d$. Then $a \leq c$ and $d \leq e$ in L. Therefore by $(\vee - \prec)$ and $(\prec - \wedge)$,

$$a \leq c \propto_\prec d \leq e \Longrightarrow a \leq c \prec g(d) \leq g(e) \Longrightarrow a \prec g(e) \Longrightarrow a \propto_\prec e.$$

□

Lemma 3.2.12. *Let* $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ *be a finite strong proximity lattice. Then for every* $a, b \in L$,

$$a \prec b \Longleftrightarrow \{x \in \mathcal{J}(L) \mid x \leq a\} \propto_\prec \{y \in \mathcal{J}(L) \mid y \not\leq b\}.$$

Proof. By Lemma A.18,

$$\bigvee\{x \in \mathcal{J}(L) \mid x \leq a\} = a \prec b = \bigwedge\{y \in \mathcal{M}(L) \mid b \leq y\}$$

$$\Longleftrightarrow \{x \in \mathcal{J}(L) \mid x \leq a\} \prec \{y \in \mathcal{M}(L) \mid b \leq y\}, \text{ by } (\vee - \prec) \text{ and } (\prec - \wedge).$$

So it remains to show that

$$\{y \in \mathcal{M}(L) \mid b \leq y\} = g(\{x \in \mathcal{J}(L) \mid x \not\leq b\}),$$

but this is immediate from the equivalence

$$x \not\leq b \Longleftrightarrow b \leq g(x)$$

mentioned above. □

Figure 3.1 gives a pictorial illustration of Lemma 3.2.12. The idea is that the element a approximates the element b if and only if the join-irreducible elements in the area A are apart from the join-irreducible elements in the areas B and C.

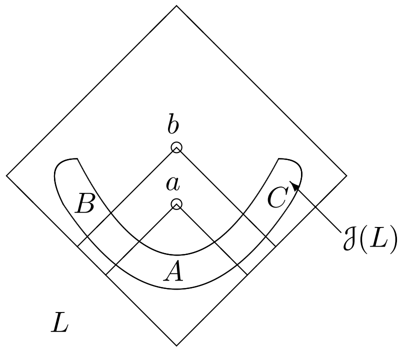


Figure 3.1: A pictorial description of Lemma 3.2.12.

Lemma 3.2.13. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a finite strong proximity lattice. Then the relation $\propto_{\prec}$ of $Pries(L)$ satisfies $(\propto\forall)$.*

Proof. For any $a, c \in \mathfrak{J}(L)$,

$$\begin{aligned} a \propto_{\prec} c &\iff a \prec g(c) \\ &\iff (\exists\, l \in L)\ a \prec l \text{ and } l \prec g(c),\ \text{by}(\prec\prec) \\ &\iff (\exists\, l \in L)\ a \propto_{\prec} \{b \in \mathfrak{J}(L) \mid b \not\leq l\} \text{ and } \{b \in \mathfrak{J}(L) \mid b \leq l\} \propto_{\prec} c \\ &\iff (\forall b \in \mathfrak{J}(L))\ a \propto_{\prec} b \text{ or } b \propto_{\prec} c. \end{aligned}$$

The second last equivalence is true by Lemma 3.2.12 and the right-to-left direction of the last equivalence is satisfied by setting $l = \bigvee\{b \in \mathfrak{J}(L) \mid b \propto_{\prec} c\}$. □

Lemma 3.2.14. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a finite strong proximity lattice. Then the relation $\propto_{\prec}$ of $Pries(L)$ satisfies $(\propto\downarrow\downarrow)$.*

Proof. Let $a, c, d \in \mathfrak{J}(L)$. We notice that $\downarrow c \cap \downarrow d$ in $\mathfrak{J}(L)$ with respect to $\leq^{\partial}$ is the same as $\uparrow c \cap \uparrow d$ in $\mathfrak{J}(L)$ with respect to $\leq$, the order inherited from the lattice. Suppose $a \propto_{\prec} (\uparrow c \cap \uparrow d)$.

For any $y \in \mathcal{M}(L)$ with $g(c) \vee g(d) \leq y$ let $x \in \mathcal{J}(L)$ be such that $y = g(x)$. Then:

$$\begin{aligned} g(c) \vee g(d) \leq g(x) &\iff x \nleq g(c) \vee g(d) \\ &\iff x \nleq g(c) \text{ and } x \nleq g(d) \\ &\iff c \leq x \text{ and } d \leq x. \end{aligned}$$

So $a \propto_{\prec} x$, i.e. $a \prec g(x) = y$. Apply $(\prec -\wedge)$ and get $a \prec g(c) \vee g(d)$.

Hence

$$\begin{aligned} a \propto_{\prec} ({\uparrow}c \cap {\uparrow}d) &\implies a \prec g(c) \vee g(d) \\ &\implies (\exists\, x', y' \in L)\; x' \prec g(c),\; y' \prec g(d) \text{ and } a \prec x' \vee y',\; \text{by } (\prec -\vee) \\ &\implies \{b \in \mathcal{J}(L) \mid b \leq x'\} \prec g(c),\; \{b \in \mathcal{J}(L) \mid b \leq y'\} \prec g(d) \text{ and} \\ & a \propto_{\prec} \{b \in \mathcal{J}(L) \mid b \nleq x' \vee y'\},\; \text{by Lemma 3.2.12.} \\ &\implies (\forall b \in \mathcal{J}(L))\; a \propto_{\prec} b,\; b \propto_{\prec} c,\; \text{or } b \propto_{\prec} d. \end{aligned}$$

□

The following lemma is proved dually to Lemma 3.2.14.

Lemma 3.2.15. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a finite strong proximity lattice. Then the relation $\propto_{\prec}$ of $Pries(L)$ satisfies $({\uparrow}{\uparrow}\propto)$.*

Theorem 3.2.16. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a finite strong proximity lattice. Then the relation $\propto_{\prec}$ of $Pries(L)$ is an apartness on the finite ordered set $\langle \mathcal{J}(L), \leq^{\partial} \rangle$.*

Proof. Conditions $({\uparrow}\propto{\downarrow})$, $(\propto\forall)$, $(\propto{\downarrow}{\downarrow})$ and $({\uparrow}{\uparrow}\propto)$ are satisfied by Lemmas 3.2.11, 3.2.13, 3.2.14, and 3.2.15, respectively. □

3.2.3 The Representation Theorem

Objects

We show that the translations of the previous two sub-sections are (essentially) inverses of each other. Since our theory is based on Birkhoff duality, only the behaviour of the strong proximity and the apartness relations need to be examined.

Definition 3.2.17. A lattice homomorphism (isomorphism) between strong proximity lattices is said to be a *proximity homomorphism (proximity isomorphism)* if it preserves (preserves in both directions) the strong proximity relation (relations).

Theorem 3.2.18. *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a finite strong proximity lattice. Then the map:*

$$\eta_L : L \longrightarrow \mathit{Prox}(\mathit{Pries}(L)); x \longmapsto \{a \in \mathcal{J}(L) \mid a \leq x\},$$

is a proximity isomorphism.

Proof. By Birkhoff's theorem, η_L is an order-isomorphism. For every $x, y \in L$:

$$\begin{aligned} x \prec y &\iff \{a \in \mathcal{J}(L) \mid a \leq x\} \propto_{\prec} \{b \in \mathcal{J}(L) \mid b \not\leq y\}, \text{ by Lemma 3.2.12} \\ &\iff \{a \in \mathcal{J}(L) \mid a \leq x\} \prec_{\propto_{\prec}} \{b \in \mathcal{J}(L) \mid b \leq y\} \\ &\iff \eta_L(x) \prec_{\propto_{\prec}} \eta_L(y). \end{aligned}$$

□

Definition 3.2.19. Let P_1 and P_2 be finite ordered sets equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. A map $f : P_1 \longrightarrow P_2$ is said to be:

- an *apartness map* from P_1 to P_2 if it is order-preserving, and for every $a, b \in P_1$,

$$f(a) \propto_2 f(b) \Longrightarrow a \propto_1 b,$$

- an *apartness isomorphism* from P_1 to P_2 if it is an order-isomorphism and for every $a, b \in P_1$,

$$a \propto_1 b \Longleftrightarrow f(a) \propto_2 f(b).$$

Theorem 3.2.20. *Let $\langle P, \leq \rangle$ be a finite ordered set equipped with apartness $\propto$. Then the map:*

$$\epsilon_P : P \longrightarrow Pries(Prox(P)); x \longmapsto \uparrow x,$$

is an apartness-isomorphism.

Proof. By Birkhoff's theorem, ϵ_P is an order-isomorphism. For every $c, d \in P$:

$$\begin{aligned} c \propto d &\Longleftrightarrow \uparrow c \propto \downarrow d, \text{ by } (\uparrow\propto\downarrow) \\ &\Longleftrightarrow \uparrow c \prec_\propto P \backslash \downarrow d \\ &\Longleftrightarrow \uparrow c \propto_{\prec_\propto} \uparrow d, \text{ because } g(\uparrow d) = P \backslash \downarrow d \\ &\Longleftrightarrow \epsilon_P(c) \propto_{\prec_\propto} \epsilon_P(d). \end{aligned}$$

□

Lemma 3.2.21. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a finite strong proximity lattice and $\langle P; \leq \rangle$ be a finite ordered set equipped with apartness $\propto$.*

1. *If $\leq$ is the lattice order and $\leq'$ is the order of $\mathfrak{J}(L)^\partial$ then*

$$\prec = \leq \Longleftrightarrow \propto_\prec = \not\leq'.$$

2. *The relation $\prec_\propto$ of $Prox(P)$ satisfies the following:*

$$\propto = \not\leq \Longleftrightarrow \prec_\propto = \leq.$$

Proof. 1. The left-to-right direction follows from Remark 3.2.10. For the other direction, suppose that $\propto_{\prec} = \not\leq'$. Then by Remark 3.2.2 $\prec_{\propto_{\prec}} = \subseteq$. For every $a, b \in L$,

$$\begin{aligned} a \prec b &\iff \eta_L(a) \prec_{\propto_{\prec}} \eta_L(b), \text{ by Theorem 3.2.18} \\ &\iff \{x \in \mathcal{J}(L) \mid x \leq a\} \subseteq \{x \in \mathcal{M}(L) \mid x \leq b\} \\ &\iff \bigvee\{x \in \mathcal{J}(L) \mid x \leq a\} \leq \bigvee\{x \in \mathcal{J}(L) \mid x \leq b\}, \text{ by Lemma } A.6 \\ &\iff a \leq b, \text{ by Lemma } A.18. \end{aligned}$$

2. The left-to-right direction follows from Remark 3.2.10. For the other direction, suppose that $\prec_{\propto} = \leq$. Then by Remark 3.2.2 $\propto_{\prec_{\propto}} = \not\supseteq$. For any $a, b \in P$,

$$\begin{aligned} a \propto b &\iff \epsilon_P(a) \propto_{\prec_{\propto}} \epsilon_P(b), \text{ by Theorem 3.2.20} \\ &\iff \uparrow a \not\supseteq \uparrow b \\ &\iff a \not\leq b. \end{aligned}$$

□

Remark 3.2.22. As we have proved that the trivial cases of apartness relations ($\propto = \not\leq$) and strong proximity relations ($\prec = \leq$) translates into each other, it is obvious that our representation theorem is a proper extension of that of Birkhoff.

Remark 3.2.23. Suppose the cardinality of L is greater than 1 and 2^L denotes the set of all lattice-homomorphisms from L to 2 ordered point-wise. Then $\mathcal{J}(L)^\partial$ is isomorphic to 2^L [20, Excersice 5.20]. A join irreducible element x of L corresponds to the characteristic function f_x of $\uparrow x$. For the binary relation $\propto_{\prec}$ (Definition 3.2.9) on $\mathcal{J}(L)^\partial$, we note that for $x, y \in \mathcal{J}(L)$:

$$\begin{aligned} x \propto_{\prec} y &\iff x \prec g(y) \\ &\iff (\exists\, a, b \in L)\; a \geq x,\; b \not\geq y \text{ and } a \prec b \\ &\iff (\exists\, a, b \in L)\; f_x(a) = 1,\; f_y(b) = 0 \text{ and } a \prec b. \end{aligned}$$

Therefore the ordered set 2^L equipped with the apartness defined as follows:

$$f \propto_{\prec} f' \overset{\text{def}}{\Longleftrightarrow} (\exists\, a, b \in L)\ f(a) = 1,\ f'(b) = 0 \text{ and } a \prec b$$

is apartness isomorphic to $Pries(L)$.

By Lemma 3.2.12, for every $x, y \in L$:

$$x \prec y \Longleftrightarrow \{f \in 2^L \mid f(x) = 1\} \propto_{\prec} \{f \in 2^L \mid f(y) = 0\}.$$

Morphisms

Lemma 3.2.24. *Let $\langle L_1, \leq_1; \prec_1 \rangle$ and $\langle L_2, \leq_2; \prec_2 \rangle$ be finite strong proximity lattices and P_1 and P_2 be finite ordered sets equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Then*

1. *For a proximity homomorphism $f : L_1 \longrightarrow L_2$, the map*

$$Pries(f) : Pries(L_2) \longrightarrow Pries(L_1); y \longmapsto \min_{L_1}\{x \in \mathcal{J}(L_1) \mid x \in f^{-1}(\uparrow y)\},$$

is an apartness map.

2. *For an apartness map $\varphi : P_1 \longrightarrow P_2$, the function*

$$Prox(\varphi) : Prox(P_2) \longrightarrow Prox(P_1); U \longmapsto \varphi^{-1}(U),$$

is a proximity homomorphism.

Proof. 1. $Pries(f)$ is well defined order preserving map by Section 2.2. Suppose $a, b \in Pries(L_2)$. We first notice that

$$f(\bigvee(L_1 \backslash \uparrow Pries(f)(b)) \leq \bigvee(L_2 \backslash \uparrow b).$$

This is true because

$$f(\bigvee(L_1 \backslash \uparrow Pries(f)(b)) = \bigvee f(L_1 \backslash \uparrow Pries(f)(b)),$$

and

$$f(L_1 \backslash {\uparrow} Pries(f)(b)) \subseteq L_2 \backslash {\uparrow} b.$$

This inclusion is proved as follows. If $c \in L_1 \backslash {\uparrow} Pries(f)(b)$ then, by definition of $Pries(f)$, $f(\{x \in \mathcal{J}(L_1) \mid x \leq c\}) \subseteq L_2 \backslash {\uparrow} b$. Therefore

$$\bigvee f(\{x \in \mathcal{J}(L_1) \mid x \leq c\}) \in L_2 \backslash {\uparrow} b.$$

But

$$\bigvee f(\{x \in \mathcal{J}(L_1) \mid x \leq c\}) = f(\bigvee \{x \in \mathcal{J}(L_1) \mid x \leq c\}) = f(c).$$

Now we have:

$$\begin{aligned} Pries(f)(a) \propto_{\prec_1} Pries(f)(b) &\implies Pries(f)(a) \prec_1 \bigvee (L_1 \backslash {\uparrow} Pries(f)(b)) \\ &\implies f(Pries(f)(a)) \prec_2 f(\bigvee (L_1 \backslash {\uparrow} Pries(f)(b))) \\ &\implies a \leq f(Pries(f)(a)) \prec_2 f(\bigvee (L_1 \backslash {\uparrow} Pries(f)(b))) \\ &\qquad \leq \bigvee (L_2 \backslash {\uparrow} b) \\ &\implies a \prec_2 \bigvee (L_2 \backslash {\uparrow} b) \\ &\implies a \propto_{\prec_2} b. \end{aligned}$$

2. By Section 2.2, $Prox(\varphi)$ is a lattice homomorphism from $\mathcal{U}(P_2)$ to $\mathcal{U}(P_1)$. For every $A, B \in \mathcal{U}(P_2)$:

$$\begin{aligned} A \prec_{\propto_2} B &\implies A \propto_2 (P_2 \backslash B) \\ &\implies \varphi^{-1}(A) \propto_1 \varphi^{-1}(P_2 \backslash B) \\ &\implies \varphi^{-1}(A) \propto_1 P_1 \backslash \varphi^{-1}(B) \\ &\implies \varphi^{-1}(A) \prec_{\propto_1} \varphi^{-1}(B) \\ &\implies Prox(\varphi)(A) \prec_{\propto_1} Prox(\varphi)(B). \end{aligned}$$

□

Remark 3.2.25. Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1 \rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2 \rangle$ be finite strong proximity lattices and $\langle P_1, \leq \rangle$ and $\langle P_2, \leq \rangle$ be finite ordered sets equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $f : L_1 \longrightarrow L_2$ and $\varphi : P_1 \longrightarrow P_2$ be a proximity homomorphism and an apartness map, respectively. Then the following is true by Birkhoff's representation theorem (Section 2.2).

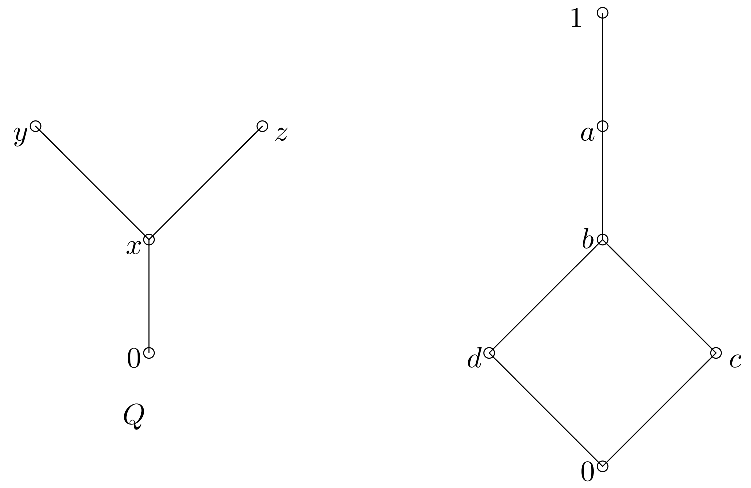
- The following diagrams commute.

$$\begin{array}{ccc} L_1 & \xrightarrow{\quad f \quad} & L_2 \\ \downarrow{\scriptstyle \eta_{L_1}} & & \downarrow{\scriptstyle \eta_{L_2}} \\ \mathit{Prox}(\mathit{Pries}(L_1)) & \xrightarrow{\mathit{Prox}(\mathit{Pries}(f))} & \mathit{Prox}(\mathit{Pries}(L_2)) \end{array}$$

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad \varphi \quad} & P_2 \\ \downarrow{\scriptstyle \epsilon_{P_1}} & & \downarrow{\scriptstyle \epsilon_{P_2}} \\ \mathit{Pries}(\mathit{Prox}(P_1)) & \xrightarrow{\mathit{Pries}(\mathit{Prox}(\varphi))} & \mathit{Pries}(\mathit{Prox}(P_2)) \end{array}$$

- The map $f \longmapsto \mathit{Pries}(f)$ establishes a one-to-one correspondence between proximity-homomorphisms from L_1 to L_2 and apartness mappings from $\mathit{Pries}(L_2)$ to $\mathit{Pries}(L_1)$. The map $\varphi \longmapsto \mathit{Prox}(\varphi)$ establishes a one-to-one correspondence between apartness mappings from P_2 to P_1 and proximity-homomorphisms from $\mathit{Prox}(P_1)$ to $\mathit{Prox}(P_2)$.

We let **FOSa** be the category whose objects are finite ordered sets equipped with apartness relations, and whose morphisms are apartness maps. **FPL** is the category of finite strong proximity lattices and proximity homomorphisms. Theorems 3.2.18 and 3.2.20 and Lemma 3.2.24 show that Birkhoff duality between finite distributive lattices and finite ordered sets can be extended to the (non-full) categories **FOSa** and **FPL**. All in all:



$L \cong \mathit{Prox}(Q) = \mathcal{U}(Q)$

Figure 3.2: An ordered set together with its corresponding distributive lattice.

Theorem 3.2.26. *The functors* Pries *and* Prox *establish a dual equivalence between the categories* **FOSa** *and* **FPL**.

3.2.4 Examples

1. The binary relation $\propto = \not\leq \setminus \{(x,0)\}$ defined on the ordered set Q in Figure 3.2 is an apartness. The lattice L in the same diagram together with the binary relation $\prec = \leq \setminus \{(a,a)\}$ is the finite strong proximity lattice corresponding to $\langle Q, \leq; \propto \rangle$.

2. Let C be the chain $\{0, \ldots, k\}$. Its corresponding finite distributive lattice, under Birkhoff duality, is $C \oplus \{\top\}$, the chain C with a new top element added. In the following table, the left-hand side column shows some apartness relations on C and the right-hand side column shows their corresponding strong proximity relations on $C \oplus \top$.

$\propto$ on C	$\prec_\propto$ on $C \oplus \top$
$>$	$\leq$
$\geq$	$n \prec_\propto m \overset{\text{def}}{\iff} n = 0$ or $n - 1 \leq m$
$\uparrow c \times \downarrow (c-1)$, for some $0 < c \in C$	$n \prec_\propto m \overset{\text{def}}{\iff} n = 0, m = \top$ or $n \leq c \leq m$

3. Suppose $\langle P; \leq \rangle$ is a bounded finite ordered set and P is a disjoint union $\downarrow x \dot{\cup} \uparrow y$ for some $x, y \in P$. Then $\propto = \uparrow y \times \downarrow x$ is an apartness relation on P. The relation $\prec_\propto$ on $Prox(P)$ can be characterised as follows:

$$A \prec_\propto B \overset{\text{def}}{\Longleftrightarrow} A = \emptyset,\ B = P \text{ or } A \subseteq \uparrow y \subseteq B.$$

Remark 3.2.27. The third case of example two is a special case of the this example.

4. Suppose $\langle P_1; \leq_1 \rangle$ and $\langle P_2; \leq_2 \rangle$ are bounded disjoint ordered sets. It is straightforward, but may be tedious, to prove that $\propto = (P_1 \times P_2) \cup (P_2 \times P_1)$ is an apartness relation on the disjoint union $\langle P_1 \dot{\cup} P_2; \leq \rangle$. Note that generally in this example $\propto$ is a proper subset of $\not\leq$ and clearly

$$\mathcal{U}(P_1 \dot{\cup} P_2) = \{A \cup B \mid A \in \mathcal{U}(P_1) \text{ and } B \in \mathcal{U}(P_2)\}.$$

Now suppose P_1 and P_2 are finite sets. We study the relation $\prec_\propto$ defined on $\mathcal{U}(P_1 \dot{\cup} P_2)$. We have

$$\begin{aligned} A \cup B \prec_\propto C \cup D &\iff (A \cup B) \propto (P_1 \cup P_2) \backslash (C \cup D) \\ &\iff (A \cup B) \propto (P_1 \backslash C \cup P_2 \backslash D) \\ &\iff (A = \emptyset \text{ or } C = P_1) \text{ and } (B = \emptyset \text{ or } D = P_2). \end{aligned}$$

5. Figure 3.3 shows an apartness map :

$$\varphi : \langle Q; \leq_1; \propto_1 = \not\leq_1 \backslash \{\langle \alpha, \delta \rangle\} \rangle \longrightarrow \langle P; \leq_2; \propto_2 = \uparrow y \times \downarrow x \rangle$$

and the associated proximity homomorphism

$$f : \langle Prox(P); \prec_{\propto_1} \rangle \longrightarrow \langle Prox(Q); \prec_{\propto_2} \rangle.$$

Note that the ordered sets $\langle P, \leq_1 \rangle$ and $\langle Q, \leq_2 \rangle$, equipped with apartness relations $\propto_1$ and $\propto_2$, respectively, and their corresponding finite strong proximity lattices are explained in detail in examples 1 and 3 above.

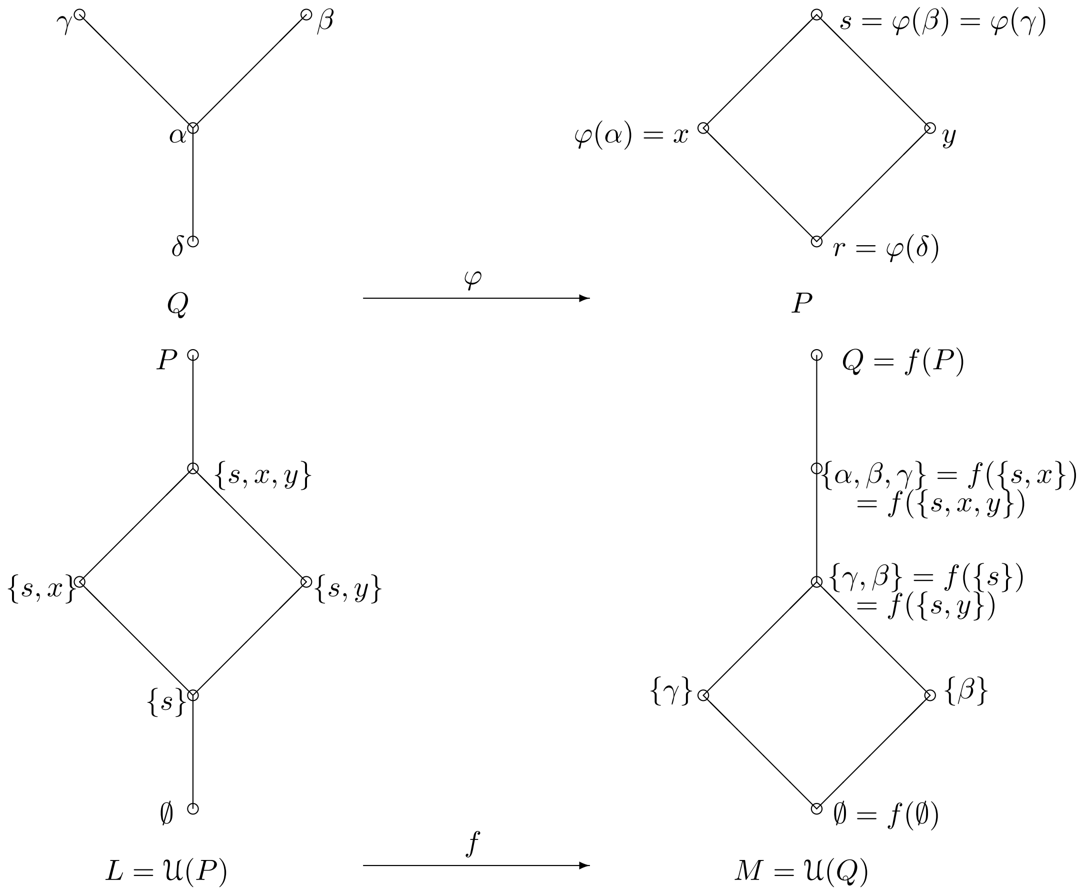


Figure 3.3: An apartness map and its corresponding proximity homomorphism

3.3 An Extension of Priestley's Representation Theorem

In this section we extend our duality from the finite to the infinite case, that is, from Birkhoff duality to Priestley duality. Recall from Section 3.1.2 that this involves two changes in our methodology: firstly, we need to equip the dual of the lattice with a topology, and secondly, we need to work with prime filters of strong proximity lattices, rather than join-irreducible elements. There is a small advantage from the second change; the order on prime filters is just inclusion, rather than the reverse of the lattice order.

3.3.1 From Priestley Spaces with Apartness to Strong Proximity Lattices

We remind the reader that we denote the sets of clopen lower and upper sets of a Priestley space X by $\mathcal{O}^{\mathcal{T}}(X)$ and $\mathcal{U}^{\mathcal{T}}(X)$, respectively. We begin with two preparatory technical results.

Lemma 3.3.1. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space. For closed lower subsets $A, B \subseteq X$ and $O \in \mathcal{T}$, if $A \cap B \subseteq O$, then there exist $V_1, V_2 \in \mathcal{O}^{\mathcal{T}}(X)$ such that $A \subseteq V_1, B \subseteq V_2$ and $V_1 \cap V_2 \subseteq O$.*

Proof. In a Priestley space every closed lower subset is the intersection of clopen lower subsets containing it. Therefore

$$\begin{aligned} A \cap B &= \bigcap\{W \in \mathcal{O}^{\mathcal{T}}(X) \mid A \subseteq W\} \cap \bigcap\{W' \in \mathcal{O}^{\mathcal{T}}(X) \mid B \subseteq W'\} \\ &= \bigcap\{W \cap W' \mid W, W' \in \mathcal{O}^{\mathcal{T}}(X), A \subseteq W \text{ and } B \subseteq W'\}. \end{aligned}$$

By the compactness of $(X \backslash O)$ and the closedness of sets $W \cap W'$, there exists a finite set

$$\{W_i \cap W_i' \mid W_i, W_i' \in \mathcal{O}^{\mathcal{T}}(X),\ A \subseteq W_i, B \subseteq W_i' \text{ and } 1 \leq i \leq n\}$$

such that $\bigcap_{1 \leq i \leq n}(W_i \cap W_i') \subseteq O$, so we can set $V_1 = \bigcap_{1 \leq i \leq n} W_i$ and $V_2 = \bigcap_{1 \leq i \leq n} W_i'$. □

Lemma 3.3.2. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space. For closed lower subsets $A_1, \ldots, A_n \subseteq X$ and $O \in \mathcal{T}$, if $\bigcap_i A_i \subseteq O$, then there exist $V_1, \ldots, V_n \in \mathcal{O}^{\mathcal{T}}(X)$ such that for every i $A_i \subseteq V_i$ and $\bigcap_i V_i \subseteq O$.*

Proof. The proof follows the same lines as that of the previous lemma. □

Lemma 3.3.3. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with a binary relation $\propto$ that satisfies condition $(\uparrow\propto\downarrow)$ and open in the product topology. For closed subsets $A, B \subseteq X$, if $A \propto B$ then there exist $U \in \mathcal{U}^{\mathcal{T}}(X)$ and $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $A \subseteq U$, $B \subseteq V$ and $U \propto V$.*

Proof. By $(\uparrow\!\propto\!\downarrow)$, $\uparrow A \propto \downarrow B$. Recall that $\uparrow A$ and $\downarrow B$ are closed subsets of X because $\mathcal{T}$ is a Priestley topology. We first show that there exist open sets O_1 and O_2 such that $\uparrow A \subseteq O_1$, $\downarrow B \subseteq O_2$ and $O_1 \propto O_2$. Fix $x \in \uparrow A$. Then for every $y \in \downarrow B$, by openness of $\propto$, there exist $O_{1y}, O_{2y} \in \mathcal{T}$ such that $x \in O_{1y}$, $y \in O_{2y}$ and $O_{1y} \propto O_{2y}$. The set $\{O_{2y} \mid y \in \downarrow B\}$ is an open cover of $\downarrow B$, and so a finite sub-cover $\{O_{2y_i} \mid 1 \leq i \leq n\}$ exists. Set $O_{1x} = \bigcap_i O_{1y_i}$ and $O_{2x} = \bigcup_i O_{2y_i}$. Then O_{1x} and O_{2x} are open sets with $x \in O_{1x}, \downarrow B \subseteq O_{2x}$ and $O_{1x} \propto O_{2x}$. Now, the set $\{O_{1x} \mid x \in \uparrow A\}$ is an open cover of $\uparrow A$ and so a finite sub-cover $\{O_{1x_i} \mid 1 \leq i \leq m\}$ exists. Set $O_1 = \bigcup_i O_{1x_i}$ and $O_2 = \bigcap_i O_{2x_i}$. Then O_1 and O_2 are open sets with $\uparrow A \subseteq O_1$, $\downarrow B \subseteq O_2$ and $O_1 \propto O_2$.

Now as $(X, \leq, \mathcal{T})$ is a Priestley space, $\uparrow A = \bigcap\{U \in \mathcal{U}^{\mathcal{T}}(X) \mid \uparrow A \subseteq U\} \subseteq O_1$, and because $X \backslash O_2$ is a compact subset of X, there exists a finite set of clopen upper sets $\{U_i \in \mathcal{U}^{\mathcal{T}}(X) \mid 1 \leq i \leq n\}$ such that $\uparrow A \subseteq \bigcap_i U_i \subseteq O_1$. Set $U = \bigcap_i U_i$. Then U is a clopen upper subset of X with $\uparrow A \subseteq U \subseteq O_1$. Similarly there exists a clopen-lower subset V of X with $\downarrow B \subseteq V \subseteq O_2$. Therefore $U \propto V$, which completes the proof. $\square$

Remark 3.3.4. Singletons are closed subsets in a Priestley space. Therefore, as a special case of Lemma 3.3.3 we have that for every $a, b \in X$, if $a \propto b$ then there exists $U \in \mathcal{U}^{\mathcal{T}}(X)$ and $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $a \in U$, $b \in V$ and $U \propto V$.

Remark 3.3.5. Clearly if $\propto$ is a binary relation $\propto$ on a Priestley space $\langle X, \mathcal{T}, \leq\rangle$ such that $\propto$ satisfies the result in Lemma 3.3.3 then $\propto$ satisfies $(\uparrow\!\propto\!\downarrow)$ and open in the product topology.

We define the dual for a Priestley space with apartness as follows:

Definition 3.3.6. Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then

$$Prox(X) = \langle \mathcal{U}^{\mathcal{T}}(X); \cup, \cap, \emptyset, X; \prec_{\propto}\rangle,$$

where $\prec_{\propto}$ is the binary relation defined on $\mathcal{U}^{\mathcal{T}}(X)$ as:

$$A \prec_{\propto} B \overset{\text{def}}{\Longleftrightarrow} A \propto (X \backslash B).$$

Remark 3.3.7. Note that if X carries the trivial apartness $\not\leq$ then the lattice $\mathcal{U}^{\mathcal{T}}(X)$ will be equipped with the trivial strong proximity $\prec_{\propto} = \subseteq$. In fact, the converse is also true: if $\prec_{\propto} = \subseteq$ then $\propto = \not\leq$.

Lemma 3.3.8. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the relation $\prec_{\propto}$ of $Prox(X)$ satisfies $(\prec\prec)$.*

Proof. Suppose $A, C \in \mathcal{U}^{\mathcal{T}}(X),\ A \prec_{\propto} C$, and $D = X \backslash C$. So $A \propto D$. Fix $a \in A$ and set $O_a = \{x \in X \mid a \propto x\}$. Then $O_a \in \mathcal{T}$ by openness of $\propto$, $a \propto O_a$ and $(X \backslash O_a) \propto D$ by $(\propto\forall)$ and the fact that $a \propto D$.

Now by Lemma 3.3.3, there exists a clopen upper subset B_a of X such that $(X \backslash O_a) \subseteq B_a$ and $B_a \propto D$. Therefore $a \propto (X \backslash B_a)$. Using Lemma 3.3.3 again, there exists $U_a \in \mathcal{U}^{\mathcal{T}}(X)$ such that $a \in U_a$ and $U_a \propto (X \backslash B_a)$. The set $\{U_a \mid a \in U\}$ is an open cover of A which is compact as it is closed. Hence a finite sub-cover $\{U_{a_i}\}_{1 \leq i \leq n}$ exists. Set $U = \bigcup_{1 \leq i \leq n} U_{a_i}$ and $B = \bigcup_{1 \leq i \leq n} B_{a_i}$. Therefore U and B are clopen upper subsets of X with $A \subseteq U \propto (X \backslash B)$ and $B \propto D$ which implies $A \prec_{\propto} B$ and $B \prec_{\propto} C$. This proves $\prec_{\propto} \subseteq \prec_{\propto}; \prec_{\propto}$.

For the other inclusion, suppose $A \prec_{\propto} B,\ B \prec_{\propto} C$ and $D = X \backslash C$. Then $A \propto (X \backslash B)$ and $B \propto D$. Pick any $a \in A$ and $d \in D$, then any $b \in X$ is either in B or in $X \setminus B$, so $a \propto b$ or $b \propto d$ from which $a \propto d$ follows by $(\propto\forall)$. Therefore $A \propto D$ which implies $A \prec_{\propto} C$. □

Remark 3.3.9. In the proof of Lemma 3.3.8 we need the openness of $\propto$, $(\uparrow\!\propto\!\downarrow)$ and $(\propto\forall)$. Furthermore if $\propto$ is a binary relation on a Priestley space X such that $\propto$ is open in the product topology on X and satisfies $(\uparrow\!\propto\!\downarrow)$ and $\prec_{\propto}$ satisfies $(\prec\prec)$ then it is not hard to prove condition $(\propto\forall)$ for $\propto$.

Lemma 3.3.10. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the relation $\prec_{\propto}$ of $Prox(X)$ satisfies $(\prec -\vee)$ and $(\wedge - \prec)$.*

Proof. Suppose $A \prec_\propto U \cup V$, $C = X \backslash U$ and $D = X \backslash V$. Then $A \propto C \cap D$ implies

$$(\forall a \in A)(\forall c \in C)(\forall d \in D)\ a \propto ({\downarrow}c \cap {\downarrow}d).$$

Fix $c \in C$ and $d \in D$. Set $O_c = \{x \in X \mid x \propto c\}$ and $O_d = \{x \in X \mid x \propto d\}$. By $({\uparrow}{\propto}{\downarrow})$ and openness of $\propto$, O_c and O_d are open upper subsets of X. Clearly $O_c \propto c$ and $O_d \propto d$. Moreover, by $({\propto}{\downarrow}{\downarrow})$, $A \propto (X \backslash (O_c \cup O_d))$.

By Lemma 3.3.3 and Lemma 3.3.1,

$$\begin{aligned} A \propto (X\backslash O_c) \cap (X \backslash O_d) \quad \text{hence} \quad & (\exists V_c, V_d \in \mathcal{O}^{\mathcal{T}}(X))\ (X\backslash O_c) \subseteq V_c, \\ & (X\backslash O_d) \subseteq V_d \text{ and } A \propto (V_c \cap V_d) \\ \text{and} \quad & (X\backslash V_c) \propto c \text{ and } (X \backslash V_d) \propto d \\ \text{hence} \quad & (\exists W_c, W_d \in \mathcal{O}^{\mathcal{T}}(X))\ c \in W_c, d \in W_d, \\ & (X\backslash V_c) \propto W_c \text{ and } (X\backslash V_d) \propto W_d. \end{aligned}$$

The sets $\{W_c \mid c \in C\}$ and $\{W_d \mid d \in D\}$ are open covers of compact subsets C and D, respectively. Therefore finite subcovers $\{W_{c_i} \mid c_i \in C \text{ and } 1 \leq i \leq n\}$ and $\{W_{d_i} \mid d_i \in D \text{ and } 1 \leq i \leq m\}$ exist.

Set $U' = \bigcap_i X\backslash V_{c_i}$ and $V' = \bigcap_i X \backslash V_{d_i}$. Then U' and V' are clopen upper subsets satisfying $U' \propto \bigcup_i W_{c_i} \supseteq C$ and so $U' \propto C$ which implies $U' \prec_\propto U$. Equally, $V' \propto D \subseteq \bigcup_i W_{d_i}$ implies $V' \prec_\propto V$. Finally $A \propto X\backslash(U' \cup V')$ implies $A \prec_\propto U' \cup V'$.

The argument for $(\wedge{-}\prec)$ is dual to this. □

Remark 3.3.11. In the proof of $(\prec -\vee)$ in Lemma 3.3.8, out of the conditions on $\propto$, we need the openness of $\propto$, $({\uparrow}{\propto}{\downarrow})$, and $({\propto}{\downarrow}{\downarrow})$. Furthermore if $\propto$ is a binary relation on a Priestley space X such that $\propto$ is open in the product topology on X and satisfies $({\uparrow}{\propto}{\downarrow})$ and $\prec_\propto$ satisfies $(\prec -\vee)$ then it is not hard to prove condition $({\propto}{\downarrow}{\downarrow})$ for $\propto$.

Theorem 3.3.12. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then $Prox(X) = \langle \mathcal{U}^{\mathcal{T}}(X); \cup, \cap, \emptyset, P; \prec_\propto\rangle$ is a strong proximity lattice.*

Proof. Clearly $\langle \mathcal{O}^{\mathcal{T}}(X); \cup, \cap, \emptyset, X\rangle$ is a bounded distributive lattice. Lemma 3.3.8 proves that $\prec_\propto$ satisfies $(\prec\prec)$. $(\vee\!-\prec)$ and $(\prec-\wedge)$ only require Boolean manipulation similar to the finite case. Conditions $(\prec-\vee)$ and $(\wedge\!-\prec)$ are proved by Lemma 3.3.10. □

3.3.2 From Strong Proximity Lattices to Priestley Spaces with Apartness

We remind the reader of the following. For a lattice L, the set of prime filters of L is denoted by $\mathit{filt}_p(L)$. This is ordered by inclusion and equipped with the *Priestley topology* $\mathcal{T}_L$ generated by the collections $U_x = \{F \in \mathit{filt}_p(L) \mid x \in F\}$ and $O_x = \{F \in \mathit{filt}_p(L) \mid x \notin F\}$. Obviously, $O_x = \mathit{filt}_p(L) \setminus U_x$ and so each U_x is a clopen upper, and each O_x a clopen lower set.

Definition 3.3.13. Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. We set

$$\mathit{Pries}(L) = \langle \mathit{filt}_p(L); \subseteq, \mathcal{T}_L; \propto_\prec\rangle,$$

where $\propto_\prec$ is the binary relation defined on $\mathit{filt}_p(L)$ as follows:

$$F \propto_\prec G \overset{\text{def}}{\iff} (\exists\, x \in F)(\exists\, y \notin G)\; x \prec y.$$

Remark 3.3.14. In the finite case, the definition of apartness relation on $\mathit{filt}_p(L)$ given in Definition 3.3.13 cuts down to that of apartness relation on $\mathcal{J}(L)$ given in Definition 3.2.9. This can be proved as follows.

Recall from Section 2.2 that for a finite distributive lattice L the set of prime filters $\mathit{filt}_p(L)$ corresponds to that of join-irreducible elements $\mathcal{J}(L)$ as follows:

$$F \in \mathit{filt}_p(L) \iff (\exists a \in \mathcal{J}(L))\; F = {\uparrow} a.$$

Suppose $F, G \in \mathit{filt}_p(L)$. Then $F = \uparrow a$ and $G = \uparrow b$ for some $a, b \in \mathcal{J}(L)$. Therefore

$$\begin{aligned} F \propto_{\prec} G &\iff (\exists\, x \in F)(\exists\, y \notin G)\; x \prec y \\ &\iff (\exists\, x \geq a)(\exists\, y \not\geq b)\; x \prec y \\ &\iff (\exists\, x \geq a)(\exists\, y \in L \backslash \uparrow b)\; x \prec y \\ &\iff a \prec \bigvee (L \backslash \uparrow b) = g(b), \text{ by } (\prec - \wedge), (\prec - \vee) \\ &\iff a \propto_{\prec} b. \end{aligned}$$

Remark 3.3.15. $\prec = \leq \quad \Longrightarrow \quad \propto_{\prec} = \not\subseteq$.

We will now show that $\propto_{\prec}$ does indeed validate the requirements for an apartness. The following preparatory result extends the definition of apartness to the basic clopen sets U_x and O_x.

Lemma 3.3.16. *Let* $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ *be a strong proximity lattice and* $x, y \in L$. *Then*

$$x \prec y \iff U_x \propto_{\prec} O_y.$$

Proof. $(\Longrightarrow)$ is clear. For the other direction, suppose $x, y \in L$ such that $x \not\prec y$. Set $\xi = \{I \in idl(L) \mid y \in I \text{ and } (\forall t \in I)\; x \not\prec t\}$. $\xi \neq \emptyset$ because $\downarrow y \in \xi$. $(\xi, \subseteq)$ is a poset. If $\{I_i\}$ is a non-empty chain in $(\xi, \subseteq)$ then clearly $\bigcup_i I_i \in \xi$. Therefore by Zorn's Lemma ξ has a maximal element J. We claim that J is prime. Suppose $a, b \in L \setminus J$ but $a \wedge b \in J$. $J_a = \downarrow\{a \vee c \mid c \in J\}$ is an ideal properly containing J. Because J is maximal in ξ, $J_a \notin \xi$. So there exists $c_a \in J$ such that $x \prec a \vee c_a$. Similarly, there exists $c_b \in J$ such that $x \prec b \vee c_b$. Now, by [54, Lemma 7], we note the following:

$$\begin{aligned} x \prec a \vee c_a \text{ and } x \prec b \vee c_b &\Longrightarrow x \prec (a \vee c_a) \vee c_b \text{ and } x \prec (b \vee c_b) \vee c_a \\ &\iff x \prec ((a \vee c_a) \vee c_b) \wedge ((b \vee c_b) \vee c_a) \\ &\iff x \prec (a \wedge b) \vee (c_a \vee c_b). \end{aligned}$$

This gives a contradiction, because $J \in \xi$ and $(a \wedge b) \vee (c_a \vee c_b) \in J$. Therefore J is a prime ideal.

Set $\zeta = \{F \in \mathit{filt}(L) \mid x \in F$ and $(\forall a \in F)(\forall b \in J)\ a \not\prec b\}$. $\zeta \neq \emptyset$ because $\uparrow x \in \zeta$. $(\zeta, \subseteq)$ is a poset. If $\{F_i\}$ is a non-empty chain in $(\zeta, \subseteq)$ then clearly $\cup_i F_i \in \zeta$. Hence by Zorn's Lemma ζ has a maximal element F. We claim that F is prime. Suppose $a, b \in L \setminus F$ but $a \vee b \in F$. Then $F_a = \uparrow\{a \wedge c \mid c \in F\}$ is a filter properly containing F. Because F is maximal in ζ, $F_a \notin \zeta$. So there exists $c_a \in F$ and $d_a \in J$ such that $a \wedge c_a \prec d_a$. Similarly, there exists $c_b \in F$ and $d_b \in J$ such that $a \wedge c_b \prec d_b$. By [54, Lemma 7], we note the following:

$$\begin{aligned}
a \wedge c_a \prec d_a \text{ and } b \wedge c_b \prec d_b \quad &\Longrightarrow \quad (a \wedge c_a) \wedge c_b \prec d_a \vee d_b \\
&\qquad \text{and } (b \wedge c_b) \wedge c_a \prec d_b \vee d_a \\
&\Longleftrightarrow \quad ((a \wedge c_a) \wedge c_b) \vee ((b \wedge c_b) \wedge c_a) \prec d_a \vee d_b \\
&\Longleftrightarrow \quad (a \vee b) \wedge (c_a \wedge c_b) \prec d_a \vee d_b.
\end{aligned}$$

The last statement is a contradiction, because $(a \vee b) \wedge (c_a \wedge c_b) \in F$, $d_a \vee d_b \in J$ and $F \in \zeta$. Hence F is a prime filter. Set $G = L \setminus J$. Then F and G are prime filters with

$$x \in F,\ y \notin G, \text{ and } F \not\propto_{\prec} G$$

which completes the proof. □

Lemma 3.3.17. *Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a strong proximity lattice. Then the relation $\propto_{\prec}$ of $\mathit{Pries}(L)$ is open in $\mathcal{T}_L \times \mathcal{T}_L$ and satisfies* $(\uparrow\propto\downarrow)$.

Proof. Clearly $\propto_{\prec}$ satisfies $(\uparrow\propto\downarrow)$. Now suppose $F \propto_{\prec} G$. Then there exist $x, y \in L$ such that $x \prec y$, $x \in F$ and $y \notin G$. Therefore

$$\{F \in \mathit{filt}_p(L) \mid x \in F\} \propto_{\prec} \{G \in \mathit{filt}_p(L) \mid y \notin J\}.$$

But these sets are open in the Priestley space which proves the openness of $\propto_{\prec}$. □

Remark 3.3.18. Let $\langle L; \vee, \wedge, 0, 1; \prec \rangle$ be a strong proximity lattice. Then the relation $\propto_{\prec}$ of $\mathit{Pries}(L)$ satisfies Lemma 3.3.3 by Lemma 3.3.17.

Lemma 3.3.19. *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the relation $\propto_\prec$ of $Pries(L)$ satisfies* $(\propto\forall)$.

Proof. For any $F, G \in \mathit{filt}_p(L)$,

$$\begin{aligned} F \propto_\prec G &\iff (\exists\, a \in F)(\exists\, c \in L\backslash G)\; a \prec c \\ &\iff (\exists\, a \in F)(\exists\, c \in L\backslash G)(\exists\, b \in L)\; a \prec b \text{ and } b \prec c,\ \text{by } (\prec\prec) \\ &\iff (\exists\, a \in F)(\exists\, c \in L\backslash G)(\exists\, b \in L)\; U_a \propto_\prec O_b \text{ and } U_b \propto_\prec O_c, \\ &\qquad \text{by Lemma 3.3.16} \\ &\iff (\forall H \in \mathit{filt}_p(L))\; F \propto_\prec H \text{ or } H \propto_\prec G. \end{aligned}$$

The right-to-left direction of the last equivalence is proved as follows. Set $O = \{H \in \mathit{filt}_p(L) \mid H \propto_\prec G\}$. Then O is an open set with $F \propto_\prec (\mathit{filt}_p(L)\backslash O)$ and $O \propto_\prec G$. Finally apply Lemma 3.3.3 to get a clopen lower set A around $\mathit{filt}_p(L) \setminus O$ with $F \propto_\prec A$ and hence $\mathit{filt}_p(L) \setminus A \propto_\prec G$. By the compactness of $\mathit{filt}_p(L) \setminus O$ the set A can be chosen to be of the form O_b. □

Lemma 3.3.20. *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice. Then the relation $\propto_\prec$ of $Pries(L)$ satisfies* $(\propto\downarrow\downarrow)$ *and* $(\uparrow\uparrow\propto)$.

Proof. Let $F, G, H \in \mathit{filt}_p(L)$ be such that $F \propto_\prec (\downarrow H \cap \downarrow G)$. Recall that $\downarrow H$ and $\downarrow G$ are closed subsets of $\mathcal{T}_L$. Recalling Remark 3.3.18, we can apply Lemma 3.3.3 and 3.3.1 to get $a, x, y \in L$ such that $F \in U_a$, $\downarrow H \subseteq O_x$, $\downarrow G \subseteq O_y$ and

$$U_a \propto_\prec O_x \cap O_y = O_{x \vee y}.$$

Hence, by Lemma 3.3.16, $a \prec x \vee y$. Now we have

$$\begin{aligned} a \prec x \vee y \quad &\text{hence} \quad (\exists\, x', y' \in L)\; x' \prec x,\ y' \prec y \text{ and } a \prec x' \vee y',\ \text{by } (\prec -\vee) \\ &\text{hence} \quad U_{x'} \propto_\prec O_x,\ U_{y'} \propto_\prec O_y \text{ and } U_a \propto_\prec O_{x' \vee y'} \text{ by Lemma 3.3.16} \\ &\text{hence} \quad (\forall K \in \mathit{filt}_p(L))\; K \propto_\prec H, K \propto_\prec G \text{ or } F \propto_\prec K. \end{aligned}$$

The argument for $(\uparrow\uparrow\propto)$ is dual. □

Lemmas 3.3.17, 3.3.19, and 3.3.20 prove:

Theorem 3.3.21. *Let* $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ *be a strong proximity lattice. Then the relation* $\propto_\prec$ *of* $Pries(L)$ *is an apartness on the Priestley space* $\langle \mathit{filt}_p(L); \subseteq, \mathfrak{T}_L\rangle$.

3.3.3 One Duality and Two Equivalences

Objects

This subsection shows that the translations of the previous two sub-sections are inverses of each other. Our representation theorem relies on the Priestley duality, therefore only the behaviour of strong proximity and apartness need to be studied.

Theorem 3.3.22. *Let* $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ *be a strong proximity lattice. Then the map*

$$\eta_L : L \longrightarrow Prox(Pries(L)); x \longmapsto U_x,$$

is a proximity isomorphism.

Proof. By Priestley duality, η_L is a lattice isomorphism. For every $x, y \in L$,

$$\begin{aligned} x \prec y &\iff U_x \propto_\prec O_y = \mathit{filt}_p(L)\backslash U_y, \text{ by Lemma 3.3.16} \\ &\iff U_x \prec_{\propto_\prec} U_y \\ &\iff \eta_L(x) \prec_{\propto_\prec} \eta_L(y). \end{aligned}$$

□

Definition 3.3.23. Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. A map $f : X_1 \longrightarrow X_2$ is said to be:

- an *apartness map* from X_1 to X_2 if it is continuous, order-preserving, and for every $a, b \in X_1$,

$$f(a) \propto_2 f(b) \Longrightarrow a \propto_1 b,$$

- an *apartness homeomorphism* from X_1 to X_2 if it is an order-isomorphism that is also a homeomorphism, and for every $a, b \in X_1$,

$$a \propto_1 b \Longleftrightarrow f(a) \propto_2 f(b).$$

Theorem 3.3.24. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the map*

$$\epsilon_X : X \longrightarrow Pries(Prox(X)); x \longmapsto \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid x \in U\},$$

is an apartness homeomorphism.

Proof. By Priestley duality, ϵ_X is an order-isomorphism and a homeomorphism from X onto $\mathit{filt}_p(\mathcal{U}^{\mathcal{T}}(X))$. For every $x, y \in X$, we have

$$\begin{aligned} x \propto y &\iff (\exists U \in \mathcal{U}^{\mathcal{T}}(X))(\exists V \in \mathcal{O}^{\mathcal{T}}(X))\ x \in U, y \in V \text{ and } U \propto V \\ &\iff (\exists U \in \mathcal{U}^{\mathcal{T}}(X))(\exists V \in \mathcal{O}^{\mathcal{T}}(X))\ x \in U, y \in V \text{ and } U \prec_{\propto} X \backslash V \\ &\iff \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid x \in U\} \propto_{\prec_\propto} \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid y \in U\} \\ &\iff \epsilon_X(x) \propto_{\prec_\propto} \epsilon_X(y). \end{aligned}$$

The first equivalence is true by Lemma 3.3.3. □

Lemma 3.3.25. *Let $\langle L; \vee, \wedge, 0, 1; \prec\rangle$ be a strong proximity lattice and $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$.*

1. *The relation $\propto_{\prec}$ of $Pries(L)$ satisfies the following:*

$$\prec = \leq \iff \propto_{\prec} = \not\leq.$$

2. *The relation $\prec_{\propto}$ of $Prox(X)$ satisfies the following:*

$$\propto = \not\leq \iff \prec_{\propto} = \leq.$$

Proof. 1. The left-to-right direction follows from Remark 3.3.15. For the other direction, suppose that $\propto_{\prec} = \not\leq$. Then by Remark 3.3.15 $\prec_{\propto_{\prec}} = \subseteq$. For any $x, y \in L$,

$$\begin{aligned} x \prec y &\iff \eta_L(x) \prec_{\propto_{\prec}} \eta_L(y), \text{ by Theorem 3.3.22} \\ &\iff \{F \in \mathit{filt}_p(L) \mid x \in F\} \subseteq \{F \in \mathit{filt}_p(L) \mid y \in F\} \\ &\iff x \leq y. \end{aligned}$$

The last equivalence is true by Lemma A.16 after identifying J and G of the lemma with $\downarrow y$ and $\uparrow x$, respectively.

2. The left-to-right direction follows from Remark 3.3.15. For the other direction, suppose that $\prec_{\propto} = \leq$. Then by Remark 3.3.15 $\propto_{\prec_{\propto}} = \not\subseteq$. For any $x, y \in L$,

$$\begin{aligned} x \propto y &\iff \epsilon_X(x) \prec_{\propto_{\prec}} \epsilon_X(y), \text{ by Theorem 3.3.24} \\ &\iff \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid x \in U\} \not\subseteq \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid y \in U\} \\ &\iff x \not\leq y, \text{ by the Priestley's separation condition .} \end{aligned}$$

□

Remark 3.3.26. Since we have shown that the trivial apartness relations ($\propto = \not\leq$) and strong proximity relations ($\prec = \leq$) get translated into each other, it is clear that our representation theorem is a proper extension of that of Priestley.

Morphisms

Morphism I: Apartness Maps and Proximity Homomorphisms

Lemma 3.3.27. *Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively, and $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices.*

1. *For a proximity homomorphism $f : L_1 \longrightarrow L_2$, the map*

$$Pries(f) : Pries(L_2) \longrightarrow Pries(L_1); F \longmapsto f^{-1}(F),$$

is an apartness map.

2. *For an apartness map $\varphi : X_1 \longrightarrow X_2$, the function*

$$Prox(\varphi) : Prox(X_2) \longrightarrow Prox(X_1); U \longmapsto \varphi^{-1}(U),$$

is a proximity homomorphism.

Proof. 1. $Pries(f)$ is a well defined continuous order preserving map by [20, Theorem 11.31]. For every $F, G \in \mathit{filt}_p(L_2)$,

$$\begin{aligned} Pries(f)(F) \propto_{\prec_1} Pries(f)(G) &\Longrightarrow (\exists\, a \in Pries(f)(F)) \\ & (\exists\, b \notin Pries(f)(G))\ a \prec_1 b \\ &\Longrightarrow f(a) \prec_2 f(b) \\ &\Longrightarrow F \propto_{\prec_2} G, \text{ because } f(a) \in F \\ & \text{ and } f(b) \notin G. \end{aligned}$$

2. $Prox(\varphi)$ is a lattice homomorphism by [20, Theorem 11.31]. We prove that it preserves the strong proximity relation. For every $U_1, U_2 \in \mathcal{U}^{\mathcal{T}}(X_2)$,

$$\begin{aligned} U_1 \prec_{\propto_2} U_2 &\Longrightarrow U_1 \propto_2 X_2 \backslash U_2 \\ &\Longrightarrow \varphi^{-1}(U_1) \propto_1 \varphi^{-1}(X_2 \setminus U_2) \\ &\Longrightarrow \varphi^{-1}(U_1) \propto_1 X_1 \setminus \varphi^{-1}(U_2) \\ &\Longrightarrow \varphi^{-1}(U_1) \prec_{\propto_1} \varphi^{-1}(U_2) \\ &\Longrightarrow Prox(\varphi)(U_1) \prec_{\propto_1} Prox(\varphi)(U_2). \end{aligned}$$

□

We define **PS** to be the category whose objects are Priestley spaces equipped with apartness relations, and whose morphisms are apartness maps. We also define **PL** to be the the category of strong proximity lattices and proximity homomorphisms. Therefore theorems 3.3.22 and 3.3.24 and Lemma 3.3.27 show that the classical Priestley duality between distributive lattices and Priestley spaces can be extended to the (non-full) categories **PL** and **PS**. To summarise:

Theorem 3.3.28. *The functors $Pries$ and $Prox$ establish a dual equivalence between the categories* **PS** *and* **PL***.*

Remark 3.3.29. All the information about $Pries(f)$ and $Prox(\varphi)$ stated in Remark 3.2.25 is still valid for the general case by Priestley's representation theorem [20, Theorem 11.31].

Morphism II: Approximable Relations and Separators

As explained in the introduction, to cover all Scott-continuous maps (representing computable programs), between stably compact spaces, Jung and Sünderhauf introduced wider classes of morphisms for lattices; strong proximity lattices were equipped with approximable and weakly approximable relations (Definition 2.7.2). **PLa** (**PLwa**) is the category whose objects are strong proximity lattices and whose morphisms are approximable (weakly approximable) relations. In this category, the composition is given by relational product.

The counterpart of **PLa** (**PLwa**) is the category **PSs** (**PSws**) of Priestley spaces equipped with apartness relations as objects, and separating (weakly separating) relations (Definition 3.1.5) as morphisms. To fill all the mathematical gaps, in the following, the first objective is to introduce the translation between separating and approximable relations. The other objective is to present identities and a definition of composition, and then show that the laws for a category are satisfied for **PSs** and **PSws**.

Definition 3.3.30. Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$

and $\propto_2$, respectively, and let $\ltimes\, \subseteq X_1 \times X_2$ be a separating relation. For $A \in \mathit{Prox}(X_1) = \mathcal{U}^{\mathcal{T}}(X_1)$ and $B \in \mathit{Prox}(X_2) = \mathcal{U}^{\mathcal{T}}(X_2)$,

$$A \vdash_{\ltimes} B \overset{\text{def}}{\iff} A \ltimes (X_2 \backslash B).$$

Definition 3.3.31. Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be an approximable relation from L_1 to L_2. For $F \in \mathit{Pries}(L_1) = \mathit{filt}_p(L_1)$ and $G \in \mathit{Pries}(L_2) = \mathit{filt}_p(L_2)$,

$$F \ltimes_{\vdash} G \overset{\text{def}}{\iff} (\exists\, x \in F)(\exists\, y \notin G)\; x \vdash y.$$

The following facts are proved similarly to their counterparts earlier in this chapter.

Lemma 3.3.32. *Let X_1 and X_2 be Priestley spaces and let $\ltimes\, \subseteq X_1 \times X_2$ be a binary relation that is open in the product topology and satisfies condition $(\uparrow_1\!\ltimes\!\downarrow_2)$. For closed subsets $A \subseteq X_1$ and $B \subseteq X_2$, if $A \ltimes B$ then there exist $U \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $V \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $A \subseteq U$, $B \subseteq V$ and $U \ltimes V$.*

Lemma 3.3.33. *Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be an approximable relation from L_1 to L_2. Then*

$$(\forall a \in L_1)(\forall b \in L_2)\; a \vdash b \iff U_a \ltimes_{\vdash} O_b.$$

Theorem 3.3.34. *Let X_1 and X_2 be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively, and let $\ltimes\, \subseteq X_1 \times X_2$ be a separating relation. Then the relation $\vdash_{\ltimes}$ satisfies $(\vdash - \prec_2)$, $(\prec_1 - \vdash)$, $(\vee - \vdash)$, $(\vdash - \wedge)$ and $(\vdash - \vee)$. So it is an approximable relation between $\mathit{Prox}(X_1)$ and $\mathit{Prox}(X_2)$.*

Theorem 3.3.35. *Let $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ and $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ be strong proximity lattices and let $\vdash$ be an approximable relation from L_1 to L_2. Then the relation $\ltimes_{\vdash}$ satisfies $(\uparrow_1\!\ltimes\!\downarrow_2)$, $(\forall\ltimes)$, $(\ltimes\forall)$ and $(\ltimes n\!\downarrow)$. Therefore it is a separating relation between $\mathit{Pries}(L_1)$ and $\mathit{Pries}(L_2)$.*

Lemma 3.3.36. *Let* $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ *and* $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ *be strong proximity lattices and let* $\vdash$ *be an approximable relation from* L_1 *to* L_2*. Let* X_1 *and* X_2 *be Priestley spaces equipped with apartness relations* $\propto_1$ *and* $\propto_2$*, respectively, and let* $\ltimes \subseteq X_1 \times X_2$ *be a separating relation. Then*

- $x \ltimes y \iff \epsilon_{X_1}(x) \ltimes_{\vdash_\ltimes} \epsilon_{X_2}(y)$.
- $a \vdash b \iff \eta_{L_1}(a) \vdash_{\ltimes_\vdash} \eta_{L_2}(b)$.

Proposition 3.3.37. *Let* $\langle X; \leq, \mathfrak{T}\rangle$ *be a Priestley space with apartness* $\propto$*. Then* $\propto$ *is a separating relation from* X *to* X*.*

Proof. $(\uparrow_1\ltimes\downarrow_2)$, $(\forall\ltimes)$ and $(\ltimes\forall)$ are clearly satisfied. $(\ltimes n\downarrow)$ is proved by induction on n as follows. The cases where $n = 0, 1$ or 2 are clear. For the induction hypothesis, suppose $(\ltimes n\downarrow)$ is true for $n = m$ and $m \geq 2$. We now prove that $(\ltimes n\downarrow)$ is true for $n = m + 1$. Suppose that for some $b \in X$ and $\{d_1 \dots, d_{m+1}\} \subseteq X$, $b \propto \bigcap_{1\leq i\leq m+1} \downarrow_i d_i$. We note the following

$$\begin{aligned}\bigcap_{1\leq i\leq m+1} \downarrow d_i &= \bigcap_{1\leq i\leq m} \downarrow d_i \cap \downarrow d_{m+1}\\ &= \bigcup\{\downarrow t \mid t \in \bigcap_{1\leq i\leq m} \downarrow d_i\} \cap \downarrow d_{m+1}\\ &= \bigcup\{\downarrow t \cap \downarrow d_{m+1} \mid t \in \bigcap_{1\leq i\leq m} \downarrow d_i\}\end{aligned}$$

Therefore

$$b \propto \bigcap_{1\leq i\leq m+1} \downarrow_i d_i \iff (\forall t \in \bigcap_{1\leq i\leq m} \downarrow d_i)\, b \propto (\downarrow t \cap \downarrow d_{m+1}).$$

Now let $c \in X$ be such that $b \not\propto c$ and $c \not\propto d_{m+1}$. Then by $(\propto\downarrow\downarrow)$, $(\forall t \in \bigcap_{1\leq i\leq m} \downarrow d_i)$ $c \propto t$ and by the induction hypothesis there exists $1 \leq j \leq m$ such that $c \propto d_j$ which completes the proof. □

Definition 3.3.38. Let X_1, X_2, and X_3 be Priestley spaces equipped with apartness relations $\propto_1, \propto_2$, and $\propto_3$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ and $\ltimes' \subseteq X_2 \times X_3$ be separating relations. The composition $\ltimes \circ \ltimes' \subseteq X_1 \times X_3$ is defined as follows:

$$(\forall - comp) \qquad a \ltimes \circ \ltimes' c \overset{\text{def}}{\iff} (\forall b \in X_2)\ a \ltimes b \text{ or } b \ltimes' c.$$

The following technical lemma is needed to show that the composition of two separators satisfies $(\ltimes n \downarrow)$:

Lemma 3.3.39. *Let X_1, and X_2 be Priestley spaces equipped with apartness relations $\propto_1$, and $\propto_2$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ be a separating relation. For every $a \in X_1$ and $\{A_i \mid 1 \leq i \leq n\} \subseteq \mathcal{O}^{\mathcal{T}}(X_2)$,*

$$a \ltimes \bigcap_i A_i \Longrightarrow (\forall x \in X_1)\ a \propto_1 x \textit{ or } (\exists i)\ x \ltimes A_i.$$

Proof. Suppose by way of contradiction that $x \in X_1$ and $d_i \in A_i$ such that $a \not\propto_1 x$ and $(\forall i)\ x \not\ltimes d_i$. On the other hand, we note that $a \ltimes \bigcap \downarrow d_i \subseteq \bigcap A_i$. Then by $(\ltimes n \downarrow)$ there exists i such that $x \ltimes d_i$, which is a contradiction. □

Lemma 3.3.40. *Let X_1, X_2, and X_3 be Priestley spaces equipped with apartness relations $\propto_1, \propto_2$, and $\propto_3$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ and $\ltimes' \subseteq X_2 \times X_3$ be (weakly) separating relations. Then the composition of $\ltimes$ and $\ltimes'$ is again a (weakly) separating relation.*

Proof. Suppose $x \ltimes \circ \ltimes' y$ and set $O = \{t \in X_2 \mid t \ltimes' y\}$. Then O is an open subset of X_2 with $x \ltimes (X_2 \backslash O)$. By Lemma 3.3.32, there exist $U \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $V \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $x \in U$, $X_2 \backslash O \subseteq V$ and $U \ltimes V$. Therefore $(X_2 \backslash V) \ltimes' y$. By applying Lemma 3.3.32 again, there exists $W \in \mathcal{O}^{\mathcal{T}}(X_3)$ such that $y \in W$ and $(X_2 \backslash V) \ltimes' W$. Hence $U \ltimes V$ and $(X_2 \backslash V) \ltimes' W$ implying $U \ltimes \circ \ltimes' W$. This proves that $\ltimes \circ \ltimes'$ is open and satisfies $(\uparrow_1 \ltimes \downarrow_2)$.

$(\forall \ltimes)$ is proved as follows. Let $x \ltimes \circ \ltimes' y$ and $z \in X_1$ such that $x \not\propto_1 z$. We claim that $z \ltimes \circ \ltimes' y$. Let $t \in X_2$ such that $z \not\ltimes t$. Therefore, by $(\forall \ltimes)$ $x \not\ltimes t$, and so $t \ltimes' y$, by definition of composition. Hence $z \ltimes \circ \ltimes' y$. $(\ltimes \forall)$ is proved similarly.

$(\ltimes n{\downarrow})$ is proved as follows. In the following let $O_i = \{t \in X_2 \mid t \ltimes' d_i\}$.

$$\begin{aligned}
b \ltimes \circ \ltimes' \bigcap {\downarrow} d_i &\implies (\forall r \in X_2)\ b \ltimes r \text{ or } r \ltimes' \bigcap {\downarrow} d_i \\
&\implies (\forall r, t \in X_2)\ b \ltimes r, r \propto_2 t \text{ or } (\exists i)\ t \ltimes' d_i,\ \text{by } (\ltimes n{\downarrow}) \\
&\implies (\forall t \in X_2)\ b \ltimes t \text{ or } (\exists i)\ t \ltimes' d_i,\ \text{by } (\ltimes \forall) \\
&\implies b \ltimes \bigcap_i (X_2 \backslash O_i) \text{ and } (\forall i)\ O_i \ltimes'\ d_i \\
&\implies (\forall c \in X_1)\ b \propto_1 c \text{ or } (\exists i)\ c \ltimes (X_2 \backslash O_i) \\
&\qquad \text{and } O_i \ltimes' d_i,\ \text{Lemma 3.3.39} \\
&\implies (\forall c \in X_1)\ b \propto_1 c \text{ or } (\exists i)\ c \ltimes \circ \ltimes' d_i
\end{aligned}$$

□

Lemma 3.3.41. **PSs** *and* **PSws** *are categories.*

Proof. For associativity of composition we compute:

$$\begin{aligned}
x\,(\ltimes \circ \ltimes') \circ \ltimes''\, y &\iff (\forall r \in X_3)\ x \ltimes \circ \ltimes' r \text{ or } r \ltimes'' y \\
&\iff (\forall r \in X_3)(\forall s \in X_2)\ x \ltimes s,\ s \ltimes' r \text{ or } r \ltimes'' y \\
&\iff (\forall s \in X_2)\ x \ltimes s \text{ or } s \ltimes' \circ \ltimes'' y \\
&\iff x \ltimes \circ (\ltimes' \circ \ltimes'')\, y.
\end{aligned}$$

Identities are given by the internal apartness relations which is a valid choice by Lemma 3.3.37. They satisfy $\propto_1 \circ \ltimes = \ \ltimes$ and $\ltimes \circ \propto_2\ =\ \ltimes$ by definition. □

Theorem 3.3.42. *The categories* **PSs** *(***PSws***) and* **PLa** *(***PLwa***) are equivalent to each other.*

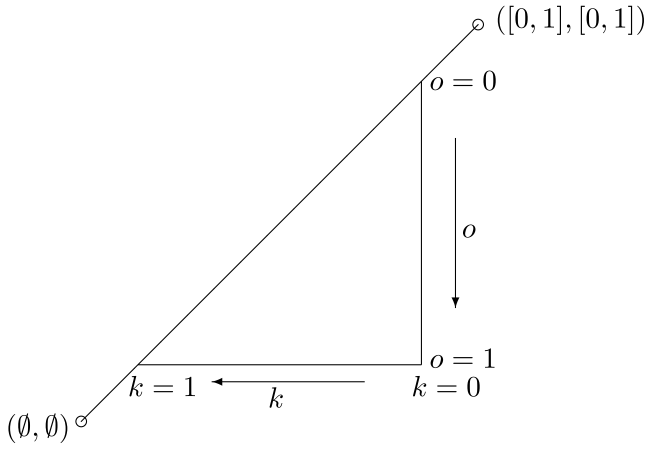


Figure 3.4: The lattice $\mathcal{B}$.

3.3.4 Examples

The Unit Interval

Consider the strong proximity lattice:

$$\mathcal{B} = \{((o,1],[k,1]) \mid 0 \leq k \leq o \leq 1\} \cup \{(\emptyset,\emptyset),([0,1],[0,1])\},$$

where the order is the component-wise inclusion and the strong proximity relation defined as follows:

$$((o,1],[k,1]) \prec ((o',1],[k',1]) \overset{\text{def}}{\Longleftrightarrow} o' < k.$$

It represents the unit interval (with the Scott topology) as a stably compact space under the Jung-Sünderhauf duality. Figure 3.4 gives a pictorial description of the lattice $\mathcal{B}$. A point (o,k) in the triangle of Figure 3.4 represents the point $((o,1],[k,1])$ of the lattice $\mathcal{B}$. Every point of the figure is greater than or equal all points to its south west and is less than or equal all points to its north east.

For a given $r \in [0,1]$, we define a horizontal line hl_r and a vertical line vl_r of the lattice $\mathcal{B}$ as follows:

$$hl_r = \{((r,1],[k,1]) \mid 0 \leq k \leq r\},\ vl_r = \{((o,1],[r,1]) \mid r \leq o \leq 1\}.$$

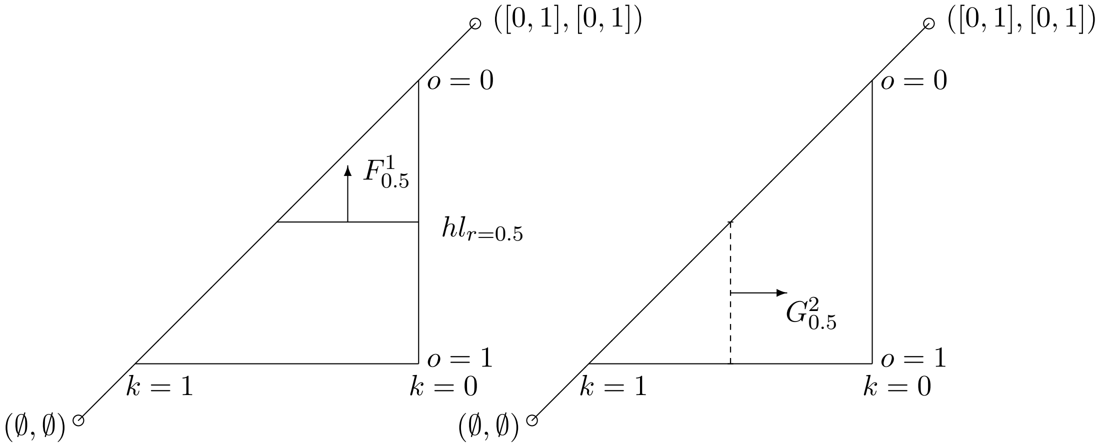


Figure 3.5: The prime filters $F^1_{0.5}$ and $G^2_{0.5}$ of the lattice $\mathcal{B}$.

The following facts describe the Priestley dual space $\langle \mathit{filt}_p(\mathcal{B}); \mathcal{T}, \leq, \propto_{\prec}\rangle$ of $\mathcal{B}$.

1. The set of prime filters $\mathit{filt}_p(\mathcal{B})$ of $\mathcal{B}$ can be described concretely as follows:

$$\mathit{filt}_p(\mathcal{B}) = \{F^1_r, F^2_r, G^1_r, \text{ and } G^2_r \mid 0 \leq r \leq 1\},$$

   where

$$F^1_r = \uparrow hl_r,\ F^2_r = \uparrow hl_r \setminus hl_r,\ G^1_r = \uparrow vl_r \text{ and } G^2_r = \uparrow vl_r \setminus vl_r.$$

   A pictorial descriptions of $F^1_{0.5}$ and $G^2_{0.5}$ is given in Figure 3.5.

   For a computational interpretation, assume some concrete representation of real numbers as finite and infinite streams of digits. A stream that begins with 0.5 and then stops explicitly (indicating that all following digits are zero) corresponds to value $G^2_{0.5}$ in that it validates all tests $x < 0.5 + \epsilon$ with $\epsilon > 0$. On the other hand, a stream that begins with 0.4 and then produces 9's forever, corresponds to $G^1_{0.5}$ in that the test $x < 0.5$ does not produce "false" in finite time. No test can distinguish between the two streams by looking at a finite initial segment of digits; the most we have is a test $(x < 0.5)$ which terminates for one and never answers for the other.

2. The order $\leq$ on $\mathit{filt}_p(\mathcal{B})$ is inclusion. Figure 3.6 shows the space $\mathit{filt}_p(\mathcal{B})$ of prime filters of $\mathcal{B}$.

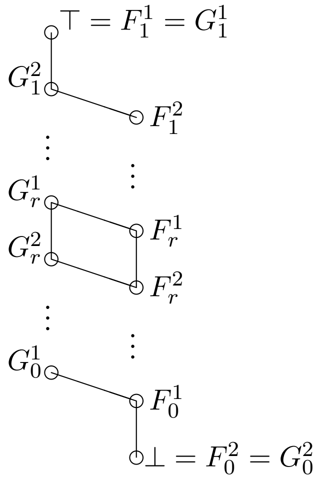


Figure 3.6: The space $\mathit{filt}_p(\mathcal{B})$.

3. The following collection is a sub-basis for $\mathcal{T}$,

$$\mathcal{S}(\mathcal{T}) = \{\downarrow F_r^2, \downarrow G_r^2, \uparrow F_r^1, \text{ and } \uparrow G_r^1 \mid 0 \leq r \leq 1\}.$$

4. For any two prime filters $A_r \in \{F_r^1, F_r^2, G_r^1, G_r^2\}$ and $B_{r'} \in \{F_{r'}^1, F_{r'}^2, G_{r'}^1, G_{r'}^2\}$, where $r, r' \in [0, 1]$:

$$A_r \propto_{\prec} B_{r'} \iff r > r'.$$

Proof. 1. It is easy to check that

$$\{F_r^1, F_r^2, G_r^1, \text{ and } G_r^2 \mid 0 \leq r \leq 1\} \subseteq \mathit{filt}_p(\mathcal{B}).$$

For the other inclusion, let $F \in \mathit{filt}_p(\mathcal{B})$. If $F = \mathcal{B} \backslash \{\bot_{\mathcal{B}}\}$ or $F = \{\top_{\mathcal{B}}\}$ then F is clearly in $\{F_r^1, F_r^2, G_r^1, \text{ and } G_r^2 \mid 0 \leq r \leq 1\}$. Therefore we assume that F does not have any of these forms. If two points (o, k) and (o', k') belong to F then their meet $(\min(o, o'), \min(k, k'))$ belongs to F. Therefore F is of the form $\uparrow(o, k)$ or $F = \uparrow(o, k) \backslash (hl_o \cup vl_k)$ for some (o, k) in $\mathcal{B}$. Now since F is prime it must belong to $\{F_r^1, F_r^2, G_r^1, \text{ and } G_r^2 \mid 0 \leq r \leq 1\}$.

2. Clearly the order on the space $\mathit{filt}_p(\mathcal{B})$ is inclusion.

3. For $((o,1],[k,1]) \in \mathcal{B}$,

   (a) $U_{((o,1],[k,1])} = \uparrow F_o^1 \cup \uparrow G_k^1$,

   (b) $O_{((o,1],[k,1])} = \downarrow F_o^2 \cup \downarrow G_k^2$, and

   (c) $U_{([0,1],[0,1])} = X$ and $U_{(\emptyset,\emptyset)} = \emptyset$.

   Hence the set $\{\downarrow F_o^2 \cup \downarrow G_k^2, \uparrow F_o^1 \cup \uparrow G_k^1 | \; 0 \leq k \leq r \leq 1\}$ is sub-basis for $\mathcal{T}$ and therefore the $\{\downarrow F_r^2, \; \downarrow G_r^2, \; \uparrow F_r^1, \; \uparrow G_r^1 \mid 0 \leq r \leq 1\}$ is a also a sub-basis for $\mathcal{T}$.

4. Let $r, r' \in [0,1]$, $A_r \in \{F_r^1, F_r^2, G_r^1, G_r^2\}$, and $B_{r'} \in \{F_{r'}^1, F_{r'}^2, G_{r'}^1, G_{r'}^2\}$. Suppose $A_r \propto B_{r'}$. Then there exist $x_1 = ((o_1,1],(k_1,1]) \in A_r$ and $x_2 = ((o_2,1],(k_2,1]) \in \mathcal{B} \backslash B_{r'}$ such that $x_1 \prec_{\mathcal{B}} x_2$. But

   $$x_1 \prec_{\mathcal{B}} x_2 \Longrightarrow [k_1, 1] \subseteq (o_2, 1] \Longrightarrow k_1 > o_2,$$

   $k_1 \leq r$ because $x_1 \in A_r$, and $o_2 \geq r'$ because $x_2 \notin B_r'$. Therefore

   $$r \geq k_1 > o_2 \geq r' \Longrightarrow r > r'.$$

   For the other direction suppose $r > r'$. Set $x_1 = ((r' + \frac{3(r-r')}{4}, 1], [r' + \frac{3(r-r')}{4}, 1])$ and $x_2 = ((r' + \frac{r-r'}{4}, 1], [r' + \frac{r-r'}{4}, 1])$. Then $x_1 \in A_r$ and $x_2 \in \mathcal{B} \backslash A_{r'}$. Moreover, $x_1 \prec x_2$ because $[r' + \frac{3(r-r')}{4}, 1]) \subseteq (r' + \frac{r-r'}{4}, 1]$. Therefore $A_r \propto B_{r'}$.

□

$\mathbb{N}_\infty$

$\mathbb{N}_\infty$ denotes the chain $\mathbb{N}$ with ∞ adjoined as a top element. We define a topology $\mathcal{T}$ on $\mathbb{N}_\infty$ as follows:

$$U \in \mathcal{T} \stackrel{\text{def}}{\Longleftrightarrow} \infty \notin U \text{ or } (\infty \in U \text{and } \mathbb{N}_\infty \backslash U \text{ is finite}).$$

$\langle \mathbb{N}_\infty, \mathcal{T} \rangle$ is a Priestley space and its lattice of clopen upper sets is isomorphic to the chain $\langle \mathbb{N} \oplus 1, \geq \rangle$ [20]. In the following table, the left-hand side column shows examples of apartness relations, on $\langle \mathbb{N}_\infty, \mathcal{T} \rangle$. Their corresponding strong proximity relations on $\mathbb{N} \oplus 1$ are shown in the right-hand side column of the table.

$\propto$ on $\mathbb{N}$	$\prec_\propto$ on $\mathbb{N} \oplus 1$
$>$	$\geq$
$\geq \backslash (\infty, \infty)$	$\geq \cup \{(n, n+1) \mid n \in \mathbb{N}\}$
$\uparrow c \times \downarrow c - 1$, for some $0 < c \in \mathbb{N}$	$n \prec_\propto m \overset{\text{def}}{\Longleftrightarrow} n = 0, m = 1$ or $n \geq c \geq m$

The Cantor 'middle third' Set

The Cantor 'middle third' set C considered as a subset of $[0, 1]$ and ordered by the natural order of reals is a Priestley space [20] and its lattice of clopen upper sets is isomorphic to the chain $B = \{\emptyset, C\} \cup F_R$, where $F_R = \{F \mid F \text{ is a removed interval}\}$, ordered as follows:

$$F_1 \leq F_2 \overset{\text{def}}{\Longleftrightarrow} F_1 \subseteq \uparrow F_2.$$

The following table shows two apartness relations on C and their corresponding strong proximity relations.

$\propto$ on C	$\prec_\propto$ on B
$>$	$\leq$
$(\uparrow r \cap C) \times (\downarrow r \cap C), r \in F \in F_R$	$F_1 \prec F_2 \overset{\text{def}}{\Longleftrightarrow} F_1 = \emptyset, F_2 = C$ or $F_1 \leq F \leq F_2$

The Algebraic Chain

Let C be an algebraic chain (a chain in which every element is the supremum of compact elements (Definition 2.3.7) below it) and $\mathcal{T}$ be the interval topology on C. Then $\langle C; \mathcal{T} \rangle$ is a Priestley space and its lattice of clopen upper sets is

$$B = \{\emptyset, C\} \cup \{\uparrow k \mid k \text{ is finite}\}.$$

Fix a finite element $f \in C$. Then the binary relation

$$\propto_f = \uparrow f \times (\downarrow f \backslash f).$$

is an apartness on C and its corresponding strong proximity relation on B is defined as follows:

$$U \prec_{\propto_k} U' \overset{\text{def}}{\iff} U = \emptyset, U' = C, \text{ or } (U = \uparrow k, U' = \uparrow k' \text{ and } k \geq f \geq k').$$

Ordinals

Let X be the set of all ordinals less than or equal to an ordinal λ ;i.e.

$$X = \{\xi \mid \xi \leq \lambda\},$$

and $\mathcal{T}$ be the order topology on λ; i.e the topology which has a sub-basis consisting of open intervals:

$$\{\xi \mid \xi < \beta\} \text{ and } \{\xi \mid \beta < \xi \leq \lambda\}, \beta \leq \lambda.$$

Then $\langle X; \mathcal{T}\rangle$ is a Priestley space [11, chapter 1] and its lattice of clopen upper sets is

$$B = \{\emptyset, \downarrow\lambda\} \cup \{\downarrow\lambda \backslash \downarrow\beta \mid \beta \leq \lambda\}.$$

Fix an ordinal $\gamma \in X$. Then the binary relation

$$\propto_\gamma = (\downarrow\lambda \backslash \downarrow\gamma) \times (\downarrow\gamma).$$

is an apartness on X and its corresponding strong proximity relation on B is defined as follows:

$$U \prec_{\propto_\gamma} U' \overset{\text{def}}{\iff} U = \emptyset, U' = \downarrow\lambda, \text{ or } (U = \downarrow\lambda \backslash \downarrow\alpha, U' = \downarrow\lambda \backslash \downarrow\alpha' \text{ and } \alpha \geq \gamma \geq \alpha').$$

Remark 3.3.43. Let β be an ordinal which is not a limit ordinal and $\mathcal{T}$ be the order topology on β. Fix an ordinal $\gamma < \beta$. Then $\propto_\gamma$ is an apartness on the Priestley space $\langle \beta; \leq, \mathcal{T}\rangle$. This so because $\beta = \lambda + 1$ for some ordinal λ and the order type of β is $\{\alpha \mid \alpha < \beta\} = \downarrow\lambda$.

The Power Set

Suppose S is a set and $\langle \wp(S), \subseteq \rangle$ is equipped with the topology $\mathfrak{T}$ that have the following subbasis:

$$\{\uparrow\{s\}, \wp(S)\backslash\uparrow\{s\} \mid s \in S\}.$$

Then $\langle \wp(S), \subseteq; \mathfrak{T} \rangle$ is a Priestley space [20]. Fix $s \in S$. Then the binary relation :

$$\propto_s = \uparrow\{s\} \times (\wp(S)\backslash\uparrow\{s\})$$

is an apartness on $\wp(S)$.

Clopen upper subsets of $\wp(S)$ are precisely the sets of the form

$$U_F = \{A \subseteq S \mid \{s_1, \ldots, s_n\} \subseteq A \text{ and } \langle s_1, \ldots, s_n \rangle \in \prod_{1 \leq i \leq n} F_i \},$$

where $F = \{F_i \subseteq_{fin} S \mid 1 \leq i \leq n\}$. Therefore $\prec_{\propto_s}$ is defined on clopen upper sets as follows:

$$U \prec_{\propto_s} U' \overset{\text{def}}{\iff} U = \emptyset, U' = \mathcal{U}^{\mathfrak{T}}(\wp(S)), \text{ or } (U = U_F, U' = U_{F'},$$
$$F_i = \{s\} \text{ for some } i \text{ and } s \notin \bigcup U'_F.$$

3.4 Indistinguishability vs. Apartness

We noted in Lemma 3.3.25 that the strong proximity relation on a strong proximity lattice is the lattice order ($\prec = \leq$) if and only if the corresponding apartness relation on the representing Priestley space is the complement of the space's order ($\propto = \not\leq$). This observation suggests representing the negation of the strong proximity relation (rather than the strong proximity relation itself as we have been doing so far). In other words, representing the negation of the strong proximity relation looks promising in terms of obtaining a simpler binary relation on the other side of the duality (the side of Priestley spaces). The important question now is the following:

How would the primary definitions of interest (apartness relations on ordered sets and Priestley spaces and (weakly) separators) look like provided that we represent the negation of the strong proximity relation rather than the strong proximity relation itself?

In the following, we introduce the answer to this question.

Definition 3.4.1. A binary relation $\rhd$ on an ordered set $\langle P, \leq\rangle$ is *co-apartness* if, for every $a, t, x, y \in P$,

1. $a \leq t \rhd x \leq y \Longrightarrow a \rhd y$.
2. $\rhd \circ \rhd = \rhd$.
3. $(\exists t \in P)\ a \rhd t \rhd x, y \Longrightarrow (\exists t' \in P)\ a \rhd t' \leq x, y$.
4. $(\exists t \in P)\ x, y \rhd t \rhd a \Longrightarrow (\exists t' \in P)\ x, y \geq t' \rhd a$.

where $A \rhd B$ is a shorthand for $a \rhd b$ for all $a \in A$, $b \in B$.

Remark 3.4.2. 1. For any ordered set $\langle P, \leq\rangle$, $\leq$ is a co-apartness.

2. $\rhd$ is a co-apartness on $\langle P; \leq\rangle$ if and only if $\rhd^{-1}$ (the complement of $\rhd$) is a co-apartness on $\langle P; \geq\rangle$.

Definition 3.4.3. A binary relation $\rhd$ on a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ is *co-apartness* if

1. $\rhd$ is closed in $\langle X; \mathcal{T}\rangle \times \langle X; \mathcal{T}\rangle$.
2. $\rhd$ is a co-apartness on the ordered set $\langle X, \leq\rangle$.

Definition 3.4.4. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with co-apartness relations $\rhd_1$ and $\rhd_2$, respectively, and let $\unrhd$ be a binary relation from X_1 to X_2. The relation $\unrhd$ is *co-separating* (or *co-separator*) if it is closed in $\mathcal{T}_1 \times \mathcal{T}_2$ and if, for every $a, b \in X_1$, $c, d \in X_2$, and $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$,

1. $a \leq_1 b \trianglerighteq c \leq_2 d \Longrightarrow a \trianglerighteq d$.

2. $\trianglerighteq = \vartriangleright_1 \circ \trianglerighteq$.

3. $\trianglerighteq = \trianglerighteq \circ \vartriangleright_2$,

4. $(\exists t \in X_1)\; a \vartriangleright_1 t \trianglerighteq d_1, \ldots, d_n \Longrightarrow (\exists t' \in X_2)\; a \trianglerighteq t' \leq d_1, \ldots, d_n$.

The relation $\trianglerighteq$ is *weakly co-separating* if it is closed and satisfies all of the above conditions, but not necessarily the last one.

From the proofs established using apartness relations, in this chapter, two facts should be clear now:

1. Priestley spaces equipped with co-apartness relations and co-separators (weak co-separators) between them are indeed the objects and morphisms, respectively, of a category. We let this category denoted by **CPSs** (**CPSws**).

2. The whole theory established in this chapter can be represented using the categories **CPSs** and **CPSws** rather than **PSs** and **PSws**. The results that we would get using the former categories are analogous to those we have proved in this chapter using the latter categories.

We prefer to work with the categories **PSs** and **PSws** rather than with **CPSs** and **CPSws** for two reasons:

1. Binary relations that are very similar to apartness relations occur naturally in constructive mathematics. Therefore our terminology suggests that apartness relations on Priestley spaces are related to, for example, Giuseppe Sambin's *pre-topologies*, [94, 95, 9]; this is indeed the case as follows. Let $\langle X; \leq, \mathcal{T} \rangle$ be a Priestley space equipped with apartness $\propto$. A lower set O in X is said to be *isolated* if

$$O = \{x \in X \mid X \backslash O \propto x\}.$$

The set of all open lower isolated subsets of X is denoted by $iso_l(X)$.

Theorem 3.4.5. *Let* $\langle X; \leq, \mathfrak{T}\rangle$ *be a Priestley space equipped with apartness* $\propto$*. Then* $iso_l(X)$ *is closed under finite intersections. Therefore* $\langle iso_l(X); \cap, X\rangle$ *is a commutative monoid. The relation* $\lhd$ *on* $iso_l(X)$*, defined by*

$$O_1 \lhd O_2 \stackrel{\text{def}}{\Longleftrightarrow} O_1 \propto (X \backslash O_2),$$

satisfies the requirements for a precover *in the sense of [95].*

2. Experiments show that working with apartness relation is mathematically more convenient; proofs (e.g. that of Lemma 3.2.5) established using apartness relations are more tidy than proofs established using the co-apartness relations.

Chapter 4

Stably Compact Spaces in Priestley Form

4.1 Introduction

In [54], the category **PLa**, of strong proximity lattices and approximable relations between them, was proved to be equivalent to the category **SCS**, of stably compact spaces and continuous maps between them. In the previous chapter, we have extended Priestley duality to cover the class of strong proximity lattices; this was done via equipping the Priestley spaces with an apartness relation. It was shown that the category **PSs**, of Priestley spaces with apartness and separating relations between them, is dual to the category **PLa**.

The facts mentioned above imply immediately that the category **PSs** is dual to the category **SCS**. But is there a direct translation between these two categories? In other words:

> *What is the direct relationship between Priestley spaces equipped with apartness and stably compact spaces ?*

The answer, which we will develop in this chapter, will take the following form:

For a Priestley space $\langle X; \leq, \mathcal{T}\rangle$ with apartness $\propto$ and $A, B \subseteq X$ we define:

1. $\propto[A] = \{x \in X \mid x \propto A\}$ and $[A]\propto = \{x \in X \mid A \propto x\}$, where, as before, $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

2. $core(X) = \{x \in X \mid [x]\propto = X \backslash \uparrow x\}$.

3. $\mathcal{T}' = \{O \cap core(X) \mid O$ is an open upper subset of $X\}$.

One of our primary results, then, is the following:

Theorem 4.1.1. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle core(X), \mathcal{T}'\rangle$ is a stably compact space. Moreover, every stably compact space can be obtained in this way.*

The second objective of this chapter is to extend the Jung-Sünderhauf representation theorem for stably compact spaces to cover stably locally compact spaces. In other words, we will remove the compactness requirement on the topological side of Jung-Sünderhauf duality.

4.1.1 Organisation

The chapter is organised as follows. Section 4.2 presents some preparatory results that are necessary for the following sections. Sections 4.2 and 4.3 investigate the direct relationship between Priestley spaces equipped with apartness relations and stably compact spaces. The relationship between frame homomorphisms, continuous maps and separators is studied in Section 4.4. This leads to the result that the categories **SCS** and **PSs** are dual equivalent. Section 4.5 uses Priestley spaces with apartness to prove some facts about the co-compact topology of a stably compact space. In section 4.6, we link the notions of isolated set and round filter. In Section 4.7, we extend Jung-Sünderhauf duality for stably compact spaces to cover stably locally compact spaces.

4.2 Preparatory Results

Definition 4.2.1. Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$ and let A be a subset of X. Then

1. $\propto[A] = \{x \in X \mid x \propto A\}$ and $[A]\propto = \{x \in X \mid A \propto x\}$, where $A \propto B$ is a shorthand for $a \propto b$ for all $a \in A$, $b \in B$.

2. An upper set $U \subseteq X$ is said to be *isolated* if $U = \propto[X \backslash U]$. A lower set $D \subseteq X$ is said to be *isolated* if $D = [X \backslash D]\propto$. The set of all upper isolated open subsets of X is denoted by $iso_u(X)$ and the set of all lower isolated open subsets of X is denoted by $iso_l(X)$.

3. $core(X) = \{x \in X \mid [x]\propto = X \backslash {\uparrow} x\} = \{x \in X \mid X \setminus {\uparrow} x \text{ is isolated}\}$.

Remark 4.2.2. For every $x \in core(X)$, $x \not\propto x$.

The following lemma follows from condition ($\propto\forall$) in the definition of apartness relation (Definition 3.1.1).

Lemma 4.2.3. *Let* $\langle X; \leq, \mathcal{T}\rangle$ *be a Priestley space with apartness* $\propto$*,* $a \in X$ *and* $V, B \subseteq X$*. Then*

$$a \propto V \textit{ and } X \backslash V \propto B \Longrightarrow a \propto B.$$

An Example

Consider the unit interval example shown in detail in Section 3.3.4. Let $0 \leq r \leq 1$. Then $A = {\uparrow} F_r^1$ and $B_r = \bigcup_{r'>r} {\uparrow} F_{r'}^1$ are open upper subsets of $\mathit{filt}_p(\mathcal{B})$. We note that A is not isolated because $F_r^2 \in \mathit{filt}_p(\mathcal{B}) \backslash A$ and $F_r^1 \not\propto F_r^2$ which means that $A \neq \propto[X \backslash A]$. On the other hand, B_r is isolated because, clearly, $B_r = \propto[\mathit{filt}_p(\mathcal{B}) \backslash B_r]$. Moreover, $\{B_r \mid 0 \leq r \leq 1\}$ is the set of isolated upper proper subsets of X.

Lemma 4.2.4. *Let* $\langle X; \leq, \mathcal{T}\rangle$ *be a Priestley space with apartness* $\propto$*. Then*

1. *For a closed subset C of X, $\propto[C] \in iso_u(X)$.*

2. *For every upper isolated open set I, $I = \bigcup^{\uparrow}\{\propto[V] \mid V \in \mathcal{O}^{\mathcal{T}}(X)$ and $I \cup V = X\}$.*

Proof. 1. By Lemma 3.3.3, $\propto[C]$ is a union of clopen upper sets and hence is an open upper set. We have

$$\begin{aligned} x \propto C &\iff (\forall y)\; x \propto y \text{ or } y \propto C \text{ by } (\propto\forall) \\ &\iff (\forall y \not\propto C)\; x \propto y \\ &\iff (\forall y \in (X \backslash \propto[C]))\; x \propto y \\ &\iff x \propto (X \backslash \propto[C]). \end{aligned}$$

This completes the proof of 1.

2. The equality in (2) follows from Lemma 3.3.3 together with the fact that I is isolated as follows. From $I = \propto[X \backslash I]$ we get for all $x \in I$, $x \propto X \backslash I$. Both $\{x\}$ and $X \backslash I$ are closed, so we can apply Lemma 3.3.3 to obtain $U \in \mathcal{U}^{\mathcal{T}}(X)$ and $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $x \in U, X \backslash I \subseteq V$ and $U \propto V$. In other words, $I \cup V = X$ and $x \in \propto[V]$. For the directedness of the union, let $V_i \in \mathcal{O}^{\mathcal{T}}(X), 1 \leq i \leq 2$, such that $V_i \cup I = X$. Then clearly $V_1 \cap V_2 \in \mathcal{O}^{\mathcal{T}}(X)$, $(V_1 \cap V_2) \cup I = X$ and $\propto[V_1], \propto[V_2] \subseteq \propto[V_1 \cap V_2]$.

$\square$

Corollary 4.2.5. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then*

$$(\forall x \in X) \propto[x] \in iso_u(X).$$

Theorem 4.2.6. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle iso_u(X); \subseteq\rangle$ forms a stably continuous frame. The following statements are true for all $I, J \in iso_u(X)$ and $\{I_k\}_{k\in K} \subseteq iso_u(X)$:*

1. $\bigwedge_{k\in K} I_k = \propto[\overline{\bigcup_{k\in K}(X \backslash I_k)}]$. *Finite infima are intersections.*

2. $\bigvee_{k\in K} I_k = \propto[\bigcap_{k\in K}(X\backslash I_k)]$. *Directed suprema are unions.*

3. $I \ll J \iff (\exists\, V \in \mathcal{O}^{\mathcal{T}}(X))\; I \propto V$ *and* $V \cup J = X$.

Proof. 1. Let A denotes the r.h.s. of 1. By Lemma 4.2.4(1), A belongs to $iso_u(X)$. It is clear that A is contained in all I_k. For the converse, let I be an isolated open upper set contained in all I_k. Then $I = \propto[X\backslash I]$ and $X\backslash I \supseteq \bigcup_k(X\backslash I_k)$. Since $X\backslash I$ is closed, also $X\backslash I \supseteq \overline{\bigcup_k(X\backslash I_k)}$. Hence $I \subseteq \propto[\overline{\bigcup_k(X\backslash I_k)}]$. If the K is finite, then $X\backslash I \supseteq \bigcup_k(X\backslash I_k)$

$$\propto[\overline{\bigcup_{k\in K}(X\backslash I_k)}] = \propto[\bigcup_{k\in K} X\backslash I_k] = \bigcap_{k\in K} \propto[X\backslash I_k] = \bigcap_{k\in K} I_k.$$

2. Let B denotes the r.h.s. of 2. Again by Lemma 4.2.4(1), B belongs to $iso_u(X)$. It is clear that B is the least upper bound of $\{I_k\}_{k\in K}$. If $\{I_k\}_{k\in K}$ is directed, then we show that $B \subseteq \bigcup_{k\in K} I_k$ as follows:

$$\begin{aligned} x \in B &\implies x \propto \bigcap_{k\in K} X\backslash I_k \\ &\implies (\exists\, V \in \mathcal{O}^{\mathcal{T}}(X))\; x \propto V \text{ and } \bigcap_{k\in K}(X\backslash I_k) \subseteq V, \text{by Lemma 3.3.3} \\ &\implies (\exists\, k \in K)\; x \in I_k \subseteq \bigcup_{k\in K} I_k. \end{aligned}$$

The last implication is true because $\bigcup_{k\in K} I_k$ is an open cover to the compact set $X\backslash V$. Hence by directedness of $\{I_k\}_{k\in K}$, $X\backslash V$ is contained in I_k for some $k \in K$. Therefore $x \propto (X\backslash I_k)$ implying $x \in I_k$.

3. The implication from left to right in (3) follows immediately from Lemma 4.2.4(2). For the converse assume the right hand side, and let $J \subseteq \bigcup_k^{\uparrow} J_k$ in $iso_u(X)$. Since $X\backslash V$ is compact, for some k, $V \cup J_k = X$, or $X\backslash J_k \subseteq V$. From this it follows that $J_k = \propto[X\backslash J_k] \supseteq \propto[V]$ and the latter is a superset of I by assumption. With this characterisation, the continuity of $iso_u(X)$ now follows from Lemma 4.2.4(2).

$X \ll X$ follows by setting $V = \emptyset$ in 3. The way-below relation on $iso_u(X)$ is multiplicative:

$$\begin{aligned} I \ll J, F &\implies (\exists\, V_1, V_2 \in \mathcal{O}^{\mathcal{T}}(X))\; V_1 \cup J = X,\; V_2 \cup F = X, I \propto V_1 \cup V_2 \\ &\implies V_1 \cup V_2 \in \mathcal{O}^{\mathcal{T}}(X),\; (V_1 \cup V_2) \cup (J \cap F) = X \text{ and } I \propto V_1 \cup V_2 \\ &\implies I \ll J \cap F. \end{aligned}$$

Finally we prove that $iso_u(X)$ is distributive.

$$\begin{aligned} x \in I \wedge (J \vee F) = I \cap (J \vee F) &\implies x \propto (X \backslash I) \text{ and } x \propto (X \backslash (J \cup F)) \\ &\implies x \propto (X \backslash (I \cap (J \cup F))) \\ &\implies x \propto (X \backslash ((I \cap J) \cup (I \cap F))) \\ &\implies x \propto ((X \backslash (I \wedge J) \cap (X \backslash (I \wedge F))) \\ &\implies x \in (I \wedge J) \vee (I \wedge F). \end{aligned}$$

The other inclusion is always true. □

The following corollary follows from Table 2.1 that is given in Section 2.4.

Corollary 4.2.7. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then the point topology $pt(iso_u(X))$ on $iso_u(X)$ is a stably compact space whose lattice of open subsets is isomorphic to $iso_u(X)$.*

In the following, *Sfilt*$(iso_u(X))$ denotes the set of Scott-open filters in $iso_u(X)$.

Lemma 4.2.8. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$, $x \in X$, C a closed subset of X, and $\mathcal{F} \in$ Sfilt$(iso_u(X))$. If $A = \{x \in X \mid \propto[x] \in \mathcal{F}\}$ then*

1. $\propto[x] \in \mathcal{F} \iff (\exists V_x \in \mathcal{O}^{\mathcal{T}}(X))\; x \in V_x, V_x \subseteq A$ *and* $\propto[V_x] \in \mathcal{F}$*, and*

2. $C \subseteq A \iff \propto[C] \in \mathcal{F}$.

Proof. 1. The right-to-left direction is obvious. For the other direction we have

$$\begin{aligned}
\propto[x] \in \mathcal{F} &\implies (\exists\, V \in \mathcal{O}^{\mathcal{T}}(X))\, \propto[x] \cup V = X \text{ and } \propto[V] \in \mathcal{F}, \text{ by Lemma } 4.2.4(2)\\
&\implies (X \backslash V) \propto x \text{ and } \propto[V] \in \mathcal{F}\\
&\implies (\exists\, V_x \in \mathcal{O}^{\mathcal{T}}(X))\, x \in V_x,\ (X\backslash V) \propto V_x,\ \propto[V] \in \mathcal{F}\\
&\implies (\forall v \in V_x)\, \propto[V] \subseteq \propto[V_x] \subseteq \propto[v], \text{ and } \propto[V] \in \mathcal{F} \text{ by Lemma } 4.2.3\\
&\implies V_x \subseteq A \text{ and } \propto[V_x] \in \mathcal{F}.
\end{aligned}$$

2. The right-to-left direction is proved as follows. For every $c \in C$, $\propto[C] \subseteq \propto[c]$ implying $\propto[c] \in \mathcal{F}$. Therefore for every $c \in C$, $c \in A$.

   The other direction is proved as follows. By the first part of the lemma for every $c \in C$, there exists a clopen lower set V_c such that $c \in V_c \subseteq A$ and $\propto[V_c] \in \mathcal{F}$. The set $\{V_c \mid c \in C\}$ is an open cover to the compact set C. Therefore a finite subcover $\{V_{c_i} \mid 1 \leq i \leq n\}$ exists. Now it is easy to check that

$$\bigcap_i (\propto[V_{c_i}]) = \propto[\bigcup_i V_{c_i}] \subseteq \propto[C] \text{ and } \bigcap_i (\propto[V_{c_i}]) \in \mathcal{F}.$$

   Therefore $\propto[C] \in \mathcal{F}$.

□

Lemma 4.2.9. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $iso_l(X) \cong$ $Sfilt(iso_u(X))$ via the mappings:*

$$\Phi : Sfilt(iso_u(X)) \longrightarrow iso_l(X); \mathcal{F} \longmapsto \{x \in X \mid \propto[x] \in \mathcal{F}\}, \textit{ and}$$

$$\Psi : iso_l(X) \longrightarrow Sfilt(iso_u(X)); F \longmapsto \{I \in iso_u(X) \mid F \cup I = X\}.$$

Proof. We show that Φ is well-defined. $\Phi(\mathcal{F})$ is open lower because it is a union of clopen lower sets by Lemma 4.2.8(1). It remains to show that $[X\backslash\Phi(\mathcal{F})]\propto = \Phi(\mathcal{F})$. This is proved as follows.

$$\begin{aligned}\propto[x] \in \mathcal{F} &\implies (\exists\, V \in \mathcal{O}^{\mathcal{T}}(X))\ (X\backslash V) \propto x \text{ and } \propto[V] \in \mathcal{F}, \text{ by Lemma } 4.2.4(2)\\ &\iff (\exists\, V \in \mathcal{O}^{\mathcal{T}}(X))\ (X\backslash V) \propto x,\ V \subseteq \Phi(\mathcal{F}), \text{ by Lemma } 4.2.8(2)\\ &\implies X\backslash\Phi(\mathcal{F}) \propto x.\end{aligned}$$

The other direction of the first implication is also true as follows. By $(\propto\forall)$

$$(\forall y \in X)\ y \propto V \Longrightarrow y \propto x, \text{ because } (X\backslash V) \propto x.$$

Therefore $\propto[V] \subseteq \propto[x]$. Hence $\propto[x] \in \mathcal{F}$ because $\propto[V] \in \mathcal{F}$. The other direction of the last implication is also true by Lemma 3.3.3.

Clearly $\Psi(F)$ is an upper set and closed under finite intersections. We show that $\Psi(F)$ is Scott-open. Let $\{I_j\}_{j\in J}$ be a directed family in $iso_u(X)$ such that $\bigvee^{\uparrow}_{j\in J} I_j \in \Psi(F)$. Then by the directness of $\{I_j\}_{j\in J}$ and compactness of $X\backslash F$ we have

$$\bigcup_{j\in J} I_j \in \Psi(F) \Longrightarrow X\backslash F \subseteq \bigcup_{j\in J} I_j \Longrightarrow (\exists\, k)\ X\backslash F \subseteq I_k \Longrightarrow I_k \in \Psi(F).$$

Clearly Φ and Ψ are monotone. We now show that they are inverses of each other. We have

$$(\Psi \circ \Phi)(\mathcal{F}) = \{I \in iso_u(X) \mid I \cup \Phi(\mathcal{F}) = X\} = \mathcal{F}.$$

This is true because by Lemma 4.2.8(2)

$$I \cup \Phi(\mathcal{F}) = X \Longleftrightarrow X\backslash I \subseteq \Phi(\mathcal{F}) \Longleftrightarrow I = \propto[X\backslash I] \in \mathcal{F}.$$

We also have

$$(\Phi \circ \Psi)(F) = \{x \in X \mid \propto[x] \cup F = X\} = F.$$

This is true because

$$x \in F \Longleftrightarrow (X\backslash F) \propto x \Longleftrightarrow (X\backslash F) \subseteq \propto[x] \Longleftrightarrow \propto[x] \cup F = X.$$

□

The following corollary follows from Table 2.2, given in Section 2.4, of correspondences between concepts on the topological and localic sides of Stone duality. In the following, for a topological space X, let $\mathcal{K}_X$ denote the set of compact saturated subsets of X.

Corollary 4.2.10. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle iso_l(X), \subseteq\rangle$ and $\langle \mathcal{K}_{pt(iso_u(X))}, \supseteq\rangle$ are isomorphic.*

The following lemma is the first step towards smoothing the way in which we get stably compact spaces from Priestley spaces equipped with apartness relations.

Lemma 4.2.11. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $core(X)$ is isomorphic to $pt(iso_u(X))$.*

Proof. By definition of $core(X)$ (Definition 4.2(3)), it is enough to show that the maps of Lemma 4.2.9 are well defined when restricted to lower isolated sets of the form $X\backslash\uparrow x$, where $x \in core(X)$, and completely prime Scott-open filters. Suppose $x \in core(X)$.

$$\Psi(X\backslash\uparrow x) = \{I \in iso_u(X) \mid (X\backslash\uparrow x) \cup I = X\} = \{I \in iso_u(X) \mid x \in I\}.$$

Let $\{I_i\}_{i\in I} \subseteq iso_u(X)$ such that $\bigvee_{i\in I}\{I_i\} \in \Psi(X\backslash\uparrow x)$. Then

$$x \in \bigvee_{i\in I}\{I_i\} \Longleftrightarrow x \propto X\backslash\bigcup_{i\in I} I_i \Longleftrightarrow X\backslash\bigcup_{i\in I} I_i \subseteq [x]\propto = X\backslash\uparrow x \Longleftrightarrow x \in \bigcup_{i\in I} I_i.$$

Hence $\Psi(F)$ is completely prime.

Suppose $\mathcal{F}$ is a completely prime element in $Sfilt(iso_u(X))$. Set $C = X\backslash\Phi(\mathcal{F})$. We first show that $C \neq \emptyset$. Suppose $X = \Phi(\mathcal{F})$. Then by Lemma 4.2.8(2), $\propto[X] \in \mathcal{F}$. For every $I \in iso_u(X)$

$$X\backslash I \subseteq X \Longrightarrow \propto[X] \subseteq \propto[X\backslash I] = I.$$

Therefore every $I \in iso_u(X)$ belongs to $\mathcal{F}$ because $\propto[X]$ does. Hence $\mathcal{F} = iso_u(X)$ which is a contradiction because $\mathcal{F}$ is a proper subset of $iso_u(X)$ as a completely prime filter. Therefore $C \neq \emptyset$. We show that $C = \uparrow x$ for some $x \in X$. For the sake of contradiction we assume that C has more than one minimal element. Let A be the set of minimal elements in C. For two different elements a, b of A, there exists a clopen upper set U_a containing a but not b. The set $\{U_a \mid a \in A\}$ is an open cover of C which is compact because it is closed. Then a finite sub-cover $\{U_{a_i}\}_{1\leq i\leq n}$ exists. Therefore $C \subseteq \bigcup_{1\leq i\leq n} U_{a_i}$. Hence $\bigcap_{1\leq i\leq n} X\backslash U_{a_i} \subseteq \Phi(\mathcal{F})$. This implies, by Lemma 4.2.8(2), $\propto[\bigcap_{1\leq i\leq n} X\backslash U_{a_i}] \in \mathcal{F}$. But we have, By $(\propto\downarrow\downarrow)$,

$$\propto[\bigcap_{1\leq i\leq n} X\backslash U_{a_i}] \subseteq \propto[X\backslash \bigcup_{1\leq i\leq n} \propto[X\backslash U_{a_i}]] = \propto[\bigcap_{1\leq i\leq n} X\backslash \propto[X\backslash U_{a_i}]] = \bigvee_{1\leq i\leq n} \propto[X\backslash U_{a_i}].$$

Therefore $\bigvee_{1\leq i\leq n} \propto[X\backslash U_{a_i}] \in \mathcal{F}$. Hence there exists i such that $\propto[X\backslash U_{a_i}] \in \mathcal{F}$. Therefore $X\backslash U_{a_i} \subseteq \Phi(\mathcal{F})$ and $C \subseteq U_{a_i}$. The last implication is a contradiction because U_{a_i} does not contain one of the minimal elements of C. Hence $\Phi(\mathcal{F}) = X\backslash\uparrow x$ for some $x \in X$. □

Remark 4.2.12. As the definitions of $iso_u(X)$ and $iso_l(X)$ are dual to each other and by Remark 3.1.4.2, every result obtained above about $iso_u(X)$ has a dual fact concerning $iso_l(X)$.

Definition 4.2.13. Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\mathcal{T}'$ is the topology on $core(X)$ defined by

$$\mathcal{T}' = \{core(X) \cap A \mid A \text{ is an open upper set in } X\}.$$

Lemma 4.2.14. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then the topology $\mathcal{T}'$ defined above on $core(X)$ can be equivalently defined as follows*

$$\mathcal{T}' = \{core(X) \cap F \mid F \in iso_u(X)\}.$$

Proof. Let A be an open upper subset of X and set $F = \propto[X\backslash A]$. Then by Lemma 4.2.4(1) $F \in iso_u(X)$. Let $x \in core(X)$. Then

$$x \in A \Longleftrightarrow \uparrow x \subseteq A \Longleftrightarrow X\backslash A \subseteq X\backslash\uparrow x = [x]\propto \Longleftrightarrow x \propto X\backslash A \Longleftrightarrow x \in F.$$

□

Theorem 4.2.15. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle core(X), \mathcal{T}'\rangle$ is homeomorphic to the point topology on $pt(iso_u(X))$.*

Proof. As proved in Lemma 4.2.11, $core(X)$ is isomorphic to $pt(iso_u(X))$ via the mappings:

$$\Delta : pt(iso_u(X)) \longrightarrow core(X); \mathcal{F} \longmapsto \min\{x \in X \mid \propto[x] \notin \mathcal{F}\}, \text{ and}$$

$$\Theta : core(X) \longrightarrow pt(iso_u(X)); x \longmapsto \{I \in iso_u(X) \mid x \in I\}.$$

We note that Δ and Θ are respectively related to the mappings Φ and Ψ of Lemma 4.2.9 as follows:

1. $\Phi(\mathcal{F}) = X \backslash \uparrow\Delta(\mathcal{F})$.

2. $\Theta(x) = \Psi(X \backslash \uparrow x)$.

Let $I \in iso_u(X)$. Then

$$\Theta^{-1}(\mathcal{O}_I) = \Theta^{-1}(\{\mathcal{F} \in pt(iso_u(X)) \mid I \in \mathcal{F}\}) = \{x \in core(X) \mid x \in I\} = I \cap core(X).$$

Let A be an open upper set.

$$\begin{aligned}
\Delta^{-1}(core(X) \cap A) &= \Delta^{-1}(\{x \in core(X) \mid x \in A\}) \\
&= \{\mathcal{F} \in pt(iso_u(X)) \mid \Delta(\mathcal{F}) \in A\} \\
&= \{\mathcal{F} \in pt(iso_u(X)) \mid \Delta(\mathcal{F}) \in \bigcup_{U \in \mathcal{U}^{\mathcal{T}}(X), U \subseteq A} U\} \\
&= \bigcup_{U \in \mathcal{U}^{\mathcal{T}}(X), U \subseteq A} \{\mathcal{F} \in pt(iso_u(X)) \mid \Delta(\mathcal{F}) \in U\} \\
&= \bigcup_{U \in \mathcal{U}^{\mathcal{T}}(X), U \subseteq A} \{\mathcal{F} \in pt(iso_u(X)) \mid X \backslash U \subseteq \Phi(\mathcal{F})\} \\
&= \bigcup_{U \in \mathcal{U}^{\mathcal{T}}(X), U \subseteq A} \{\mathcal{F} \in pt(iso_u(X)) \mid \propto[X \backslash U] \in \mathcal{F}\} \\
&= \bigcup_{U \in \mathcal{U}^{\mathcal{T}}(X), U \subseteq A} \mathcal{O}_{\propto[X \backslash U]}.
\end{aligned}$$

The second last equality is true because by Lemma 4.2.8(2),

$$\propto[X\backslash U] \in \mathcal{F} \iff X\backslash U \subseteq \Phi(\mathcal{F}).$$

□

Remark 4.2.16. Let $\langle X, \leq, \mathcal{T}\rangle$ be a Priestley space with the trivial apartness $\propto = \not\leq$. Then $core(X) = X$ and $\mathcal{T}'$ is the set of open upper sets of X. Therefore the set of open upper subsets of any Priestley space is a stably compact topology on the same space.

We have already noted the following but we recall it in a remark as we will need to refer to it later in the chapter.

Remark 4.2.17. The mappings Δ and Θ of Theorem 4.2.15 are respectively related to the mappings Φ and Ψ of Lemma 4.2.9 as follows:

1. $\Phi(\mathcal{F}) = X\backslash\uparrow\Delta(\mathcal{F})$.

2. $\Theta(x) = \Psi(X\backslash\uparrow x)$.

4.3 From Stably Compact Spaces to Priestley Spaces with Apartness

In the previous section we have presented a fairly nice way of obtaining stably compact spaces from Priestley spaces equipped with apartness relations. In this section we show that every stably compact space can be obtained in this way and is a retract of a Priestley space with apartness.

Definition 4.3.1. Suppose $\langle Y, \mathcal{T}\rangle$ is a stably compact space and $\mathcal{K}_Y$ is the set of compact saturated sets in Y. Then

$$\mathcal{B}_Y = \{(o, k) \in \mathcal{T} \times \mathcal{K}_Y \mid o \subseteq k\}.$$

Remark 4.3.2. As shown in [54, Theorem 23], $\langle \mathcal{B}_Y; \wedge, \vee, (\emptyset, \emptyset), (Y, Y); \prec \rangle$ where $\wedge$ and $\vee$ are pair-wise intersection and union, respectively, and

$$(o, k) \prec (o', k') :\stackrel{\text{def}}{\Longleftrightarrow} k \subseteq o',$$

is a strong proximity lattice.

Remark 4.3.3. $\mathit{Pries}(\mathcal{B}_Y) = \langle \mathit{filt}_p(\mathcal{B}_Y); \subseteq, \mathcal{T}_P, \propto_{\prec} \rangle$ is the Priestley dual of $\mathcal{B}_Y$ equipped with the apartness $\propto_{\prec}$. $\mathit{filt}_p(\mathcal{B}_Y)$ is the set of prime filters of $\mathcal{B}_Y$. $\mathcal{T}_P$ is the Priestley topology generated by the sets $U_{(o,k)} = \{F \in \mathit{filt}_p(\mathcal{B}_Y) \mid (o, k) \in F\}$ and $O_{(o,k)} = \{F \in \mathit{filt}_p(\mathcal{B}_Y) \mid (o, k) \notin F\}$. Obviously, $O_{(o,k)} = \mathit{filt}_p(\mathcal{B}_X) \backslash U_{(o,k)}$ and so each $U_{(o,k)}$ is a clopen upper, and each $O_{(o,k)}$ a clopen lower set. Moreover, every clopen upper (lower) has the form $U_{(o,k)}$ ($O_{(o,k)}$). $\propto_{\prec}$ is the apartness on $\mathit{filt}_p(\mathcal{B}_Y)$ defined as follows

$$F \propto_{\prec} G \stackrel{\text{def}}{\Longleftrightarrow} (\exists (o, k) \in F)(\exists (o', k') \notin G)\; k \subseteq o'.$$

Lemma 4.3.4. *Let $\langle Y, \mathcal{T} \rangle$ be a stably compact space. Then $\langle Y, \mathcal{T} \rangle$ is homeomorphic to $\langle \mathit{core}(\mathit{filt}_p(\mathcal{B}_Y)), \mathcal{T}'_P \rangle$.*

Proof. Because $\mathit{core}(\mathit{filt}_p(\mathcal{B}_Y))$ and Y are sober, it is sufficient to prove that the topologies are isomorphic. The following mappings accomplish this task:

$$\Psi : \langle \mathcal{T}, \subseteq \rangle \longrightarrow \langle \mathit{iso}_u(\mathit{filt}_p(\mathcal{B}_Y)), \subseteq \rangle; \mathcal{U} \longmapsto \propto_{\prec}[O_{(\mathcal{U},Y)}], \mathit{and}$$

$$\Phi : \langle \mathit{iso}_u(\mathit{filt}_p(\mathcal{B}_Y)), \subseteq \rangle \longrightarrow \langle \mathcal{T}, \subseteq \rangle; \mathcal{U} \longmapsto \bigcup \{o \mid U_{(o,k)} \subseteq \mathcal{U})\}.$$

Φ is clearly well-defined and Ψ is well-defined by Lemma 4.2.4(1) and they are clearly monotone. We show that they are inverses of each other.

$$\Phi(\Psi(\mathcal{U})) = \bigcup \{o \mid U_{(o,k)} \subseteq \propto_{\prec}[O_{(\mathcal{U},Y)}]\} = \mathcal{U}.$$

This is proved as follows:

$$\begin{aligned}
x \in \Phi(\Psi(\mathcal{U})) &\iff (\exists o \in \mathcal{T})(\exists k \in \mathcal{K}_Y)\ x \in o \text{ and } U_{(o,k)} \subseteq \propto[O_{(\mathcal{U},Y)}] \\
&\iff (\exists o \in \mathcal{T})(\exists k \in \mathcal{K}_Y)\ x \in o \text{ and } U_{(o,k)} \propto_{\prec} O_{(\mathcal{U},Y)} \\
&\overset{\text{Lemma 3.3.16}}{\iff} (\exists o \in \mathcal{T})(\exists k \in \mathcal{K}_Y)\ x \in o \text{ and } (o,k) \prec (\mathcal{U},Y) \\
&\iff (\exists o \in \mathcal{T})(\exists k \in \mathcal{K}_Y)\ x \in o \subseteq k \subseteq \mathcal{U} \\
&\iff x \in \mathcal{U}.
\end{aligned}$$

The last equivalence is true because X is locally compact.

Also we have

$$\Psi(\Phi(\mathcal{U})) = \propto_{\prec}[O_{(\Phi(\mathcal{U}),Y)}] = \mathcal{U}.$$

This is true because

$$\begin{aligned}
G \propto_{\prec} O_{(\Phi(\mathcal{U}),Y)} &\iff G \in U_{(o,k)} \propto_{\prec} O_{(\Phi(\mathcal{U}),Y)}, \text{by Lemma 3.3.3} \\
&\iff (\exists\, (o,k) \in G)\ (o,k) \prec (\Phi(\mathcal{U}),Y) \\
&\iff (\exists\, (o,k) \in G)\ o \subseteq k \subseteq \Phi(\mathcal{U}) \\
&\iff (\exists\, (o,k),\ (o',k') \in \mathcal{B}_X)\ G \in U_{(o,k)},\ U_{(o',k')} \subseteq \mathcal{U}, \\
&\qquad \text{and } o \subseteq k \subseteq o',\ \text{because } k \text{ is compact} \\
&\iff (\exists\, (o,k),\ (o',k') \in \mathcal{B}_X)\ G \in U_{(o,k)},\ U_{(o',k')} \subseteq \mathcal{U}, \\
&\qquad \text{and } U_{(o,k)} \propto_{\prec} O_{(o',k')} \\
&\iff G \propto_{\prec} \mathit{filt}_p(\mathcal{B}_X)\backslash\mathcal{U},\ \text{by Lemma 3.3.3} \\
&\iff G \in \mathcal{U}.
\end{aligned}$$

The left-to-right direction of the second last equivalence is true because

$$\mathit{filt}_p(\mathcal{B}_Y)\backslash\mathcal{U} \subseteq \mathit{filt}_p(\mathcal{B}_Y)\backslash U_{(o',k')} = O_{(o',k')}.$$

□

Theorem 4.3.5. *The cores of Priestley spaces equipped with apartness relations are precisely the stably compact spaces.*

Example

The unit interval $[0, 1]$ equipped with the Scott topology is a stably compact space. Its corresponding strong proximity lattice and Priestley space with apartness were explained in detail in Section 3.3.4.

4.4 Morphisms: Separators and Continuous Maps

In this section we prove the one-to-one correspondence between separators and frame homomorphisms. This is sufficient to prove the equivalence between the category **PSs**, of Priestley spaces equipped with apartness relations and separators, and the category **SCS** of stably compact spaces and continuous maps. The equivalence of these categories follows because, as we have mentioned before in several occasions, the category of stably continuous frames and frame homomorphisms is equivalent to the category **SCS** [4, Section 7].

We also present the direct relationship between separators, between Priestley spaces equipped with apartness relations, and continuous maps, between stably compact spaces.

We begin by reminding the reader of the following lemma which was presented in the previous chapter (Lemma 3.3.32) and which is referred to in several occasions in this section.

Lemma 4.4.1. *Let $\langle X_1; \leq_1, \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ be a separator. Then for closed subsets $A \subseteq X_1$ and $B \subseteq X_2$, if $A \ltimes B$ then there exist $U \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $V \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $A \subseteq U$, $B \subseteq V$ and $U \ltimes V$.*

Lemma 4.4.2. *Let $\langle X_1; \leq_1, \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ be a separator. Then*

$$h_{\ltimes} : iso_u(X_2) \longrightarrow iso_u(X_1); I \longmapsto \ltimes[X_2 \backslash I].$$

is a frame homomorphism.

Proof. Similarly to Lemma 4.2.4.1, $h_{\ltimes}(I)$ is proved to be in $iso_u(X_1)$ and so $h_{\ltimes}$ is well defined. Clearly, $h_{\ltimes}$ is a monotone and preserves finite infima. Now we show that $h_{\ltimes}$ preserves arbitrary suprema. Let $\{I_j\}_{j\in J} \subseteq iso_u(X_2)$. Then

$$\begin{aligned} a \in h_{\ltimes}(\bigvee_{j\in J} I_j) &\iff a \ltimes X_2 \backslash \bigvee_{j\in J} I_j \\ &\iff a \ltimes (X_2 \backslash \propto_2 [X_2 \backslash \bigcup_{j\in J} I_j]) \\ &\iff a \ltimes (X_2 \backslash \bigcup_{j\in J} I_j), \text{ by } (\ltimes\forall) \\ &\iff a \ltimes \bigcap_{j\in J} X_2 \backslash I_j \\ &\iff a \propto_1 (X_1 \backslash \bigcup_{j\in J} \ltimes[X_2 \backslash I_j]) \\ &\iff a \in \bigvee_{j\in J} h_{\ltimes}(I_j). \end{aligned}$$

The right-to-left direction of the second last equivalence is by $(\forall\ltimes)$ and the fact that

$$(\bigcup_{j\in J} \ltimes[X_2 \backslash I_j]) \ltimes (\bigcap_{j\in J} X_2 \backslash I_j).$$

The other direction is proved as follows. By Lemma 4.4.1, there exists $V \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $a \ltimes V$ and $(X_2 \backslash \bigcup_{j\in J} I_j) \subseteq V$. Then by compactness of $X_2 \backslash V$ the collection $\{I_j\}_{j\in J}$ has a finite subsets $\{I_j\}_{1\leq j\leq n}$ such that $(X_2 \backslash \bigcup_{1\leq j\leq n} I_j) \subseteq V$. Therefore $a \ltimes \bigcap_{1\leq j\leq n} X_2 \backslash I_j$. Hence by $(\ltimes n\downarrow)$, $a \propto_1 (X_1 \backslash \bigcup_{1\leq j\leq n} \ltimes[X_2 \backslash I_j])$ implying $a \propto_1 (X_1 \backslash \bigcup_{j\in J} \ltimes[X_2 \backslash I_j])$ as required. □

Lemma 4.4.3. *Let $\langle X_1; \leq_1, \mathcal{T}_1 \rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2 \rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let h be a frame homomorphism from $iso_u(X_2)$ to $iso_u(X_1)$, then the binary relation $\ltimes_h \subseteq X_1 \times X_2$ defined as follows:*

$$a \ltimes_h \ b :\stackrel{\text{def}}{\Longleftrightarrow} a \in h(\propto_2[b]).$$

is a separator.

Proof. $\ltimes_h$ is open and satisfies $(\uparrow_1 \ltimes \downarrow_2)$ as follows.

$$\begin{aligned}
a \ltimes_h b &\implies a \in h(\propto_2[b]) \\
&\implies (\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))\ a \in U \subseteq h(\propto_2[b]),\ \text{because } h(\propto_2[b]) \text{ is open upper in } X_1 \\
&\implies (\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ a \in U \subseteq h(\propto_2[V]) \text{ and } b \in V \\
&\implies (\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ a \in U,\ b \in V \text{ and } U \ltimes_h V.
\end{aligned}$$

That the third implication is seen as follows. By Lemma 4.2.4.2, the fact that h preserves joins, and the compactness of U, there exist a clopen lower set V' in X_2 such $U \subseteq h(\propto_2[V'])$ and $X_2 \backslash V' \propto_2 b$. Then by Lemma 4.4.1, there exists a clopen-lower set V such that $X_2 \backslash V' \propto_2 V$ and $b \in V$. Then by $(\propto\forall)$, $\propto_2[V'] \subseteq \propto_2[V]$ and hence $h(\propto_2[V']) \subseteq h(\propto_2[V])$. Therefore $a \in U \subseteq h(\propto_2[V])$ and $b \in V$.

$(\forall\ltimes)$ is satisfied because:

$$\begin{aligned}
a \ltimes_h b &\iff a \in h(\propto_2[b]) \\
&\iff a \propto_1 X_1 \backslash h(\propto_2[b]) \\
&\iff (\forall x \in X_1)\ a \propto_1 x \text{ or } x \propto_1 X_1 \backslash h(\propto_2[b]),\ \text{by } (\propto\forall) \\
&\iff (\forall x \in X_1)\ a \propto_1 x \text{ or } x \in h(\propto_2[b]) \\
&\iff (\forall x \in X_1)\ a \propto_1 x \text{ or } x \ltimes_h b.
\end{aligned}$$

$(\ltimes\forall)$ is also satisfied because

$$\begin{aligned} a \in h(\propto_2[b]) &\iff a \in h(\propto_2[X_2 \backslash \propto_2[b]]) \\ &\iff (\forall x \in X_2 \backslash \propto_2[b])\ a \in h(\propto_2[x]) \\ &\iff (\forall x \in X_2)\ a \ltimes_h x \text{ or } x \propto_2 b. \end{aligned}$$

The right-to-left direction of the second equivalence is true because, as proved above, $a \in h(\propto_2[x])$ implies there exists $V_x \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $x \in V_x$ and $a \in h(\propto_2[V_x])$. $\{V_x \mid x \in (X_2 \backslash \propto_2[b])\}$ is an open cover to $X_2 \backslash \propto[b]$ which is closed and hence is compact. Therefore a finite subcover $\{V_{x_i} \mid 1 \leq i \leq n\}$ exists and

$$a \in \bigcap_i h(\propto_2[V_{x_i}]) = h(\bigcap_i \propto_2[V_i]) = h(\propto_2[\bigcup_i V_{x_i}]) \subseteq h(\propto_2[X_2 \backslash \propto[b_2]]).$$

$(\ltimes n\downarrow)$ is satisfied because:

$$\begin{aligned} a \ltimes_h \cap\!\downarrow\! d_i &\implies a \in h(\propto_2[\bigcap_i \downarrow d_i]),\ \text{as proved in the previous section} \\ &\implies a \in h(\propto_2[X_2 \backslash \bigcup_i \propto_2[d_i]]),\ \text{by } (\propto\forall) \\ &\implies a \in h(\bigvee_i \propto_2[d_i]) = \bigvee_i h(\propto_2[d_i]) \\ &\implies a \propto_1 X_1 \backslash \bigcup_i h(\propto_2[d_i]) \\ &\implies (\forall x \in X_1)\ a \propto_1 x \text{ or } (\exists i)\ x \in h(\propto_2[d_i]) \\ &\implies (\forall x \in X_1)\ a \propto_1 x \text{ or } (\exists i)\ x \ltimes_h d_i. \end{aligned}$$

□

Lemma 4.4.4. *The maps between separators and frame homomorphisms defined in Lemma 4.4.2 and 4.4.3 are inverses of each other.*

Proof.

$$h_{\ltimes_h}(I) = \ltimes_h[X_2 \backslash I] = \{a \in X_1 \mid a \in h(\propto_2[X_2 \backslash I])\} = h(I).$$

$$\begin{aligned} a \ltimes_{h_\ltimes} b &\iff a \in h_\ltimes(\propto_2[b]) = \{x \in X_1 \mid x \ltimes (X_2 \backslash \propto_2[b])\} \\ &\iff a \ltimes (X_2 \backslash \propto_2[b]) \\ &\iff (\forall x \in X_2)\ a \ltimes x \text{ or } x \propto_2 b \\ &\iff a \ltimes b. \end{aligned}$$

□

Theorem 4.4.5. *The category* **PSs***, of Priestley spaces with apartness and separators, is equivalent to the category* **SCS** *of stably compact spaces and continuous maps.*

In the following we present a direct way of getting the continuous map corresponding to a separator and vice versa.

Lemma 4.4.6. *Let $\langle X_1; \leq_1, \mathcal{T}_1 \rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2 \rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes \subseteq X_1 \times X_2$ be a separator. Then*

$$f_\ltimes : core(X_1) \longrightarrow core(X_2) : x \longmapsto \min(X_2 \backslash [x]\ltimes),$$

is the continuous map (with respect to the stably compact topologies $\mathcal{T}_1'$ and $\mathcal{T}_2'$) corresponding to the separator $\ltimes$.

Proof. By Lemma 4.4.2 the separator $\ltimes$ corresponds to frame homomorphism:

$$h_\ltimes : iso_u(X_2) \longrightarrow iso_u(X_1); I \longmapsto \ltimes[X_2 \backslash I].$$

By Section 2.4, this frame homomorphism corresponds to the following continuous map:

$$f_{h_\ltimes} : pt(iso_u(X_1)) \longrightarrow pt(iso_u(X_2)) : \mathcal{F} \longmapsto h_\ltimes^{-1}(\mathcal{F}).$$

Let $x \in core(X_1)$. By Lemma 4.2.15, $\{I \in iso_u(X_1) \mid x \in I\}$ is the completely prime filter in $pt(iso_u(X_1))$ corresponding to x. Under $f_{h_\ltimes}$, $\{I \in iso_u(X_1) \mid x \in I\}$ is sent to $\{I \in iso_u(X_2) \mid x \in h_\ltimes(I)\}$ which is the same as $\{I \in iso_u(X_2) \mid x \ltimes X_2 \backslash I\}$. Again by

Lemma 4.2.15, the completely prime filter $\{I \in iso_u(X_2) \mid x \ltimes X_2 \backslash I\}$ corresponds to the following point of $core(X_2)$:

$$\min\{y \in X_2 \mid x \not\ltimes X_2 \backslash \propto_2[y]\}$$

By condition $(\ltimes\forall)$ (in the definition of separators) this is the same as

$$\min\{y \in X_2 \mid x \not\ltimes y\} = \min(X_2 \backslash [x]\ltimes).$$

□

For the other direction, according to the theory presented in [54], a continuous function $f : Y_1 \longrightarrow Y_2$ between stably compact spaces $(Y_1, \mathcal{T}_1)$ and $(Y_2, \mathcal{T}_2)$ corresponds to the approximable relation $\vdash_f \subseteq \mathcal{B}_{Y_1} \times \mathcal{B}_{Y_2}$ defined as follows:

$$(o, k) \vdash_f (o', k') \stackrel{\text{def}}{\iff} f(k) \subseteq o'.$$

As we have proved in the previous chapter, the approximable relation $\vdash_f$ corresponds to the separator $\ltimes_f \subseteq \mathit{filt}_p(\mathcal{B}_{Y_1}) \times \mathit{filt}_p(\mathcal{B}_{Y_2})$ defined as follows:

$$F \ltimes_f F' \stackrel{\text{def}}{\iff} (\exists (o, k) \in F)(\exists (o', k') \notin F')\ f(k) \subseteq o'.$$

4.4.1 Computational Reading

We now present a computational reading of the separator $\ltimes_f$ corresponding to a continuous map f. Let f be a continuous map (computable program) from a stably compact space (data type) Y_1 to another one Y_2. Let $\langle o, k\rangle \in \mathcal{B}_{Y_1}$ and $\langle o', k'\rangle \in \mathcal{B}_{Y_2}$ be such that $f(k) \subseteq o'$. As we have mentioned in the introduction of the thesis, the elements of the strong proximity lattices $\mathcal{B}_{Y_2}$ and $\mathcal{B}_{Y_2}$, which are pairs of the form $\langle o, k\rangle$, have the following computational interpretation. Each pair $\langle o, k\rangle$ represents a property that is

- satisfied by members of the open set o,

- unsatisfied by the elements in the complement of the compact saturated set k, and
- unobservable by members of $k \backslash o$.

Therefore the condition $f(k) \subseteq o'$ is interpreted as follows. If the property $\langle o, k\rangle$ is satisfied or unobservable by the input, of the program f then the corresponding output will satisfy the property $\langle o', k'\rangle$. Hence a property $\langle o, k\rangle$ is related to a property $\langle o', k'\rangle$ in the approximable relation $\vdash_f$ if and only if the satisfaction and un-observability of the former property implies the satisfaction of the latter property. Let us fix this as a definition:

Definition 4.4.7. Let $f : Y_1 \longrightarrow Y_2$ be a continuous map between stably compact spaces Y_1 and Y_2. Let $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$, as defined above, be their lattices of observable properties. Let $\langle o, k\rangle$ and $\langle o', k'\rangle$ be two properties (elements) of $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$, respectively. Then we say that the property $\langle o, k\rangle$ *implies* the property $\langle o', k'\rangle$ under the program (map) f if and only of $f(k) \subseteq o'$.

Lemma 4.4.8. *Let $f : Y_1 \longrightarrow Y_2$ be a continuous map between stably compact spaces Y_1 and Y_2. Let $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$ be their lattices of properties. Let $\langle o, k\rangle$ and $\langle o', k'\rangle$ be two properties (elements) of $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$, respectively. Let $\vdash_f$ be the approximable relation corresponding to f. Then*

$$\langle o, k\rangle \vdash_f \langle o', k'\rangle \iff \langle o, k\rangle \textit{ implies } \langle o', k'\rangle, \textit{under } f.$$

The separator $\ltimes_f$, corresponding to f and $\vdash_f$, relates prime filters, which are in the language of logic, models of properties, of the strong proximity lattice $\mathcal{B}_{Y_1}$ to prime filters of the strong proximity lattice $\mathcal{B}_{Y_2}$. Under $\ltimes_f$ a model $F \in \mathit{filt}_p(\mathcal{B}_{Y_1})$ is related to a model $F' \in \mathit{filt}_p(\mathcal{B}_{Y_2})$ if and only if there is a property in the model F that implies a property in the complement of the model F'. This has the following computational consequence.

Lemma 4.4.9. *Let $f : Y_1 \longrightarrow Y_2$ be a continuous map between stably compact spaces Y_1 and Y_2. Let*

- *$\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$ be their lattices of properties, respectively, and $filt_p(\mathcal{B}_{Y_1})$ and $filt_p(\mathcal{B}_{Y_1})$ be their corresponding Priestley spaces, respectively,*
- *$\ltimes_f$ be the separator corresponding to f, and*
- *F be a model in $filt_p(\mathcal{B}_{Y_1})$ that is not related, under the relation $\ltimes_f$, to a model F' in $filt_p(\mathcal{B}_{Y_2})$.*

Then the set of all properties implied by a property in F is contained in F'.

4.4.2 Examples

Consider the unit interval $[0,1]$ together with its Scott topology. It is a stably compact space and its corresponding Priestley space X was explained in detail in Section 3.3.4. In the following we show the separators that correspond to two simple continuous (with respect to Scott topology) maps on $[0,1]$.

1. The function

$$\begin{aligned} S:[0,1] &\rightarrow [0,1] \\ x &\mapsto \begin{cases} 0 & : \quad 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & : \quad x > \frac{1}{2}, \end{cases} \end{aligned}$$

corresponds to the separator $\ltimes_s$ defined as follows

$$\begin{aligned} \ltimes_S = {} & \{(x_r, y_{r'}) \mid x_r \in \{F_r^1, F_r^2, G_r^1, G_r^2\}, y_{r'} \in \{F_{r'}^1, F_{r'}^2, G_{r'}^1, G_{r'}^2\}, \\ & \frac{1}{2} < r \leq 1, 0 \leq r' < \frac{1}{2}\}. \end{aligned}$$

2. The function

$$\begin{aligned} T:[0,1] &\rightarrow [0,1] \\ x &\mapsto x^2, \end{aligned}$$

corresponds to the separator $\ltimes_T$ defined as follows

$$\begin{aligned}\ltimes_T &= \{(x_t, y_{r^2}) \mid x_t \in \{F_t^1, F_t^2, G_t^1, G_t^2\}, y_{r^2} \in \{F_{r^2}^1, F_{r^2}^2, G_{r^2}^1, G_{r^2}^2\},\\ &\quad r < t \leq 1, 0 \leq r < 1\}.\end{aligned}$$

4.5 The Co-compact Topology

This section has two objectives. The first one is to prove in full detail the isomorphism in Corollary 4.2.10. The second objective is to show that co-compact topology on $core(X)$, for a Priestley space X equipped with apartness $\propto$, is homeomorphic to the core of X^∂ equipped with $\propto^{-1}$. In other words, this section investigates the direct relationship between Priestley spaces equipped with apartness relations, their duals, their cores and co-compact topologies of their cores. This is important because the co-compact topology plays a pivot role in the relationship between stably compact spaces and compact ordered spaces. The later relationship was discussed in detail in Section 2.6.

Theorem 4.5.1. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle iso_l(X), \subseteq\rangle$ and $\langle \mathcal{K}_{core(X))}, \supseteq\rangle$ are isomorphic via the following mappings:*

$$comp : iso_l(X) \longrightarrow \mathcal{K}_{core(X)}; F \longmapsto core(X)\backslash F, \text{ and}$$

$$S : \mathcal{K}_{core(X)} \longrightarrow iso_l(X); K \longmapsto X\backslash{\uparrow}K.$$

Proof. Let $F \in iso_l(X)$. Then

$$\begin{aligned} F &\longmapsto \Psi(F), \text{via Theorem } 4.2.9\\ &\longmapsto \{\mathcal{F} \in pt(iso_u(X)) \mid \Psi(F) \subseteq \mathcal{F}\}, \text{via Hofmann-Mislove Theorem[4]}\\ &= \{\mathcal{F} \in pt(iso_u(X)) \mid F \subseteq \Phi(\mathcal{F})\},\ \text{because } \Phi = \Psi^{-1}\\ &\longmapsto \{\Delta(\mathcal{F}) \mid \mathcal{F} \in pt(iso_u(X)) \text{ and } F \subseteq \Phi(\mathcal{F})\},\ \text{by Theorem } 4.2.15\\ &\longmapsto \{\Delta(\mathcal{F}) \mid \mathcal{F} \in pt(iso_u(X)) \text{ and } \Delta(\mathcal{F}) \notin F\},\ \text{by Remark } 4.2.17\\ &= core(X)\backslash F,\ \text{because } \Delta \text{ is onto.}\end{aligned}$$

Let $K \in \mathcal{K}_{core(X)}$. Then

$$\begin{aligned}
K &\longmapsto \{\Theta(x) \mid x \in K\}, \text{by Theorem 4.2.15} \\
&\longmapsto \bigcap\{\Theta(x) \mid x \in K\}, \text{via Hofmann-Mislove Theorem[4]} \\
&\longmapsto \Phi(\bigcap\{\Theta(x) \mid x \in K\}), \text{via } 4.2.9 \\
&= \bigcap\{\Phi(\Theta(x)) \mid x \in K\}, \text{because } \Phi \text{ preserves intersection} \\
&= \bigcap\{\Phi(\Psi(X\backslash{\uparrow}x)) \mid x \in K\}, \text{ by } Remark\ 4.2.17 \\
&= \bigcap\{X\backslash{\uparrow}x \mid x \in K\} \\
&= X\backslash{\uparrow}K.
\end{aligned}$$

□

Lemma 4.5.2. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then $\langle core(X), (\mathcal{T}_{core(X)})_c\rangle$ is homoeomorphic to $\langle core(X^\partial), \mathcal{T}_{core(X^\partial)}\rangle$ via the mappings:*

$$f : core(X) \longrightarrow core(X^\partial); x \longmapsto \max(X\backslash{\propto}[x]), \textit{ and}$$

$$g : core(X^\partial) \longrightarrow core(X); y \longmapsto \min(X\backslash[y]{\propto}).$$

Furthermore, the frame isomorphism between $iso_l(X) \cong (\mathcal{T}_{core(X)})_c$ and $iso_u(X^\partial) \cong \mathcal{T}_{core(X^\partial)}$ that arises from this homeomorphism is the identity.

Proof. Let $x \in core(X)$. Then by Corollary 4.2.5, ${\propto}[x] \in iso_u(X)$. Suppose c, d are two distinct maximal elements in $X\backslash{\propto}[x]$. Then $({\uparrow}c \cap {\uparrow}d) \propto x$. Therefore by $({\uparrow}{\uparrow}{\propto})$, we have $c \propto x,\ d \propto x$ or $x \propto x$. But $x \not\propto x$ because $x \in core(X)$. Hence $c \propto x$ or $d \propto x$; i.e. $c \in {\propto}[x]$ or $d \in {\propto}[x]$, which is a contradiction. Therefore $X\backslash{\propto}[x] = {\downarrow}y$ for some $y \in X$. Therefore ${\propto}[x] = X\backslash{\downarrow}y$ and hence $X\backslash{\downarrow}y$ is isolated. So $\max(X\backslash{\propto}[x]) = y \in core(X^\partial)$. Dually, g is well-defined. We show that

$$x = \min(X\backslash[\max(X\backslash{\propto}[x])]{\propto}).$$

We first prove that

$$[\max(X\backslash\propto[x])]\propto = [x]\propto.$$

One inclusion is proved as follows:

$$\begin{aligned} x \propto t &\implies x \propto \max(X\backslash\propto[x]) \text{ or } \max(X\backslash\propto[x]) \propto t, \text{ by } (\propto\forall) \\ &\implies \max(X\backslash\propto[x]) \propto t, \text{ because } x \in X\backslash\propto[x]. \end{aligned}$$

The other inclusion is proved as follows.

$$\begin{aligned} \max(X\backslash\propto[x]) \propto t &\implies \max(X\backslash\propto[x]) \propto x \text{ or } x \propto t, \text{ by } (\propto\forall) \\ &\implies x \propto t. \end{aligned}$$

Since $X\backslash\uparrow x = [x]\propto$,

$$\uparrow x = X\backslash[\max(X\backslash\propto[x])]\propto.$$

Therefore

$$x = \min(X\backslash[\max(X\backslash\propto[x])]\propto).$$

Dually for $y \in \mathit{core}X^{\partial}$,

$$y = \max(X\backslash\propto[\min(X\backslash[x]\propto)]).$$

By Theorem 4.2.15, every open set on $\mathit{core}(X^{op})$ is of the form

$$\mathcal{O}_A^{op} = \{y \in \mathit{core}(X^{op}) \mid y \in A\}$$

for some open lower set $A \in \mathit{iso}_u(X^{\partial}) = \mathit{iso}_l(X)$. Let $x \in \mathit{core}(X)$. Then

$$\begin{aligned} \max(X\backslash\propto[x]) \in \mathcal{O}_A^{op} &\iff \max(X\backslash\propto[x]) \in A \\ &\iff X\backslash\propto[x] \subseteq A \\ &\iff X\backslash A \subseteq \propto[x] \\ &\iff x \in [X\backslash A]\propto = A \\ &\iff x \in \mathit{core}(X)\backslash\mathit{comp}(A), \end{aligned}$$

where comp is the function defined in Theorem 4.5.1. □

4.6 Isolated Sets and Round Ideals and Filters

As we have seen so far, the concepts of lower and upper isolated subsets of Priestley spaces equipped with apartness relations play a pivot role in our theory, in general, and in this chapter, in particular. In this section, we introduce yet another meaning for isolated sets; we show a one-to-one correspondence between lower (upper) isolated subsets of a Priestley space equipped with apartness and round filters (ideals) of the corresponding strong proximity lattice.

Theorem 4.6.1. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then the lattice $\langle iso_u(X); \subseteq\rangle$ is isomorphic to the lattice $\langle ridl(Pries(X)); \subseteq\rangle$ via the mappings:*

$$\Psi : iso_u(X) \longrightarrow ridl(Pries(X)); O \longmapsto \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid U \subseteq O\}, \text{ and}$$

$$\Phi : ridl(Pries(X)) \longrightarrow iso_u(X); I \longmapsto \cup I.$$

Proof. We first show that Ψ and Φ are well defined.

1. $\downarrow_{\prec_\propto} \Psi(O) = \Psi(O)$ because for any $U \in \mathcal{U}^{\mathcal{T}}(X)$,

$$\begin{aligned} U \in \Psi(O) &\iff U \propto (X\backslash O) \\ &\iff (X\backslash[U]\propto) \propto (X\backslash O) \text{ by } (\propto\forall) \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X))\ (X\backslash[U]\propto) \subseteq U' \text{ and } U' \propto (X\backslash O) \text{ by Lemma 3.3.3} \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X))\ U \propto (X\backslash U') \text{ and } U' \subseteq O \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X))\ U \prec_\propto U' \text{ and } U' \in \Psi(O). \end{aligned}$$

For the second equivalence, we note that $[U]\propto$ is open by the dual of Lemma 4.2.4(1). Also, $\Psi(O)$ is clearly lower and closed under finite unions. Therefore Ψ is well-defined.

2. We notice that

$$\begin{aligned} x \in \Phi(I) &\iff (\exists\, U \in \mathcal{U}^{\mathcal{T}}(X))\; x \in U \in I \\ &\iff (\exists\, U, U' \in \mathcal{U}^{\mathcal{T}}(X))\; x \in U \prec_{\propto} U' \in I \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X))\; x \propto (X \backslash U') \text{ and } U' \in I, \text{ by Lemma 3.3.3} \\ &\iff x \propto (X \backslash \Phi(I)). \end{aligned}$$

For the the right-to-left direction of the last equivalence, by Lemma 3.3.3, there exists $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $x \propto V$ and $X\backslash\Phi(I) \subseteq V$. Then U' exists because $\Phi(I)$ is an open cover to the compact set $X\backslash V$. $\Phi(I)$ is clearly an open upper set and hence Φ is well-defined.

Ψ and Φ are clearly monotone and they are inverses of each other by Priestley's representation theorem. □

Corollary 4.6.2. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then the set of prime round ideals of $Pries(X)$ is isomorphic to the $core(X^{\partial})$.*

Proof. The statement follows from Theorem 4.6.1, Remark 4.2.2 and [20, exercises 11.17]. □

Remark 4.6.3. Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space with apartness $\propto$. Then, dually to Theorem 4.6.1 and Corollary 4.6.2, the lattice $\langle iso_l(X); \subseteq\rangle$ is isomorphic to the lattice $\langle rfilt(Pries(X)); \subseteq\rangle$ and the set of prime round filters of $Pries(X)$ is isomorphic to the $core(X)$.

4.7 An Extension of Jung-Sünderhauf Representation Theorem

In this section we extend the Jung-Sünderhauf representation theorem [54] (reviewed in Section 2.7) between strong proximity lattices and stably compact spaces. We show that removing the compactness condition from the stably compact spaces side is equivalent to removing the condition of having a top element from the strong proximity lattices side. Removing the latter condition must be associated with removing any use of the empty meet which is the top element. The resulting definition is the following.

Definition 4.7.1. A binary relation $\prec$ on a distributive lattice $\langle L; \vee, \wedge \rangle$ with 0 is called a *strong proximity* if, for every $a, x, y \in L$ and $M \subseteq_{\mathit{fin}} L$,

$$
\begin{array}{rl}
(\prec\prec) & \prec \circ \prec = \prec, \\
(\vee - \prec) & M \prec a \Longleftrightarrow \bigvee M \prec a, \\
(\prec - \wedge)' & a \prec x \text{ and } a \prec y \Longleftrightarrow a \prec x \wedge y, \\
(\prec - \vee) & a \prec x \vee y \Longrightarrow (\exists\, x', y' \in L)\; x' \prec x,\; y' \prec y \text{ and } a \prec x' \vee y', \\
(\wedge - \prec) & x \wedge y \prec a \Longrightarrow (\exists\, x', y' \in L)\; x \prec x',\; y \prec y' \text{ and } x' \wedge y' \prec a.
\end{array}
$$

A *pointed-strong proximity lattice* is a distributive lattice $\langle L; \vee, \wedge \rangle$ with 0 together with a strong proximity relation $\prec$ on L.

On the morphism level the definition of approximable relations need to be adjusted to get the following definition.

Definition 4.7.2. Let $\langle L_1; \vee, \wedge, 0; \prec_1 \rangle$ and $\langle L_2; \vee, \wedge, 0; \prec_2 \rangle$ be pointed-strong proximity lattices and let $\vdash$ be a binary relation from L_1 to L_2. The relation $\vdash$ is called *pointed-approximable* if for every $a \in L_1, b \in L_2$, $M_1 \subseteq_{\mathit{fin}} L_1, b, c \in L_2$ and $M_2 \subseteq_{\mathit{fin}} L_2$,

$$
\begin{array}{rl}
(\vdash - \prec_2) & \vdash \circ \prec_2 = \vdash, \\
(\prec_1 - \vdash) & \prec_1 \circ \vdash = \vdash, \\
(\vee - \vdash) & M_1 \vdash b \Longleftrightarrow \bigvee M_1 \vdash b, \\
(\vdash - \wedge)' & a \vdash b \text{ and } a \vdash c \Longleftrightarrow a \vdash b \wedge c, \\
(\vdash - \vee) & a \vdash \bigvee M_2 \Longrightarrow (\exists\, N \subseteq_{\mathit{fin}} L_1)\; a \prec_1 \bigvee N \text{ and } (\forall n \in N) \\
 & (\exists\, m \in M_2)\; n \vdash m.
\end{array}
$$

The relation $\vdash$ is called *weakly pointed-approximable* if it satisfies all of the above conditions but not necessarily ($\vdash -\vee$).

Therefore we prove that the category of stably locally compact spaces, which are those topological spaces that are sober and locally compact and in which binary intersections of compact saturated subsets are compact, and continuous maps between them is equivalent to the category of pointed-strong proximity lattices and pointed-approximable relations between them.

One interesting point about this extension is that stably locally compact spaces cover all coherent domains in their Scott-topologies whereas stably compact spaces cover only coherent domains which are compact in their Scott-topologies.

4.7.1 From Pointed-Strong Proximity Lattices to Stably Locally Compact Spaces

We get stably locally compact spaces from pointed-strong proximity lattice in exactly the same way we get stably compact spaces from strong proximity lattices. All proofs still work perfectly for the general case in this section.

4.7.2 From Stably Locally Compact Spaces to Pointed-Strong Proximity Lattices

The proof of the basic result (Theorem 2.7.9) in the direction from stably compact spaces to strong proximity lattices is not obviously true for the general case of stably locally compact spaces and pointed-strong proximity lattices. The reason is that the proof relies on the fact that for any stably compact space $\langle Y, \mathcal{T} \rangle$, the ordered set $\langle \mathcal{K}_Y, \supseteq \rangle$ is a stably continuous frame and hence for every $K \in \mathcal{K}_Y$,

$$K = \bigcap \{K' \in \mathcal{K}_Y \mid K' \ll K\}.$$

This fact is not obvious for stably locally compact spaces; it needs to be proved for achieving the more general result.

In the following we first prove this for stably locally compact spaces and then we review in full detail how we get pointed-strong proximity lattices from stably locally compact spaces.

We remind the reader that for a stably locally compact space $\langle Y, \mathcal{T}\rangle$, $\mathcal{K}_Y$ is the collection of compact saturated subsets of Y ordered by $\supseteq$.

Theorem 4.7.3. *For a stably locally compact space $\langle Y, \mathcal{T}\rangle$,*

1. $(\forall K, K' \in \mathcal{K}_Y)\ K \ll K' \iff (\exists O \in \mathcal{T})\ K' \subseteq O \subseteq K.$
2. *for every $K \in \mathcal{K}_Y$, the set $\{K' \in \mathcal{K}_Y \mid K' \ll K\}$ is filtered and its intersection equals K.*

Proof. 1. By the Hofmann-Mislove Theorem [59, 29] and local compactness,

$$K' = \bigcap \{N \mid N \text{ is compact saturated and } K' \subseteq N^\circ\},$$

where N° denotes the interior of N (Definition B.2.2). The set in the right hand side is filtered and hence the intersection is a directed join in the lattice of compact saturated subsets of Y. Suppose $K \ll K'$. Then there exists N such that $K \leq N \leq K'$ i.e. $K' \subseteq N \subseteq K$. Therefore $K' \subseteq N^\circ \subseteq N \subseteq K$.

For the other direction, suppose $K' \subseteq O \subseteq K$. If $K' \leq \bigcap\{K_i\}_{i\in I}$ i.e. $\bigcap\{K_i\}_{i\in I} \subseteq K'$ then $\bigcap\{K_i\}_{i\in I} \subseteq O$. Hence by Hofmann-Mislove Theorem, $K_i \subseteq O,$ for some $i \in I$. Therefore $K_i \subseteq K$ i.e. $K \leq K_i$.

2. Let $K \in \mathcal{K}_Y$. From 1, it is clear that

$$\{K' \in \mathcal{K}_Y \mid K' \ll K\},$$

is filtered and $K \subseteq \bigcap\{K' \in \mathcal{K}_Y \mid K' \ll K\}$. For the converse, suppose x belongs to the intersection and O is an open set which contains K. By local compactness, there exists a compact saturated subset $N \subseteq O$ such that $K \subseteq N^{\circ}$. Therefore $N \ll K$ and hence it contains x. So $x \in O$. Therefore

$$x \in \bigcap \{O \mid O \in \mathcal{T} \text{ and } K \subseteq O\} = K.$$

□

Theorem 4.7.4. *For a stably locally compact space $\langle Y, \mathcal{T}\rangle$, the algebra $\langle \mathcal{B}_Y, \vee_Y, \wedge_Y, 0_Y; \prec_Y\rangle$, where*

- $\mathcal{B}_Y := \{(O, K) \in \mathcal{T} \times \mathcal{K}_Y \mid O \subseteq K\}$,
- $(O, K) \vee_Y (O', K') := (O \cup O', K \cup K')$,
- $(O, K) \wedge_Y (O', K') := (O \cap O', K \cap K')$,
- $0_Y := (\emptyset, \emptyset)$,
- $(O, K) \prec_Y (O', K') \overset{\text{def}}{\Longleftrightarrow} K \subseteq O'$.

is a pointed-strong proximity lattice. Moreover, $\langle Y, \mathcal{T}\rangle$ and spec$(\mathcal{B}_Y)$ equipped with the canonical topology are homeomorphic.

Proof. Clearly, $\langle \mathcal{B}_Y, \vee_Y, \wedge_Y\rangle$ is a distributive lattice with 0_Y as its bottom element and $\prec_Y \circ \prec_Y \subseteq \prec_Y$. The other inclusion ($\prec_Y \subseteq \prec_Y \circ \prec_Y$) follows from local compactness. It is easy to prove $(\vee\!- \prec)$ and $(\prec -\wedge)$.

$(\prec -\vee)$ follows from compactness. To show $(\wedge\!- \prec)$, suppose $K_1 \cap K_2 \subseteq O$, where $O \in \mathcal{T}, K_1, K_2 \in \mathcal{K}_Y$. Since

$$K_i = \bigcap \{K_i' \in \mathcal{K}_Y \mid K_i' \ll K_i\}, \text{ for } i = 1, 2,$$

$K_1 \cap K_2 = \bigcap\{K_1' \cap K_2' \mid K_1', K_2' \in \mathcal{K}_Y, K_1' \ll K_1, K_2' \ll K_2\}$. The latter intersection is a filtered intersection of compact saturated subsets of Y. Therefore by Hofmann-Mislove Theorem [59, 29], there exists $K_1' \cap K_2' \subseteq O$.

Now we show that $\langle Y, \mathcal{T}\rangle$ and $\mathrm{spec}(\mathcal{B}_Y)$ equipped with the canonical topology are homeomorphic. It is enough to show that the topologies are isomorphic, because the topologies are sober. To show that, we define:

$$\Psi : \langle \mathcal{T}, \subseteq\rangle \longrightarrow \langle ridl(\mathcal{B}), \subseteq\rangle; O \longmapsto \{(U, K) \in \mathcal{B} \mid U \subseteq K \subseteq O\},$$

and

$$\Phi : \langle ridl(\mathcal{B}), \subseteq\rangle \longrightarrow \langle \mathcal{T}, \subseteq\rangle; I \longmapsto \bigcup\{O \mid (\exists K \in \mathcal{K}_Y)\ (O, K) \in I\}.$$

It is easy to check that Ψ, Φ are well defined and monotone. Now, we show that Ψ, Φ are inverses to each other.

$$\begin{aligned}
\Phi(\Psi(O)) &= \bigcup\{O' \in \mathcal{T} \mid (\exists K' \in \mathcal{K}_Y)\ (O', K') \in \Psi(O)\} \\
&= \bigcup\{O' \in \mathcal{T} \mid (\exists K' \in \mathcal{K}_Y)\ O' \subseteq K' \subseteq O\} \\
&= O, \text{ by local compactness.}
\end{aligned}$$

$$\begin{aligned}
(O, K) \in \Psi(\Phi(I)) &\iff O \subseteq K \subseteq \Phi(I) \\
&\iff O \subseteq K \subseteq \bigcup\{O' \mid (\exists K' \in \mathcal{K}_Y)\ (O', K') \in I\} \\
&\iff K \subseteq O', \text{ for some } (O', K') \in I, \text{ because } K \text{ is compact} \\
&\qquad \text{and } I \in ridl(\mathcal{B}) \\
&\iff (O, K) \in I.
\end{aligned}$$

□

The Jung-Sünderhauf machinery works perfectly (without any modifications) for the new morphism, i.e under Jung-Sünderhauf duality, there is a one-to-one correspondence between pointed-approximable relations and continuous maps between stably locally compact spaces.

Therefore we have the following result:

Theorem 4.7.5. *The category of pointed-strong proximity lattices and pointed-approximable relations between them is equivalent to the category of stably locally compact spaces and continuous functions between them.*

Chapter 5

Priestley Semantics for MLS

5.1 Introduction

In Chapter 3, the notion of apartness relation (Definitions 3.1.1 and 3.1.3) on Priestley spaces was introduced to accomplish the task of extending the Priestley duality of bounded distributive lattices to a duality theorem for strong proximity lattices (Definition 1.1.2). Therefore the category **PSws**, of Priestley spaces equipped with apartness relations as objects and weakly separating relations (Definition 3.1.5) between them as morphisms was introduced in Chapter 3 to represent the category **PLwa** of strong proximity lattices as objects and weakly approximable relations between them as morphisms (Definition 2.7.2). The objects of the latter category were introduced, by Achim Jung and Philipp Sünderhauf in [54] (reviewed in Section 2.7) to provide a finitary representation for stably compact spaces (Section 2.6) which are topological spaces that capture most semantics domains in the mathematical theory of computation. Therefore it was essential to study the direct links between Priestley spaces equipped with apartness relations and stably compact spaces; this was done in Chapter 4.

The category **PLwa** and the duality established in [54], then, are the basis for a logical description (expounded in [50, 52] and reviewed in Section 2.8), of stably compact spaces, similar to Samson Abramsky's *domain theory in logical form*, [2]. This logical description is via the category **MLS** (Multi Lingual Sequents) which has logical systems, the so-called coherent sequent calculi, as objects and consequence relations, the so-called compatible consequence relations (Definition 2.8.1), between them as morphisms. It was shown, in [52], that the categories **MLS** and **PLwa** are equivalent. Therefore it is obvious that the category **MLS** is a logical description to the category **PSws**.

Hence, once again it is our duty to explore the direct links between the categories **PSws** and **MLS**. This is the first goal of this chapter whose second goal is to show that **PSws** provides a useful semantics for **MLS**. The latter goal is achieved via establishing Priestley semantics (in **PSws**) for different **MLS**'s concepts and facts, i.e. via translating **MLS**'s concepts and facts into corresponding concepts and facts, respectively, in the category **PSws**. The third object of this chapter is to further our study of Priestley spaces equipped with apartness relations. This is be done via linking the category **PSws** to other categories in mathematics.

In the previous chapter we have shown how the duality, in [54], between strong proximity lattices and stably compact spaces can be extended to cover stably locally compact spaces. For this end, we have introduced the notion of *pointed-strong proximity lattice*. Therefore the fourth goal of this chapter is to show how **MLS** can be extended to describe stably locally compact spaces. The fifth and the final goal of this chapter is to show how domain constructions (like lifting, sum, product and Smyth power domain) can be done in the Priestley form.

5.1.1 Organisation

The chapter is organised as follows. Section 5.2 presents preparatory results that are necessary for the following sections. Sections 5.3 and 5.4 investigate the direct relationship between the categories **MLS** and **PLwa**. The equivalence of these categories is proved directly in Section 5.5.

Section 5.6 has two objectives:

1. Introducing Priestley semantics (in **PSws**) for **MLS**'s concepts and facts (such as compatibility, Gentzen's cut rule, round ideals and filters, and consistency).

2. Introducing a full and faithful functor from the category **PSws** to the category **SL** of directed-complete meet semilattices and Scott-continuous semilattice homomorphisms. This results in proving that the category **PSws** is equivalent to the image of this functor and therefore the full subcategory consisting of the image of the functor is self-dual as **PSws** is. The self-duality of this full sub-category was first noticed and proved in [65].

In Section 5.7, we show that the category **PSws** is equivalent to two other categories:

1. The Kleisli category $\mathbf{SCS}_{\mathcal{K}}$ of the Smyth power monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$ where $\mathcal{K}$ [52, section 6] is an endofunctor $\mathcal{K}$ on **SCS**. For an object $X \in$ **SCS**, $\mathcal{K}(X)$ (or $\mathcal{K}_X$) is the set of compact saturated subsets of X equipped with the Scott topology and for a morphism $f : X \longrightarrow Y$ in **SCS**, $\mathcal{K}(f)$ assigns to each compact saturated subset A of X the saturation of $f(A)$.

2. The category of stably compact spaces as objects and upper relations of the form $R \subseteq X \times Y_c$ as morphisms, where Y_c is the co-compact topology on Y.

In Section 5.8, we show how the category **MLS** can be modified (in a very simple way) to provide a logical description for the class of stably locally compact spaces which includes coherent domains in their Scott topologies. Section 5.9 shows how domain constructions can be done in the Priestley form.

5.2 Preparatory Results

This section presents technical results that are necessary for the following sections.

Lemma 5.2.1. *Let $\langle X, \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the following statements are true for every $a, c, d \in X$:*

$$(R\!-\propto) \qquad a \propto (\downarrow c \cap \downarrow d) \Longrightarrow (\forall b \in X)\; a \propto (\downarrow b \cap \downarrow d) \textit{ or } b \propto c,$$

$$(L\!-\propto) \qquad (\uparrow c \cap \uparrow d) \propto a \Longrightarrow (\forall b \in X)\; c \propto b \textit{ or } (\uparrow b \cap \uparrow d) \propto a.$$

Proof. We have

$$\begin{aligned} a \propto (\downarrow c \cap \downarrow d) &\Longrightarrow (\forall x \in X)\; a \propto x,\; x \propto c \text{ or } x \propto d, \text{ by } (\propto\downarrow\downarrow) \\ &\Longrightarrow (\forall x, b \in X)\; a \propto x,\; x \propto b,\; b \propto c \text{ or } x \propto d, \text{ by } (\propto\forall) \\ &\Longrightarrow (\forall x, b \in X)\; a \propto x,\; x \propto (\downarrow b \cap \downarrow d) \text{ or } b \propto c, \text{ by } (\uparrow\propto\downarrow) \\ &\Longrightarrow (\forall b \in X)\; a \propto (\downarrow b \cap \downarrow d) \text{ or } b \propto c, \text{ by } (\propto\forall). \end{aligned}$$

The argument for $(L\!-\propto)$ is dual. □

Lemma 5.2.2. *Let $\langle X, \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the following statements are true for every $a, c_1, \ldots, c_n \in X$:*

$$a \propto (\downarrow c_1 \cap \ldots \cap \downarrow c_n) \Longrightarrow (\forall b \in X)\; a \propto (\downarrow b \cap \downarrow c_2 \cap \ldots \cap \downarrow c_n) \textit{ or } b \propto c_1,$$

$$(\uparrow c_1 \cap \ldots \cap \uparrow c_n) \propto a \Longrightarrow (\forall b \in X)\; c_1 \propto b \textit{ or } (\uparrow b \cap \uparrow c_2 \cap \ldots \cap \uparrow c_n) \propto a.$$

Proof. The proof follows the same lines as that of Lemma 5.2.1. We will just have to use $(\ltimes n\downarrow)$ instead of $(\propto\downarrow\downarrow)$. Recall that $\propto$ satisfies $(\ltimes n\downarrow)$ because $\propto$ is a separator. □

Lemma 5.2.3. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Let $U_1, U_2 \in \mathcal{U}^{\mathcal{T}}(X)$ and $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $U_1 \cap U_2 \propto V$. If $O = \{x \in X \mid U_1 \propto x\}$ then*

$$((X \backslash O) \cap U_2) \propto V.$$

Proof. By$(\uparrow\propto\downarrow)$ O is a lower set and is open by Lemma 3.3.3. Obviously, $U_1 \propto O$. Let $x \in (X\backslash O) \cap U_2$ and $a \in V$. Then there exist $e \in X\backslash O$ and $d \in U_2$ such that $x \in {\uparrow}e \cap {\uparrow}d$. By definition of O there exists $c_e \in U_1$ such that $c_e \not\propto e$. But $({\uparrow}c_e \cap {\uparrow}d) \propto a$ because $U_1 \cap U_2 \propto V$. Then by condition $(L\!- \propto)$ of Lemma 5.2.1 $({\uparrow}e \cap {\uparrow}d) \propto a$ which implies $x \propto a$. □

Lemma 5.2.4. *Let $\langle X_1; \leq_1; \mathfrak{T}_1\rangle$ and $\langle X_2; \leq_2, \mathfrak{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weakly separating relation from X_1 to X_2. Then the following statements are true for every $a, e \in X_1,\ c, d \in X_2$:*

$$(R\!-\ltimes) \qquad a \ltimes ({\downarrow}c \cap {\downarrow}d) \Longleftrightarrow (\forall b \in X_2)\ a \ltimes ({\downarrow}b \cap {\downarrow}d) \textit{ or } b \propto_2 c,$$

$$(L\!-\ltimes) \qquad ({\uparrow}a \cap {\uparrow}e) \ltimes c \Longleftrightarrow (\forall b \in X_1)\ a \propto_1 b \textit{ or } ({\uparrow}b \cap {\uparrow}e) \ltimes c.$$

Proof. We have

$$\begin{aligned} a \ltimes ({\downarrow}c \cap {\downarrow}d) &\Longrightarrow (\forall t \in X_1)\ a \ltimes t \text{ or } t \propto_2 ({\downarrow}c \cap {\downarrow}d),\ \text{by } (\ltimes\forall) \\ &\Longrightarrow (\forall t, b \in X_1)\ a \ltimes t,\ t \propto_2 ({\downarrow}b \cap {\downarrow}d),\ \text{or } b \propto_2 c,\ \text{by Lemma 5.2.1} \\ &\Longrightarrow (\forall b \in X_1)\ a \ltimes ({\downarrow}b \cap {\downarrow}d) \text{ or } b \propto_2 c,\ \text{by } (\ltimes\forall). \end{aligned}$$

The other direction is proved as follows:

$$\begin{aligned} (\forall b \in X_2)\ a \ltimes ({\downarrow}b \cap {\downarrow}d) \text{ or } b \propto_2 c &\Longrightarrow (\forall b, x \in X_2)\ a \ltimes ({\downarrow}b \cap {\downarrow}x),\ x \propto_2 d \text{ or } b \propto_2 c \\ &\Longrightarrow (\forall b \in X_2)\ a \ltimes b,\ b \propto_2 c \text{ or } b \propto_2 d \\ &\Longrightarrow (\forall b \in X_2)\ a \ltimes b \text{ or } b \propto_2 ({\downarrow}c \cap {\downarrow}d) \text{ by } (\uparrow\propto\downarrow) \\ &\Longrightarrow a \ltimes ({\downarrow}c \cap {\downarrow}d) \text{ by } (\ltimes\forall). \end{aligned}$$

The first implication is true by the left-to-right direction, proved above, and the second is true by taking $x = b$. The argument for $(L\!-\ltimes)$ is dual. □

Lemma 5.2.5. *Let $\langle X; \leq, \mathfrak{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Let $V \in \mathcal{O}^{\mathfrak{T}}(X)$ and $U, U_1, \ldots, U_n \in \mathcal{U}^{\mathfrak{T}}(X)$ such that $(\bigcap_i U_i \cap U) \propto V$. If for every $1 \leq i \leq n$*

$O_i = \{x \in X \mid U_i \propto x\}$ *then*

$$(\bigcap_i (X \backslash O_i) \cap U) \propto V.$$

Proof. By($\uparrow\propto\downarrow$) every O_i is a lower set and is open by Lemma 3.3.3. Obviously, $U_i \propto O_i$. Let $x \in \bigcap_i (X \backslash O_i) \cap U$ and $a \in V$. Then there exists $d \in U$ and for every i there exists $e_i \in X \backslash O_i$ such that $x \in \bigcap_i \uparrow e_i \cap \uparrow d$. For every i, by definition of O_i there exists $c_e^i \in U_1$ such that $c_e^i \not\propto e_i$. But $(\bigcap_i \uparrow c_e^i \cap \uparrow d) \propto a$ because $(\bigcap_i U_i \cap U) \propto V$. Then by Lemma 5.2.2 $(\bigcap_i \uparrow e_i \cap \uparrow d) \propto a$ which implies $x \propto a$. □

Corollary 5.2.6. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Let $V \in \mathcal{O}^{\mathcal{T}}(X)$ and $U_1, \ldots, U_n \in \mathcal{U}^{\mathcal{T}}(X)$ such that $\bigcap_i U_i \propto V$. If for every $1 \leq i \leq n$, $O_i = \{x \in X \mid U_i \propto x\}$ then*

$$\bigcap_i X \backslash O_i \propto V.$$

Proof. The proof follows from Lemma 5.2.5 because $X \in \mathcal{U}^{\mathcal{T}}(X)$ and $\bigcap_i U_i = \bigcap_i U_i \cap X$. □

Lemma 5.2.7. *Let $\langle X, \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the following is true for every upper set D in X, $c, d \in X$, and $A, B, C \subseteq X$:*

1. $A \subseteq B \Longrightarrow [B]\propto \subseteq [A]\propto$.

2. $([A \cap B]\propto) \cap ([A \cap C]\propto) = [A \cap (B \cup C)]\propto$.

3. $c \propto (X \backslash D) \Longrightarrow [D \cap \uparrow d]\propto \subseteq [\uparrow c \cap \uparrow d]\propto$;

4. $c \propto (X \backslash D) \Longrightarrow [D \cap \uparrow c]\propto = [c]\propto$;

Proof. (1) and (2) are obvious. (3) is proved as follows.

$$\begin{aligned} c \propto (X \backslash D),\ (D \cap \uparrow d) \propto a &\Longrightarrow (\forall b \in X)\ c \propto b \text{ or } (\uparrow b \cap \uparrow d) \propto a \\ &\Longrightarrow (\forall b, t \in X)\ c \propto b,\ b \propto t,\ d \propto t \text{ or } t \propto a, \text{ by } (\uparrow\uparrow\propto) \\ &\Longrightarrow (\forall t \in X)\ c \propto t,\ d \propto t \text{ or } t \propto a, \text{ by } (\propto\forall) \\ &\Longrightarrow (\forall t \in X)\ (\uparrow c \cap \uparrow d) \propto t \text{ or } t \propto a, \text{ by } (\uparrow\propto\downarrow) \\ &\Longrightarrow (\uparrow c \cap \uparrow d) \propto a, \text{ by } (\propto\forall). \end{aligned}$$

(4) follows from $(1),\ (3)$ and $(\uparrow\propto\downarrow)$. □

5.3 From PSws to MLS

This section explores the direct way of obtaining coherent sequent calculi and compatible consequence relations from Priestley spaces equipped with apartness relations and weakly separating relations, respectively.

Definition 5.3.1. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weakly separating relation from X_1 to X_2. Then a consequence relation $\vdash_\ltimes$ from the algebra $\langle \mathcal{U}^\mathcal{T}(X_1); \cap, \cup, \emptyset, X_1\rangle$ to the algebra $\langle \mathcal{U}^\mathcal{T}(X_2); \cap, \cup, \emptyset, X_2\rangle$ is defined as follows:

$$\Gamma \vdash_\ltimes \Delta \overset{\text{def}}{\Longleftrightarrow} (\cap\Gamma) \ltimes (X_2 \backslash \cup \Delta).$$

Lemma 5.3.2. *Let X_1, X_2, and X_3 be Priestley spaces equipped with apartness relations $\propto_1, \propto_2$, and $\propto_3$, respectively. Let $\ltimes_1 \subseteq X_1 \times X_2$ and $\ltimes_2 \subseteq X_2 \times X_3$ be weak separators. Then*

$$\vdash_{\ltimes_1 \circ \ltimes_2} = \vdash_{\ltimes_1} \dagger \vdash_{\ltimes_2}.$$

Proof. Suppose $\Gamma \vdash_{\ltimes_1 \circ \ltimes_2} \Delta$ and $O = \{x \in X_2 \mid (\cap\Gamma) \ltimes_1 x\}$. Then, by Lemma 3.3.32, O is open, and by $(\forall - \mathit{comp})$ $(X_2 \backslash O) \ltimes_2 (X_3 \backslash \cup \Delta)$. Again by Lemma 3.3.32, there exists

$W \in \mathcal{U}^{\mathcal{T}}(X_2)$ such that $(\cap\Gamma) \ltimes_1 (X_2 \backslash W)$ and $W \ltimes_2 (X_3 \backslash \cup \Delta)$. Hence

$$\frac{\Gamma \vdash_{\ltimes_1} W \qquad W \vdash_{\ltimes_2} \Delta}{\Gamma \vdash_{\ltimes_1} \dagger \vdash_{\ltimes_2} \Delta} (Cut - Comp).$$

For the other direction, suppose $\Gamma \vdash_{\ltimes_1} \dagger \vdash_{\ltimes_2} \Lambda$. Then there exist $\Delta_1, \ldots, \Delta_n, \Theta_1, \ldots, \Theta_m$, finite subsets of $\mathcal{U}^{\mathcal{T}}(X_2)$, such that

$$\Gamma \vdash_{\ltimes_1} \Delta_1,\ \Gamma \vdash_{\ltimes_1} \Delta_2, \ldots, \Gamma \vdash_{\ltimes_1} \Delta_n \text{ and } \Theta_1 \vdash_{\ltimes_2} \Lambda, \Theta_2 \vdash_{\ltimes_2} \Lambda, \ldots, \Theta_m \vdash_{\ltimes_2} \Lambda.$$

Hence

$$(\cap\Gamma) \ltimes_1 (X_2 \backslash \cup \Delta_1),\ (\cap\Gamma) \ltimes_1 (X_2 \backslash \cup \Delta_2), \ldots, (\cap\Gamma) \ltimes_1 (X_2 \backslash \cup \Delta_n),$$

and

$$(\cap\Theta_1) \ltimes_2 (X_3 \backslash \cup \Lambda),\ (\cap\Theta_2) \ltimes_2 (X_3 \backslash \cup \Lambda), \ldots, (\cap\Theta_m) \ltimes_2 (X_3 \backslash \cup \Lambda).$$

Therefore

$$(\cap\Gamma) \ltimes_1 (\cup_i (X_2 \backslash \cup \Delta_i)) \text{ and } (\cup_i (\cap\Theta_i)) \ltimes_2 (X_3 \backslash (\cup\Lambda)).$$

But we have $\cup_i (X_2 \backslash \cup \Delta_i) = X_2 \backslash (\cap_i \cup \Delta_i)$ and $\cap_i \cup \Delta_i = \cup_f \cap X_f$, where $X_f \in \Pi_i \Delta_i$. By the side condition of $(Cut - Comp)$ each X_f covers Θ_j, for some j. Therefore for every X_f, $\cap X_f \subseteq \cap\Theta_j$, for some j. Consequently, $\cup_f \cap X_f \subseteq \cup_i (\cap\Theta_i)$. This implies

$$(\cap\Gamma) \ltimes_1 (X_2 \backslash (\cup_f (\cap X_f))) \text{ and } (\cup_f (\cap X_f)) \ltimes_2 (X_3 \backslash \cup \Lambda),$$

which implies that

$$(\forall x \in X_2)\ (\cap\Gamma) \ltimes_1 x \text{ or } x \ltimes_2 (X_3 \backslash \cup \Lambda).$$

Therefore

$$(\cap\Gamma) \ltimes_1 \circ \ltimes_2 (X_3 \backslash \cup \Lambda),$$

which implies $\Gamma \vdash_{\ltimes_1 \circ \ltimes_2} \Lambda$. □

The following corollary which follows from the previous two lemmas makes it clear that $(\forall - comp)$ provides a Priestley semantics for $(Cut - Comp)$ of consequence relations.

Corollary 5.3.3. *Let $\ltimes_1$ and $\ltimes_2$ be weak separators. Then*

$$\Delta \vdash_{\ltimes_1} \dagger \vdash_{\ltimes_2} \Gamma \Longleftrightarrow (\cap\Delta) \ltimes_1 \circ \ltimes_2 (X_3 \backslash \cup \Gamma).$$

Lemma 5.3.4. *Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then $\vdash_\propto$ is a consequence relation which has interpolants on the algebra $\langle \mathcal{U}^{\mathcal{T}}(X); \cap, \cup, \emptyset, X\rangle$.*

Proof. We prove that $\vdash_\propto$ satisfies condition $(L - \mathit{Int})$ of Definition 2.8.1. Suppose $\phi, \Gamma \vdash_\propto \Delta$. Then $(\phi \cap (\cap\Gamma)) \propto (X\backslash \cup \Delta)$. Set $O = \{x \in X \mid \phi \propto x\}$. Then by Lemma 5.2.3 $((X\backslash O) \cap (\cap\Gamma)) \propto (X\backslash \cup \Delta)$. Now by Lemma 3.3.3 and the dual of Lemma 3.3.1, there exists $\phi' \in \mathcal{U}^{\mathcal{T}}(X)$ such that $(X\backslash O) \subseteq \phi'$ and $(\phi' \cap (\cap\Gamma)) \propto (X\backslash \cup \Delta)$. Therefore $\phi \vdash_\propto \phi'$ and $\phi', \Gamma \vdash_\propto \Delta$. The argument for $(R - \mathit{Int})$ is dual. □

Remark 5.3.5. Using Lemma 3.3.3 and the dual of Lemma 3.3.1, it is not hard to prove that if $\vdash_\propto$ has interpolants then $\propto$ satisfies $(L\!-\propto)$ and $(R\!-\propto)$. Therefore $(L\!-\propto)$ and $(R\!-\propto)$ provide a Priestley semantics for the concept of interpolation in the category **MLS**.

Definition 5.3.6. Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then

$$\mathit{coh}(X) = \langle \mathcal{U}^{\mathcal{T}}(X); \cap, \cup, \emptyset, X; \vdash_\propto\rangle,$$

where $\vdash_\propto$ is the binary relation defined on $\mathcal{U}^{\mathcal{T}}(X)$ as follows:

$$\Gamma \vdash_\propto \Delta \overset{\text{def}}{\Longleftrightarrow} (\cap\Gamma) \propto (X_2 \backslash \cup \Delta).$$

Corollary 5.3.7. *Let $\langle X, \leq, \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then $\mathit{coh}(X)$ is a coherent sequent calculus.*

Proof. Recall that a consequence relation on an algebra with interpolants is closed under (Cut) if and only if it is closed under $(\mathit{Cut} - \mathit{Comp})$ (Lemma 2.8.3). The closedness of $\vdash_\propto$ under $(\mathit{Cut} - \mathit{Comp})$ follows from Corollary 5.3.3 and the fact that $\propto$ is closed under $(\forall - \mathit{comp})$. □

Corollary 5.3.8. *Let* $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ *and* $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ *be Priestley spaces equipped with apartness relations* $\propto_1$ *and* $\propto_2$*, respectively. Let* $\ltimes$ *be a weak separator from* X_1 *to* X_2*. Then* $coh(\ltimes) = \vdash_{\ltimes}$ *is a compatible consequence relation from the algebra* $\langle \mathcal{U}^{\mathcal{T}}(X_1); \cap, \cup, \emptyset,$
$X_1\rangle$ *to the algebra* $\langle \mathcal{U}^{\mathcal{T}}(X_2); \cap, \cup, \emptyset, X_2\rangle$.

Proof. The compatibility of $\vdash_{\ltimes}$ with $\vdash_{\propto_1}$ and $\vdash_{\propto_2}$ follows from Corollary 5.3.3 and the fact that $\propto_1 \circ \ltimes = \ltimes = \ltimes \circ \propto_2$. □

5.4 From MLS to PSws

Definition 5.4.1. For a coherent sequent calculus $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$, we let $\equiv$ stands for the least congruence [73] such that $L/\equiv$ is a bounded distributive lattice.

We remind the reader that $L/\equiv$ is ordered by the following order:

$$[\phi] \leq [\psi] \overset{\text{def}}{\Longleftrightarrow} \phi \wedge \psi \equiv \phi \Longleftrightarrow \phi \vee \psi \equiv \psi.$$

Joins and meets in $L/\equiv$ are defined as follows:

$$[\phi] \wedge [\psi] = [\phi \wedge \psi] \text{ and } [\phi] \vee [\psi] = [\phi \vee \psi].$$

Definition 5.4.2. Let $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$ be a coherent sequent calculus. Then a non-empty subset $I \subseteq L$ is said to be an *ideal* if it satisfies the following conditions:

1. $a \in I$ and $a \wedge b \equiv b \Longrightarrow b \in I$.

2. $a, b \in I \Longrightarrow a \vee b \in I$.

If an ideal I additionally satisfies the condition:

$$a \wedge b \in I \Longrightarrow a \in I \text{ or } b \in I,$$

then I is said to be a *prime ideal*.

Filters and prime filters of coherent sequent calculi are defined dually to ideals and prime ideals, respectively. We let $\mathit{filt}(L)$ and $idl(L)$ denote the partial orders of filters and ideals, respectively, ordered by inclusion. We also let $\mathit{filt}_p(L)$ and $idl_p(L)$ denote the partial orders of prime filters and prime ideals, respectively.

Lemma 5.4.3. *Let $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$ be a coherent sequent calculus. Then $\mathit{filt}(L)$ and $\mathit{filt}(L/\equiv)$ are isomorphic via the following maps:*

$$\Psi : \mathit{filt}(L) \longrightarrow \mathit{filt}(L/\equiv); F \longmapsto \{[\phi] \mid \phi \in F\}, \textit{ and}$$

$$\Phi : \mathit{filt}(L/\equiv) \longrightarrow \mathit{filt}(L); \mathcal{F} \longmapsto \bigcup \mathcal{F}.$$

Proof. Let $F \in \mathit{filt}(L)$. Then Ψ is well-defined because

$$[\phi] \in \Psi(F) \text{ and } [\psi] \geq [\phi] \Longrightarrow \phi \in F \text{ and } \phi \vee \psi \equiv \psi \Longrightarrow \psi \in F \Longrightarrow [\psi] \in \Psi(F), \text{ and}$$

$$[\phi], [\psi] \in \Psi(F) \Longrightarrow \phi, \psi \in F \Longrightarrow \phi \wedge \psi \in F \Longrightarrow [\phi \wedge \psi] \in \Psi(F).$$

Let $\mathcal{F} \in \mathit{filt}(L/\equiv)$. Then Φ is well-defined because

$$\phi \in \Phi(\mathcal{F}), \phi \vee \psi \equiv \psi \Longrightarrow [\psi] \geq [\phi] \in \mathcal{F} \Longrightarrow [\psi] \in \mathcal{F} \Longrightarrow \psi \in \Phi(\mathcal{F}), \text{ and}$$

$$\phi, \psi \in \Phi(\mathcal{F}) \Longrightarrow [\phi], [\psi] \in \mathcal{F} \Longrightarrow [\phi] \wedge [\psi] = [\phi \wedge \psi] \in \mathcal{F} \Longrightarrow \phi \wedge \psi \in \Phi(\mathcal{F}).$$

It is also true that $\mathcal{F} = \Psi(\Phi(\mathcal{F}))$ and $F = \Phi(\Psi(F))$ because

$$[\phi] \in \mathcal{F} \Longleftrightarrow \phi \in \Phi(\mathcal{F}) \Longleftrightarrow [\phi] \in \Psi(\Phi(\mathcal{F})), \text{ and}$$

$$\phi \in F \Longleftrightarrow [\phi] \subseteq F \Longleftrightarrow [\phi] \in \Psi(F) \Longleftrightarrow \phi \in \Phi(\Psi(F)).$$

□

Corollary 5.4.4. *Let $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$ be a coherent sequent calculus. Then $\mathit{filt}_p(L)$ and $\mathit{filt}_p(L/\equiv)$ are isomorphic via the mappings of Lemma 5.4.3.*

Definition 5.4.5. Let $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$ be a coherent sequent calculus. Then

$$Pries(L) = \langle \mathit{filt}_p(L), \subseteq, \mathfrak{T}_L, \propto_\vdash \rangle, \text{ where}$$

$\mathfrak{T}$ is the topology generated by the collections $U_\phi = \{F \in \mathit{filt}_p(L) \mid \phi \in F\}$ and $O_\phi = \{F \in \mathit{filt}_p(L) \mid \phi \notin F\}$. Furthe $\propto_\vdash$ is a binary relation on $\mathit{filt}_p(L)$ defined as follows:

$$F_1 \propto_\vdash F_2 \overset{\text{def}}{\Longleftrightarrow} (\exists\, \phi \in F_1,\ \psi \notin F_2)\, \phi \vdash \psi.$$

Lemma 5.4.6. *Let $\langle L; \vee, \wedge, \bot, \top; \vdash\rangle$ be a coherent sequent calculus. Then $\langle \mathit{filt}_p(L), \subseteq, \mathfrak{T}_L\rangle$, of $Pries(L)$, is a Priestley space and $\propto_\vdash$, of $Pries(L)$, is an apartness.*

Proof. The bounded distributive lattice $L/\equiv$ together with the binary relation $\prec$ defined as follows:

$$[\phi] \prec [\psi] \overset{\text{def}}{\Longleftrightarrow} \phi \vdash \psi,$$

is a strong proximity lattice. Therefore it enough to show that $Pries(L/\equiv)$ is apartness homeomorphic to $Pries(L)$ (see Section 2.8). It is easy to check that the mappings of Lemma 5.4.3 satisfy

$$U_{[\phi]} = \Psi(U_\phi) \text{ and } U_\phi = \Phi(U_{[\phi]}).$$

Therefore the Priestley topology on $\mathit{filt}_p(L/\equiv)$ is homeomorphic to $\mathfrak{T}_L$ on $\mathit{filt}_p(L)$ via these mappings. It remains to show that

$$(\forall F_1, F_2 \in \mathit{filt}_p(L))\ F_1 \propto_\vdash F_2 \Longleftrightarrow \Psi(F_1) \propto_\prec \Psi(F_2).$$

This is proved as follows:

$$\begin{aligned} F_1 \propto_\vdash F_2 &\Longleftrightarrow (\exists\, \phi \in F_1,\ \psi \notin F_2)\, \phi \vdash \psi \\ &\Longleftrightarrow (\exists\, [\phi] \in \Psi(F_1),\ [\psi] \notin \Psi(F_2))\, [\phi] \prec [\psi] \\ &\Longleftrightarrow \Psi(F_1) \propto_\prec \Psi(F_2). \end{aligned}$$

□

5.5 Equivalence of PSws and MLS

Lemma 5.5.1. *Let* $(L; \vee, \wedge, \bot, \top; \vdash)$ *be a coherent sequent calculus and* $\phi, \psi \in L$*. Then*

1. $\phi \vdash \psi \Longleftrightarrow U_\phi \propto_\vdash O_\psi$.

2. $U_\phi \cap U_\psi = U_{\phi\wedge\psi}$ *and* $U_\phi \cup U_\psi = U_{\phi\vee\psi}$.

Proof. 1. We have

$$\phi \vdash \psi \Longleftrightarrow [\phi] \prec [\psi] \Longleftrightarrow U_{[\phi]} \propto_\prec O_{[\phi]} \Longleftrightarrow U_\phi \propto_\vdash O_\phi.$$

The second equivalence is true because $L/\equiv$ is a bounder distributive lattice. The third equivalence follows from the proof of Lemma 5.4.6.

2. Let $F \in \mathit{filt}(L)$ and $\phi \wedge \psi \in F$. Then $\phi \vee (\phi \wedge \psi) \equiv \phi$. Therefore, by the dual of Definition 5.4.2(1), $\phi \in F$. Similarly

$$\phi \wedge \psi \in F \Longrightarrow \psi \in F.$$

This proves $U_{\phi\wedge\psi} \subseteq U_\phi \cap U_\psi$. The other inclusion is clear. The argument for $U_\phi \cup U_\psi = U_{\phi\vee\psi}$ is dual.

□

Theorem 5.5.2. *The categories* **MLS** *and* **PSws** *are equivalent.*

Proof. We prove that the functor coh introduced in Corollaries 5.3.7 and 5.3.8 is faithful as follows:

$$\begin{aligned}
a \ltimes b &\Longleftrightarrow (\exists\, U_1 \in \mathcal{U}^{\mathcal{T}}(X_1), U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\; a \in U,\; b \notin U_2 \text{ and } U_1 \ltimes (X\backslash U_2) \\
&\Longleftrightarrow (\exists\, U_1 \in \mathcal{U}^{\mathcal{T}}(X_1), U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\; a \in U,\; b \notin U_2 \text{ and } U_1 \vdash_\ltimes U_2 \\
&\Longleftrightarrow (\exists\, U_1 \in \mathcal{U}^{\mathcal{T}}(X_1), U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\; a \in U,\; b \notin U_2 \text{ and } U_1 \vdash_{\ltimes'} U_2 \\
&\Longleftrightarrow (\exists\, U_1 \in \mathcal{U}^{\mathcal{T}}(X_1), U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\; a \in U,\; b \notin U_2 \text{ and } U_1 \ltimes' (X\backslash U_2) \\
&\Longleftrightarrow a \ltimes' b.
\end{aligned}$$

The first and the last equivalences are true by Lemma 3.3.32. Now we prove that coh is full. Let $\vdash$ be an arbitrary compatible consequence relation between objects in the image of the functor coh. Set

$$\ltimes_{\vdash} = \bigcup\{U \times (X_2 \backslash U') \mid U \in \mathcal{U}^{\mathcal{T}}(X_1),\ U' \in \mathcal{U}^{\mathcal{T}}(X_2),\ U \vdash U'\}.$$

It is easy to check that $\ltimes_{\vdash}$ is a weak separator.

Before we show that $coh(\ltimes_{\vdash}) = \vdash$, we need to notice that for $U \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $U' \in \mathcal{U}^{\mathcal{T}}(X_2)$,

$$U \ltimes_{\vdash} (X_2 \backslash U') \Longleftrightarrow U \vdash U'.$$

The right-to-left direction is obvious. The other direction is proved as follows. Fix $x \in U$. Then

$$(\forall y \in X_2 \backslash U')(\exists\, U_y \in \mathcal{U}^{\mathcal{T}}(X_1),\ U'_y \in \mathcal{U}^{\mathcal{T}}(X_2))\ x \in U_y,\ y \in X_2 \backslash U'_y \text{ and } U_y \vdash U'_y.$$

Therefore $\bigcap\{U'_y \mid y \in U\} \subseteq U'$. Hence, by co-compactness of U', a finite intersection $\bigcap\{U'_{y_i} \mid 1 \leq i \leq n\}$ is contained in U'. Set $U_x = \bigcap_i U_{y_i}$. By (W), $(L\wedge)$ and $(R\wedge)$, $U_x \vdash U'$. The set $\{U_x \mid x \in U\}$ forms an open cover to U. Hence a finite subcover $\{U_{x_j} \mid 1 \leq j \leq m\}$ exists. By $(L\vee)$, $\bigcup_j U_{x_j} \vdash U'$ and hence by (W) we have $U \vdash U'$.

Now we show that $coh(\ltimes_{\vdash}) = \vdash$ as follows:

$$\begin{aligned} \Gamma \vdash_{\ltimes_{\vdash}} \Delta &\Longleftrightarrow \bigcap \Gamma \ltimes_{\vdash} (X_2 \backslash \bigcup \Delta) \\ &\Longleftrightarrow \bigcap \Gamma \vdash \bigcup \Delta \\ &\Longleftrightarrow \Gamma \vdash \Delta. \end{aligned}$$

Finally, we show that every equivalence class (up to isomorphism) of objects in **MLS** meets the image of the functor coh. Let $\langle L; \wedge, \vee, 0, 1; \vdash\rangle$ be a coherent sequent calculus. We show that L and $coh(Pries(L))$ are isomorphic in **MLS**. To this end we define:

$$\begin{aligned} \Gamma \vdash_1 U_{\delta_1}, \ldots, U_{\delta_n} &\overset{\text{def}}{\Longleftrightarrow} \Gamma \vdash \delta_1, \ldots, \delta_n \\ U_{\delta_1}, \ldots, U_{\delta_n} \vdash_2 \Gamma &\overset{\text{def}}{\Longleftrightarrow} \delta_1, \ldots, \delta_n \vdash \Gamma. \end{aligned}$$

Obviously $\vdash_1$ and $\vdash_2$ are well-defined compatible consequence relations. It is straightforward to check $\vdash_1 \dagger \vdash_2 = \vdash$ and $\vdash_2 \dagger \vdash_1 = \vdash_{\propto_\vdash}$ using that facts that $\vdash = \vdash \dagger \vdash$ and

$$\begin{aligned} U_{\delta_1}, \ldots, U_{\delta_n} \vdash_{\propto_\vdash} U_{\phi_1}, \ldots, U_{\phi_m} &\iff \bigcap_i U_{\delta_i} \propto_\vdash \bigcap_j O_{\phi_j} \\ &\iff U_{\wedge_i \delta_i} \propto_\vdash O_{\vee_j \phi_j}, \text{ by Lemma 5.5.1.2} \\ &\iff \wedge_i \delta_i \vdash \vee_j \phi_j, \text{ by Lemma 5.5.1.1} \\ &\iff \delta_1, \ldots, \delta_n \vdash \phi_1, \ldots, \phi_m. \end{aligned}$$

$\square$

5.6 Semantics

This section has two goals; the first one is to establish Priestley semantics (in **PSws**) for concepts in the category **MLS**. The other goal is to introduce a full and faithful functor (called $ISOu$) from the category **PSws** to the category **SL** of directed-complete meet semilattices and Scott-continuous semilattice homomorphisms. As it is well known from category theory, this proves that the category **PSws** is equivalent to the image of the functor $ISOu$. Therefore the image of this functor, which is a full subcategory of **SL**, is self-dual as **PSws** is. The self-duality of this full subcategory was noted first by J. D. Lawson in [65].

5.6.1 Compatibility

Lemma 5.6.1. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a binary relation from X_1 to X_2 that is open in $\mathcal{T}_1 \times \mathcal{T}_2$ and satisfies $(\uparrow_1 \ltimes \downarrow_2)$. Then the following statements are equivalent:*

1. *$\vdash_\ltimes$ is compatible.*

2. *$\ltimes$ satisfies $(\forall \ltimes)$ and $(\ltimes \forall)$ and hence is a weak separator.*

3. $\ltimes$ *satisfies conditions* $(L-\ltimes)$ *and* $(R-\ltimes)$ *of Lemma 5.2.4.*

Proof. We first prove that 1 implies 2. The condition $(\forall\ltimes)$ is proved as follows.

$$\begin{aligned} b \ltimes d &\iff (\exists U_1 \in \mathcal{U}^{\mathcal{T}}(X_1))(U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\ b \in U_1,\ d \notin U_2 \text{ and } U_1 \ltimes (X \backslash U_2) \\ &\iff (\exists U_1 \in \mathcal{U}^{\mathcal{T}}(X_1))(U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\ b \in U_1,\ d \notin U_2 \text{ and } U_1 \vdash_{\ltimes} U_2 \\ &\iff (\exists U_1, U \in \mathcal{U}^{\mathcal{T}}(X_1))(U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\ b \in U_1,\ d \notin U_2,\ U_1 \vdash_{\propto_1} U \\ &\qquad \text{and } U \vdash_{\ltimes} U_2, \text{ by Lemma 2.8.2} \\ &\iff (\exists U_1, U \in \mathcal{U}^{\mathcal{T}}(X_1))(U_2 \in \mathcal{U}^{\mathcal{T}}(X_2))\ b \in U_1,\ d \notin U_2,\ U_1 \propto_1 (X_1 \backslash U) \\ &\qquad \text{and } U \ltimes (X_2 \backslash U_2) \\ &\iff (\forall c \in X_1)\ b \propto_1 c \text{ or } c \ltimes d. \end{aligned}$$

The left-to-right direction of the last equivalence is true as follows. Set $O = \{x \in X_1 \mid x \ltimes d\}$. Then by Lemma 3.3.32 O is open and $b \propto_1 (X_1 \backslash O)$. Now we apply Lemmas 3.3.3 and 3.3.32 to get the required clopen upper sets. The argument for $(\ltimes\forall)$ is dual.

(2) implies (3) is Lemma 5.2.4. Finally we show that (3) implies (1). We do this via proving the conditions of Lemma 2.8.2. We first prove $(L - Int')$ as follows. Suppose $\phi, \Gamma \vdash_{\ltimes} \Delta$. Then $\phi \cap (\cap\Gamma) \ltimes (X_2 \backslash \cup \Delta)$. Set $O = \{x \in X_1 \mid \phi \propto_1 x\}$. Then by Lemma 3.3.3 O is open lower, and by Lemma 5.2.3 $(X \backslash O) \cap (\cap\Gamma) \ltimes (X_2 \backslash \cup \Delta)$. Now by Lemmas 3.3.32 and 3.3.1 there exists $\phi' \in \mathcal{U}^{\mathcal{T}}(X_1)$ such that $X \backslash O \subseteq \phi'$ and $\phi' \cap (\cap\Gamma) \ltimes (X_2 \backslash \cup \Delta)$. Therefore $\phi \vdash_{\propto_1} \phi'$ and $\phi', \Gamma \vdash_{\ltimes} \Delta$. The condition $(L - Cut)$ is proved as follows. Suppose $\Gamma \vdash_{\propto_1} \phi$ and $\phi, \Theta \vdash_{\ltimes} \Lambda$. Therefore $\cap\Gamma \propto_1 (X_1 \backslash \phi)$ and $\phi \cap (\cap\Theta) \ltimes (X_2 \backslash \cup \Lambda)$. Fix $c \in \cap\Gamma$ and $a \notin \cup\Lambda$. Then for every $b \in X_1$ and $d \in \cap\Theta$, we have $c \propto_1 b$ or $(\uparrow b \cap \uparrow d) \ltimes a$. Therefore, by condition $(L-\ltimes)$ of Lemma 5.2.4, $(\uparrow c \cap \uparrow d) \ltimes a$. Since this is true for every $d \in \cap\Theta$, we have $(\uparrow c \cap (\cap\Theta)) \ltimes a$. Again since this is true for every $c \in \cap\Gamma$, $a \notin \cup\Lambda$, we have $((\cap\Gamma) \cap (\cap\Theta)) \ltimes (X_2 \backslash \cup \Lambda)$. Therefore $\Gamma, \Theta \vdash_{\ltimes} \Lambda$. The arguments for $(R - Int')$ and $(R - Cut)$ are, respectively, dual to that of

$(L - Int')$ and $(L - Cut)$. Therefore $\vdash_{\ltimes}$ is compatible.

$\square$

5.6.2 Gentzen's Cut Rule

Lemma 5.6.2. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ be a Priestley space equipped with apartness relations $\propto$. Then the following statements are equivalent:*

1. *$\vdash_\propto$ is closed under Gentzen's cut rule (or (Cut)) .*
2. *$\propto$ is closed under $(\forall - comp)$.*

Proof. $(1) \implies (2)$ is proved as follows. Suppose $a \propto \circ \propto b$ and set $O = \{x \mid a \propto x\}$. Then $a \propto O$ and $(X\backslash O) \propto b$. Now by Lemma 3.3.3 there exits $C \in \mathcal{U}^{\mathcal{T}}(X)$ such that $a \propto X\backslash C$ and $C \propto b$. By the same lemma again there exist $U \in \mathcal{U}^{\mathcal{T}}(X)$ and $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $a \in U$, $b \in V$, $U \propto X\backslash C$ and $C \propto V$. Therefore $U \vdash_\propto C$ and $C \vdash_\propto X\backslash V$. By the assumption, $U \vdash_\propto X\backslash V$ and hence $U \propto V$ implying $a \propto b$.

For the other direction, suppose $\Gamma \vdash_\propto \Delta, \phi$ and $\phi, \Theta \vdash_\propto \Lambda$ be two sequents with $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $\Theta = \{\theta_1, \ldots, \theta_m\}$. We claim that $(\bigcap \Gamma \cap \bigcap \Theta) \propto (X\backslash(\bigcup \Delta \cup \bigcup \Lambda))$. Let $a \in (\bigcap \Gamma \cap \bigcap \Theta)$ and $b \in X\backslash(\bigcup \Delta \cup \bigcup \Lambda)$, we claim that $a \propto \circ \propto b$. From the first sequent $\cap\Gamma \propto (\cap_i(X\backslash\delta_i) \cap (X\backslash\phi))$. Set $C_i = \propto[X\backslash\delta_i]$. Then by the dual of Lemma 5.2.5 $a \propto (\cap_i X\backslash C_i) \cap (X\backslash\phi)$. Similarly if we set $D_i = [\theta_i]\propto$, then by Lemma 5.2.5 $(\cap_i X\backslash D_i) \cap \phi \propto b$. Moreover, by the choice of a and b we have $a \propto \cup_i D_i$ and $\cup_i C_i \propto b$. Let $x \in X$ be such that $a \not\propto x$ we claim that $x \propto b$. Then $x \in X\backslash \cup_i D_i$ because $a \propto \cup_i D_i$. If $x \in \phi$ then $x \propto b$ because $(\cap_i X\backslash D_i) \cap \phi \propto b$. If $x \notin \phi$ then $x \in \cup_i C_i$ because $a \propto (\cap_i X\backslash C_i) \cap (X\backslash\phi)$. Therefore $x \in b$. Hence for every $x \in X$, $a \propto x$ or $x \propto b$. Therefore $a \propto \circ \propto b$ which implies by assumption $a \propto b$. $\square$

5.6.3 Round Ideals and Filters

Recall the notions of round ideal and filter of coherent sequent calculus is reviewed in Section 2.8.

Lemma 5.6.3. *Let $\langle X; \leq; \mathcal{T}\rangle$ be a Priestley space equipped with apartness relations $\propto$. Then*

1. *$iso_u(X)$ and round ideals of the coherent sequent calculus $\langle \mathcal{U}^{\mathcal{T}}(X); \cap, \cup, \phi, X; \vdash_\propto\rangle$ are isomorphic via the following mappings:*

$$\Psi : iso_u(X) \longrightarrow ridl(\mathcal{U}^{\mathcal{T}}(X)); I \longmapsto \{U \subseteq I \mid U \in \mathcal{U}^{\mathcal{T}}(X)\},\ \textit{and}$$

$$\Phi : ridl(\mathcal{U}^{\mathcal{T}}(X)) \longrightarrow iso_u(X); \mathcal{I} \longmapsto \bigcup \mathcal{I}.$$

2. *$iso_l(X)$ and round filters of the coherent sequent calculus $\langle \mathcal{U}^{\mathcal{T}}(X); \cap, \cup, \phi, X; \vdash_\propto\rangle$ are isomorphic via the following mappings:*

$$\Psi : iso_l(X) \longrightarrow \textit{rfilt}(\mathcal{U}^{\mathcal{T}}(X)); F \longmapsto \{U \in \mathcal{U}^{\mathcal{T}}(X) \mid X\backslash F \subseteq U\},\ \textit{and}$$

$$\Phi : \textit{rfilt}(\mathcal{U}^{\mathcal{T}}(X)) \longrightarrow iso_l(X); \mathcal{F} \longmapsto X\backslash \bigcap \mathcal{F}.$$

Proof. 1. We show that Ψ is well defined by proving that $\Psi(I) = [\vdash_\propto]\Psi(I)$.

$$\begin{aligned} U \in \Psi(I) &\iff U \propto (X\backslash I) \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X))\ U' \subseteq I \text{ and } U \propto (X\backslash U'),\ \text{by Lemma 3.3.3} \\ &\iff (\exists\, \Delta \subseteq_{fin} \Psi(I))\ U \propto (X\backslash \cup \Delta) \\ &\iff (\exists\, \Delta \subseteq_{fin} \Psi(I))\ U \vdash_\propto \Delta \\ &\iff U \in [\vdash_\propto]\Psi(I). \end{aligned}$$

Ψ is clearly an order-preserving map. Now we show that Φ is well-defined.

$$
\begin{aligned}
x \in \bigcup \mathfrak{I} &\iff (\exists\, U \in \mathfrak{I})\; x \in U \\
&\iff (\exists\, U \in \mathfrak{I}, \Delta \subseteq_{fin} \mathfrak{I})\; x \in U \vdash_{\propto} \Delta \\
&\iff (\exists\, U \in \mathfrak{I}, \Delta \subseteq_{fin} \mathfrak{I})\; x \in U \propto (X \backslash \cup \Delta) \\
&\iff (\exists\, U, U' \in \mathfrak{I})\; x \in U \propto (X \backslash U') \\
&\iff x \propto (X \backslash \bigcup \mathfrak{I}), \text{by Lemma 3.3.3.}
\end{aligned}
$$

The right-to-left direction of the last equivalence is proved as follows. By Lemma 3.3.3 there exists $U, U' \in \mathfrak{U}^{\mathfrak{T}}(X)$ such that $x \in U$, $U' \subseteq \bigcup \mathfrak{I}$ and $U \propto (X \backslash U')$. Because $\mathfrak{I}$ is an open cover of U', there exists $V \in \mathfrak{I}$ such that $U' \subseteq V$ and this implies $U' \in \mathfrak{I}$.

$\Psi(\Phi(\mathfrak{I})) = \mathfrak{I}$ and $\Phi(\Psi(I)) = I$ follows from the Priestley's representation theorem, Section 2.2.2 (pp 37).

2. The argument for this is dual to that of (1).

□

For the rest of this section the reader needs to recall Theorem 4.2.6 and its dual.

Lemma 5.6.4. *Let $\langle X; \leq; \mathfrak{T}\rangle$ be a Priestley space equipped with apartness relations $\propto$. Then the mappings*

$$\mathfrak{T} \longrightarrow iso_l(X); O \longmapsto [X\backslash O]\propto \text{ and } \mathfrak{T} \longrightarrow iso_u(X); O \longmapsto \propto[X\backslash O]$$

are Scott-continuous retractions on $\mathfrak{T}$.

Proof. The maps are clearly monotone. Idempotence follows from the definition of isolated sets. Scott-continuity is proved as follows. Let $\{O_i \mid i \in I\}$ be a directed subset of $\mathfrak{T}$. We claim that $\bigcup_{i\in I}(\propto[X\backslash O_i]) = \propto[X\backslash(\bigcup_{i\in I} O_i)]$.

$$\begin{aligned} x \in \bigcup_{i\in I}(\propto[X\backslash O_i]) &\iff (\exists\, i \in I)\; x \propto (X\backslash O_i) \\ &\iff x \propto (X\backslash \bigcup_{i\in I} O_i). \end{aligned}$$

The right-to-left direction of the last equivalence is proved as follows. By Lemma 3.3.3, there exists $U \in \mathcal{U}^{\mathcal{T}}(X)$ such that $x \propto X\backslash U$ and $U \subseteq \bigcup_{i\in I} O_i$. Then by the compactness of U and the directedness of $\{O_i \mid i \in I\}$, there exits $i \in I$ such that $x \propto X\backslash O_i \subseteq X\backslash U$. $\square$

The following Lemma is a straightforward generalisation of Lemma 4.2.4.

Lemma 5.6.5. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weak separator from X_1 to X_2. Then for closed subsets $A \subseteq X_1$ and $B \subseteq X_2$, $[A]\ltimes \in iso_l(X_2)$ and $\ltimes[B] \in iso_u(X_1)$.*

Theorem 5.6.6. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weak separator from X_1 to X_2. Then the mappings:*

$$\begin{aligned} X &\longmapsto iso_l(X) \\ \ltimes &\longmapsto (F \longmapsto [X\backslash F]\ltimes) \\ Y &\longmapsto iso_l(Y) \end{aligned}$$

establish a functor $ISOl$ from **PSws** *to the category* **SL** *of directed-complete meet semilattices and Scott-continuous semilattice homomorphisms. Dually, The mappings*

$$\begin{aligned} X &\longmapsto iso_u(X) \\ \ltimes &\longmapsto (\ltimes[X\backslash I] \longleftarrow I) \\ Y &\longmapsto iso_u(Y) \end{aligned}$$

establish a contravariant functor $ISOu$ from **PSws** *to* **SL**.

Proof. From Lemma 5.6.5, $[X\backslash(.)]\ltimes$ is well-defined. Also $[X\backslash(.)]\propto$ is the identity on $iso_l(X)$ by definition of $iso_l(X)$. Similarly to Lemma 5.6.4, the maps above are proved to be Scott-continuous. Clearly, $[X\backslash X]\ltimes = [\emptyset]\ltimes = Y$. For the binary meet we have

$$[X\backslash(F\cap G)]\ltimes = [(X\backslash F)\cup(X\backslash G)]\ltimes = ([X\backslash F]\ltimes)\cap([X\backslash G]\ltimes).$$

It remains to show that $\mathit{ISOl}(\ltimes_1 \circ \ltimes_2) = \mathit{ISOl}(\ltimes_1)\circ \mathit{ISOl}(\ltimes_2)$.

$$\begin{aligned}
x \in (\mathit{ISOl}(\ltimes_1)\circ \mathit{ISOl}(\ltimes_2))(F) &\iff x \in [X_2\backslash[X_1\backslash F]\ltimes_1]\ltimes_2\\
&\iff (X_2\backslash[X_1\backslash F]\ltimes_1)\ \ltimes_2\ x\\
&\iff (\exists\, U \in \mathcal{U}^{\mathcal{T}}(X_2))\ (X_2\backslash[X_1\backslash F]\ltimes_1) \subseteq U\\
&\qquad \text{and } U \ltimes_2 x\\
&\iff (\exists\, U \in \mathcal{U}^{\mathcal{T}}(X_2))\ (X_2\backslash U) \subseteq [X_1\backslash F]\ltimes_1\\
&\qquad \text{and } U \ltimes_2 x\\
&\iff (\exists\, U \in \mathcal{U}^{\mathcal{T}}(X_2))\ (X_1\backslash F)\ \ltimes_1\ (X\backslash U) \text{ and } U \ltimes_2 x\\
&\iff (X_1\backslash F)\ \ltimes_1 \circ \ltimes_2\ x\\
&\iff x \in [X_1\backslash F](\ltimes_1 \circ \ltimes_2)\\
&\iff x \in (\mathit{ISOl}(\ltimes_1\circ\ltimes_2))(F).
\end{aligned}$$

The left-to-right direction of the third last equivalence is proved as follows. Set $O = \{y \in X_2 \mid y \ltimes_2 x\}$. Obviously $O \ltimes_2 x$. Moreover, by Lemma 3.3.32 O is an open upper set, and by $(\forall - comp)$ $(X_1\backslash F)\ \ltimes_1\ (X_2\backslash O)$. Then we can apply Lemma 3.3.32 to get U. □

The next two lemmas are essential for the following subsection.

Lemma 5.6.7. *Let $\langle X;\leq;\mathcal{T}\rangle$ be a Priestley space equipped with apartness relations $\propto$. Then for every $X \neq F \in iso_l(X)$, the following statements are equivalent where $\{V_1,\ldots,V_n\} \subseteq \mathcal{O}^{\mathcal{T}}(X)$:*

1. *F is meet-prime in the lattice $iso_l(X)$;*

2. $(X\backslash F) \propto (\bigcap_i V_i) \Longrightarrow (\exists\, i)\ V_i \subseteq F$;

3. $F = X\backslash{\uparrow}x$, *for some* $x \in X$.

Proof. (1) $\Longrightarrow$ (2): Suppose $(X\backslash F) \propto (\bigcap_i V_i)$. Set $O_i = \propto[V_i]$. Then by the dual of Corollary 5.2.6 $(X\backslash F) \propto (X\backslash \cup_i O_i)$. Each O_i is open therefore we can apply Lemma 3.3.2, to get $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}^{\mathcal{T}}(X)$ such that $U_i \subseteq O_i$ and $(X\backslash F) \propto (X\backslash \cup_i U_i)$. We notice that

$$\begin{aligned} \bigcap_i [(X\backslash F) \cap U_i)]\propto &= [(X\backslash F) \cap (\cup_i U_i)]\propto, \text{ by Lemma } 5.2.7(2) \\ &\subseteq [X\backslash F]\propto = F, \text{ by Lemma } 5.2.7(3) \end{aligned}$$

By (1), there exists $1 \leq i \leq n$ such that

$$V_i \subseteq [U_i]\propto \subseteq [(X \setminus F) \cap U_i]\propto \subseteq F.$$

(2) $\Longrightarrow$ (3): Suppose a, b are distinct minimal points in $X\backslash F$. Then by Priestley separation condition, there exist clopen upper sets $U_a, U_b \in \mathcal{U}^{\mathcal{T}}(X)$ such that $a \in U_a$, $b \in U_b$, $a \notin U_b$, and $b \notin U_a$. Therefore the set

$$\{U \mid U \in \mathcal{U}^{\mathcal{T}}(X), X\backslash F \not\subseteq U\}$$

is an open cover to the compact set $X\backslash F$. Therefore a finite subcover $\{U_i \mid 1 \leq i \leq n\}$ exists. Hence $\cap_i(X\backslash U_i) \subseteq F$ which implies $(X\backslash F) \propto \cap_i(X\backslash U_i)$. Therefore by (2) there exists i such that $X\backslash U_i \subseteq F$ which implies $X\backslash F \subseteq U_i$. But this is a contradiction because U_i does not contain all elements of $X\backslash F$.

(3) $\Longrightarrow$ (1): Suppose $F_1, F_2 \in iso_l(X)$ and they are not contained in F. Then $x \in F_1$ and $x \in F_2$. So $x \in F_1 \cap F_2$ and hence $F_1 \cap F_2 \not\subseteq F$. □

Lemma 5.6.8. *Let* $\langle X; \leq; \mathcal{T}\rangle$ *be a Priestley space equipped with apartness relations* $\propto$. *Then if* $I \in iso_u(X)$ *and* $F \in iso_l(X)$ *are maximal with respect to the property* $F \cup I \neq X$, *then* F *is meet-prime in the lattice* $iso_l(X)$.

Proof. Suppose F is not meet-prime, then by Lemma 5.6.7 there exists $\{V_1 \ldots, V_n\} \subseteq \mathcal{O}^{\mathcal{T}}(X)$ such that $(X \backslash F) \propto (\cap_i V_i)$ and $V_i \nsubseteq F$ for every $1 \leq i \leq n$. Set $O_i = \propto[V_i]$. Then by the dual of Corollary 5.2.6 $(X \backslash F) \propto (X \backslash \cup_i O_i)$. Each O_i is open therefore we can apply Lemma 3.3.2, to get $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}^{\mathcal{T}}(X)$ such that $U_i \subseteq O_i$ and $(X \backslash F) \propto (X \backslash \cup_i U_i)$. Therefore $V_i \subseteq [U_i]\propto$ and hence

$$(\forall 1 \leq i \leq n)\ [U_i]\propto \nsubseteq F.$$

Therefore

$$\begin{aligned}
& (\forall 1 \leq i \leq n)(\exists x \in X)\ U_i \propto x \text{ and } x \notin F \\
\Longrightarrow\ & (\forall 1 \leq i \leq n)(\exists x \in X)\ U_i \propto x \text{ and } X \backslash F \not\propto x \\
\Longrightarrow\ & (\forall 1 \leq i \leq n)(\exists x \in X)\ ((X \backslash F) \cap U_i) \propto x \text{ and } x \notin [X \backslash F]\propto \\
\Longrightarrow\ & (\forall 1 \leq i \leq n)\ F = [X \backslash F]\propto \subsetneqq [(X \backslash F) \cap U_i]\propto.
\end{aligned}$$

Hence for every i, $I \cup [(X \backslash F) \cap U_i]\propto = X$ because F is maximal with respect this property. We notice that

$$\begin{aligned}
(\forall 1 \leq i \leq n)\ I \cup [(X \backslash F) \cap U_i]\propto = X \quad & \Longrightarrow \quad I \cup \bigcap_i [(X \backslash F) \cap U_i]\propto = X \\
& \Longrightarrow \quad I \cup [(X \backslash F) \cap \bigcup_i U_i]\propto = X \\
& \Longrightarrow \quad I \cup F = X.
\end{aligned}$$

The second implication is true by Lemma 5.2.7(2). The last implication is true because for every $c \in (X \backslash F)$, $c \propto (X \backslash \bigcup_i U_i)$ and hence by Lemma 5.2.7(4),

$$[\uparrow c \cap \bigcup_i U_i]\propto = [c]\propto.$$

Therefore

$$\begin{aligned}
[(X\backslash F)\cap\bigcup_i U_i]\propto &= [\bigcup_{c\in X\backslash F}\uparrow c\cap\bigcup_i U_i]\propto \\
&= [\bigcup_{c\in X\backslash F}(\uparrow c\cap\bigcup_i U_i)]\propto \\
&= \bigcap_{c\in X\backslash F}[(\uparrow c\cap\bigcup_i U_i)]\propto \\
&= \bigcap_{c\in X\backslash F}[c]\propto \\
&= [X\backslash F]\propto = F.
\end{aligned}$$

□

5.6.4 Consistency

Definition 5.6.9. Let $\langle X_1;\leq_1;\mathcal{T}_1\rangle$ and $\langle X_2;\leq_2,\mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weak separator from X_1 to X_2. A pair of sets $A\subseteq X_1$ and $B\subseteq X_2$ is said to be $\ltimes-$*close* if $\neg(A\ltimes B)$.

Lemma 5.6.10. *Let $\langle X_1;\leq_1;\mathcal{T}_1\rangle$ and $\langle X_2;\leq_2,\mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weak separator from X_1 to X_2. For every $F\in iso_l(X_1)$ and $I\in iso_u(X_2)$, The following statements are equivalent:*

1. *$\langle X_1\backslash F,\ X_2\backslash I\rangle$ is $\ltimes-$close.*

2. *$\langle\Psi(F)),\ \Psi(I)\rangle$ is $\vdash_{\ltimes}-$consistent, where the functions Ψ are defined in Lemma 5.6.3.*

Proof. Suppose $\langle X_1\backslash F,\ X_2\backslash I\rangle$ is $\ltimes-$close. This is equivalent to $(X_1\backslash F)\not\ltimes(X_2\backslash I)$ and, by Lemma 3.3.32, the last is equivalent to

$$(\forall U\in\mathcal{U}^{\mathcal{T}}(X_1))(\forall U'\in\mathcal{U}^{\mathcal{T}}(X_2))\ X_1\backslash F\subseteq U \text{ and } U'\subseteq I\Longrightarrow U\not\ltimes X_2\backslash U'.$$

Again this is equivalent to

$$(\forall\Delta\subseteq_{fin}\Psi(F)) \text{ and } (\forall\Gamma\subseteq_{fin}\Psi(I)))\ (\cap\Delta)\not\ltimes(X_2\backslash\cup\Gamma)$$

which is equivalent to

$$(\forall \Delta \subseteq_{fin} \Psi(F)) \text{ and } (\forall \Gamma \subseteq_{fin} \Psi(I))) \ \Delta \nvdash_{\ltimes} \Gamma$$

which means that $\langle \Phi(F), \Psi(I) \rangle$ is $\vdash_{\ltimes}$-consistent. $\square$

Lemma 5.6.11. *Let $\langle X_1; \leq_1; \mathfrak{T}_1 \rangle$ and $\langle X_2; \leq_2, \mathfrak{T}_2 \rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes$ be a weak separator from X_1 to X_2. For a pair of closed subsets $A \subseteq X_1$ and $B \subseteq X_2$, the following statements are equivalent.*

1. *$\langle A,\ B \rangle$ is $\ltimes-$close.*
2. *$\langle A,\ X_2 \backslash \propto_2 [B] \rangle$ is $\ltimes-$close.*
3. *$\langle X_1 \backslash [A]\propto_1,\ B \rangle$ is $\ltimes-$close.*
4. *$\langle X_2 \backslash [A]\ltimes,\ B \rangle$ is $\propto_2-$close.*
5. *$\langle A,\ X_1 \backslash \ltimes [B] \rangle$ is $\propto_1-$close.*
6. *$[A]\ltimes \cup \propto_2[B] \neq X_2$.*
7. *$[A]\propto_1 \cup \ltimes[B] \neq X_1$.*
8. *$(\exists\, a \in A,\ y \in X_2,\ b \in B)\ a \not\ltimes y \not\propto_2 b$.*
9. *$(\exists\, a \in A,\ x \in X_1,\ b \in B)\ a \not\propto_1 x \not\ltimes b$.*
10. *$(\exists\, a \in A,\ y \in core(X_2^{\partial}),\ b \in B)\ a \not\ltimes y \not\propto_2 b$.*
11. *$(\exists\, a \in A,\ x \in core(X_1),\ b \in B)\ a \not\propto_1 x \not\ltimes b$.*

Proof. (1) $\Longrightarrow$ (3): If $(X_1 \backslash [A]\propto_1) \ltimes B$ then by $(\forall\ltimes)$ $A \ltimes B$ because $A \propto_1 ([A]\propto_1)$.

(3) $\Longrightarrow$ (5): If $A \propto_1 (X_1 \backslash \ltimes[B])$ then by $(\forall\ltimes)$ $A \ltimes B$ because $(\ltimes[B]) \ltimes B$. By the same condition this implies that $(X_1 \backslash [A]\propto_1) \ltimes B$.

$(5) \Longrightarrow (7)$: If $[A]\propto_1 \cup \ltimes[B] = X_1$ then $X_1 \backslash \ltimes[B] \subseteq [A]\propto_1$ which is equivalent to $A \propto_1 (X_1 \backslash \ltimes[B])$.

$(7) \Longrightarrow (1)$: If $A \ltimes B$, then, by $(\forall\ltimes)$, for every, $x \in X_1$, $A \propto_1 x$ or $x \ltimes B$. Therefore $[A]\propto_1 \cup \ltimes[B] = X_1$.

Clearly (9) is equivalent to (7), Therefore (1), (3), (5), (7) and (9) are equivalent. Moreover, (11) implies (9) is obvious.

$(7) \Longrightarrow (11)$: Suppose $[A]\propto_1 \cup \ltimes[B] \neq X_1$. Set

$$S = \{F \in iso_l(X_1) \mid [A]\propto_1 \subseteq F \text{ and } F \cup \ltimes[B] \neq X_1\}.$$

By the dual of Theorem 4.2.6, the directed joins in $iso_l(X_1)$ are unions. For a directed subset $D \subseteq S$, it is true that

$$(\forall d \in D)\ d \cup \ltimes[B] \neq X_1 \Longrightarrow (\bigcup_{d \in D} d) \cup \ltimes[B] \neq X_1,$$

otherwise $\bigcup_{d \in D} d$ is an open cover to the compact set $X_1 \backslash \ltimes[B]$ which implies by directedness of D that $d \cup \ltimes[B] = X_1$, for some $d \in D$. By Zorn's Lemma S has a maximal element say M and by Lemma 5.6.8, M is a meet-prime in $iso_l(X_1)$. Therefore, by Lemma 5.6.7, $M = X \backslash \uparrow x = [\uparrow x]\propto_1 = [x]\propto_1$ for some $x \in X$. Therefore $x \in core(X_1)$ and $[A]\propto_1 \subseteq X_1 \backslash \uparrow x$ which implies $x \notin [A]\propto_1$ because $[A]\propto_1$ is lower by the dual of Lemma 4.2.4(1). Hence there exists $a \in A$ such that $a \not\propto_1 x$. Finally $[x]\propto_1 \cup \ltimes[B] \neq X_1$ implies that there exists $t \in X_1$ and $b \in B$ such that $x \not\propto_1 t \not\ltimes b$. This by $(\forall\ltimes)$ implies $x \not\ltimes b$.

The argument that (1), (2), (4), (6), (8) and (10) are equivalent is dual. □

Lemma 5.6.12. *The functors ISOl and ISOu of Theorem 5.6.6 are full and faithful.*

Proof. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $\ltimes_1$ and $\ltimes_2$ be weak separators from X_1 to X_2 satisfying

$[X_1 \backslash F]\ltimes_1 = [X_1 \backslash F]\ltimes_2$ for every $F \in iso_l(X_1)$. Then

$$\begin{aligned} a \ltimes_1 b &\iff \langle \{a\}, \{b\} \rangle \text{ is not } \ltimes_1 - \text{close} \\ &\iff \langle X_1 \backslash [a]\propto_1, \{b\} \rangle \text{ is not } \ltimes_1 - \text{close, by Lemma } 5.6.11(1), (3) \\ &\iff \langle X_2 \backslash [X_1 \backslash [a]\propto_1]\ltimes_1, \{b\} \rangle \text{ is not } \propto_2 - \text{close, by Lemma } 5.6.11(1), (4) \\ &\iff \langle X_2 \backslash [X_1 \backslash [a]\propto_1]\ltimes_2, \{b\} \rangle \text{ is not } \propto_2 - \text{close, by assumption} \\ &\iff \langle X_1 \backslash [a]\propto_1, \{b\} \rangle \text{ is not } \ltimes_2 - \text{close, by Lemma } 5.6.11(1), (4) \\ &\iff \langle \{a\}, \{b\} \rangle \text{ is not } \ltimes_2 - \text{close, by Lemma } 5.6.11(1), (3) \\ &\iff a \ltimes_2 b. \end{aligned}$$

Hence the functor $ISOl$ is faithful.

Let $f : iso_l(X_1) \longrightarrow iso_l(X_2)$ be a Scott-continuous semilattice homomorphism. We define a binary relation $\ltimes_f \subseteq X_1 \times X_2$ by

$$\ltimes_f = \bigcup \{U \times V \mid U \in \mathcal{U}^{\mathcal{T}}(X_1),\ V \in \mathcal{O}^{\mathcal{T}}(X_2) \text{ and } f([U]\propto_1) \cup \propto_2[V] = X_2\}.$$

We show that $\ltimes_f$ is a weak separator. Obviously $\ltimes_f$ is open in $\mathcal{T}_1 \times \mathcal{T}_2$ and satisfies condition $(\uparrow_1\ltimes\downarrow_2)$. The condition $(\forall\ltimes)$ is proved as follows. Suppose $a \ltimes_f b$. This is equivalent to

$$(\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ a \in U, b \in V \text{ and } f([U]\propto_1) \cup \propto_2[V] = X_2.$$

Moreover,

$$\begin{aligned} f([U]\propto_1) \cup \propto_2[V] = X_2 &\iff X_2 \backslash \propto_2[V] \subseteq f([U]\propto_1) \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X_1))\ X_2 \backslash \propto_2[V] \subseteq f([U']\propto_1) \text{ and } U \propto_1 X_1 \backslash U' \\ &\iff (\exists\, U' \in \mathcal{U}^{\mathcal{T}}(X_1))\ U \propto_1 X_1 \backslash U' \text{ and } f([U']\propto_1) \cup \propto_2[V] = X_2 \\ &\iff (\forall x \in X_1)\ a \propto_1 x \text{ or } x \ltimes_f b. \end{aligned}$$

The left-to-right direction of the second equivalence is true because by the dual of Lemma 4.2.4(1):

$$\begin{aligned} f([U]\propto_1) &= f(\cup^{\uparrow}\{[K]\propto_1 \mid K \in \mathcal{U}^{\mathcal{T}}(X_1) \text{ and } K \cup [U]\propto_1 = X_1\}) \\ &= \cup^{\uparrow}\{f([K]\propto_1) \mid K \in \mathcal{U}^{\mathcal{T}}(X_1) \text{ and } K \cup [U]\propto_1 = X_1\}, \end{aligned}$$

is an open cover to the compact set $X_2 \backslash ([V]\propto_2)$ taking into consideration the directedness of the union. The other direction of the same equivalence is proved as follows. It is true that $U \ \propto_1 \ X_1 \backslash U'$ and $U' \ \propto_1 \ [U']\propto_1$. Therefore by $(\propto\forall)$ $U \propto_1 [U']\propto_1$ implying $[U']\propto_1 \subseteq [U]\propto_1$. Hence $X_2 \backslash \propto_2[V] \subseteq f([U']\propto_1) \subseteq f([U]\propto_1)$ implying $X_2 \backslash \propto_2[V] \subseteq f([U]\propto_1)$.

The right-to-left direction of the last equivalence is proved as follows. Set $O = \{x \in X_1 \mid a \propto_1 x\}$. Hence $a \propto O$ and O is a lower open subset of X_1. Moreover, for every $x \in X_1 \backslash O$, there exists $U_x \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $V_x \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $x \in U_x$, $b \in V_x$, and $f([U_x]\propto_1) \cup \propto_2[V_x] = X_2$. The set $\{U_x \mid x \in X_1 \backslash O\}$ is an open cover to $X \backslash O$ which is compact and hence a finite subcover $\{U_{x_i} \mid 1 \leq i \leq n\}$ exists. Set $U' = \cup_i U_{x_i}$ and $V = \cap_i V_{x_i}$. Therefore $a \propto_1 X_1 \backslash U'$ because $X_1 \backslash U' \subseteq O$. We prove $f([U']\propto_1) \cup \propto_2[V] = X_2$ as follows. For every $1 \leq i \leq n$, $V \subseteq V_{x_i}$ and hence $\propto_2[V_{x_i}] \subseteq \propto_2[V]$. Therefore

$$\begin{aligned} & (\forall 1 \leq i \leq n)\ f([U_{x_i}]\propto_1) \cup \propto_2[V] = X_2 \\ \Longrightarrow\ & \bigcap_{1 \leq i \leq n} f([U_{x_i}]\propto_1) \cup \propto_2[V] = X_2 \\ \Longrightarrow\ & f(\bigcap_{1 \leq i \leq n} [U_{x_i}]\propto_1) \cup \propto_2[V] = X_2, \text{ because } f \text{ is semilattice homomorphism} \\ \Longrightarrow\ & f([\bigcup_{1 \leq i \leq n} U_{x_i}]\propto_1) \cup \propto_2[V] = X_2 \\ \Longrightarrow\ & f([U']\propto_1) \cup \propto_2[V] = X_2. \end{aligned}$$

By Lemma 3.3.3, U exists.

The condition $(\ltimes\forall)$ is proved as follows. Suppose $a \ltimes_f b$. This is equivalent to

$$(\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ a \in U, b \in V \text{ and } f([U]\propto_1) \cup \propto_2[V] = X_2.$$

Moreover,

$$\begin{aligned} f([U]\propto_1) \cup \propto_2[V] = X_2 &\iff (X_2\backslash f([U]\propto_1)\ \propto_2 V \\ &\iff (\exists\, T \in \mathcal{U}^{\mathcal{T}}(X_2))\ (X_2\backslash f([U]\propto_1)\ \propto_2 X_2\backslash T \text{ and } T \propto_2 V \\ &\iff (\exists\, T \in \mathcal{U}^{\mathcal{T}}(X_2))\ f([U]\propto_1) \cup \propto_2[X_2\backslash T] = X_2 \text{ and } T \propto_2 V \\ &\iff (\forall x \in X_1)\ a \ltimes_f x \text{ or } x \propto_2 b. \end{aligned}$$

The second equivalence is true by $(\propto\forall)$ and Lemma 3.3.3. The prove of the last equivalence is similar to its counter part in the prove of $(\forall\ltimes)$ above.

In the following we show that $ISOl(\ltimes_f) = f$.

$$\begin{aligned} [X_1\backslash F]\ltimes_f &= \{x \in X_2 \mid X_1\backslash F \ltimes_f x\} \\ &= \cup^{\uparrow}\{V \mid V \in \mathcal{O}^{\mathcal{T}}(X_2), U \in \mathcal{U}^{\mathcal{T}}(X_1), (X_1\backslash F) \subseteq U \text{ and } f([U]\propto_1) \cup \propto_2[V] = X_2\} \\ &= \cup^{\uparrow}\{V \mid V \in \mathcal{O}^{\mathcal{T}}(X_2), V \subseteq f([U]\propto_1), U \in \mathcal{U}^{\mathcal{T}}(X_1) \text{ and } (X_1\backslash F) \subseteq U\} \\ &= \cup^{\uparrow}\{f([U]\propto_1) \mid U \in \mathcal{U}^{\mathcal{T}}(X_1), (X_1\backslash F) \subseteq U\} \\ &= f(\cup^{\uparrow}\{[U]\propto_1 \mid U \in \mathcal{U}^{\mathcal{T}}(X_1), F \cup U = X_1\}) \\ &= f(F), \text{by the dual of } 4.2.4(2). \end{aligned}$$

The second equality is proved as follows. For every $a \in X_1\backslash F$, there exists $U_a \in \mathcal{U}^{\mathcal{T}}(X_1)$ and $V_a \in \mathcal{O}^{\mathcal{T}}(X_2)$ such that $a \in U_a$, $x \in V_a$, and $f([U_a]\propto_1) \cup \propto_2[V_a] = X_2$. The set $\{U_a \mid a \in X_1\backslash F\}$ is an open cover to $X\backslash F$ which is compact and hence a finite sub-cover $\{U_{a_i} \mid 1 \leq i \leq n\}$ exists. Set $U' = \cup_i U_{a_i}$ and $V = \cap_i V_{a_i}$. As proved above $f([U']\propto_1) \cup \propto_2[V] = X_2$. The union is directed as follows. Suppose V_1 and V_2 are clopen lower subsets of X_2 that belong to the collection on the right hand side of the second equality. Then there exist clopen upper subsets U_1 and U_2 of X_1 such that for $i = 1, 2$,$(X_1\backslash F) \subseteq$

U_i and $f([U_i]\propto_1) \cup \propto_2[V_i] = X_2$. Therefore $(X_1 \backslash F) \subseteq U_1 \cap U_2$. Moreover for $i = 1, 2$, $f([U_1 \cap U_2]\propto_1) \cup \propto_2[V_i] = X_2$ because $f([U_1]\propto_1), f([U_2]\propto_1) \subseteq f([U_1 \cap U_2]\propto_1)$. Therefore $f([U_1 \cap U_2]\propto_1) \cup (\propto_2[V_1] \cap \propto_2[V_2]) = X_2$. Hence $f([U_1 \cap U_2]\propto_1) \cup \propto_2[V_1 \cup V_2] = X_2$ implying $V_1 \cup V_2$ belongs to the collection on the right hand side of the second equality. The third equality is proved as follows. Because $f([U]\propto_1) \in iso_l(X_2)$, we have

$$\begin{aligned} V \subseteq f([U]\propto_1) &\iff X_2 \backslash f([U]\propto_1) \propto_2 V \\ &\iff X_2 \backslash f([U]\propto_1) \subseteq \propto_2[V] \\ &\iff f([U]\propto_1) \cup \propto_2[V] = X_2. \end{aligned}$$

The fourth equality is true because $f([U]\propto_1)$ as an open lower subset of a Priestley space is the union of all clopen lower sets contained in it. The second last equality is true because f is Scott-continuous. □

5.7 Two More Equivalences of Categories

In [52, Section 6], the category **SCS** of stably compact spaces and continuous functions between them is equipped with an endofunctor $\mathcal{K}$. For an object $X \in$ **SCS** $\mathcal{K}(X)$ (or $\mathcal{K}_X$) is the set of compact saturated subsets of X equipped with the Scott topology, and for a morphism $f : X \longrightarrow Y$ in **SCS** $\mathcal{K}(f)$ assigns to each compact saturated subset A of X the saturation of $f(A)$. The endofunctor $\mathcal{K}$ defines a monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$ on the category **SCS** where the unit $\uparrow$ assigns to each object X in **SCS** the function $X \longrightarrow \mathcal{K}_X; x \longmapsto \uparrow x$ (the upper closure with respect to the specialisation order) and the multiplication $\bigcup$ assigns to each object X in **SCS** the function $\mathcal{K}_{\mathcal{K}_X} \longrightarrow \mathcal{K}_X; A \longmapsto \bigcup A$. The monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$ is called the *Smyth power monad*.

In this section, we study the Kleisli category $\mathbf{SCS}_{\mathcal{K}}$ of the Smyth power monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$ and show that it is equivalent to the category **PSws**. This is the first goal of this section.

The second object of the section is to show that the category **PSws** is equivalent to the category of stably compact spaces as objects and upper relations of the form $R \subseteq X \times Y_c$ as morphisms where Y_c is the set underlying Y equipped with co-compact topology on Y.

Lemma 5.7.1. *Let $\langle X; \leq; \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Let $I \in iso_u(X)$ and $F \in iso_l(X)$ be such that $F \cup I \neq X$. Then there exists $x \in core(X)\backslash I$ such that $x \propto F$. Moreover,*

$$F = \bigcap\{[x]\propto \mid x \in core(X),\ x \propto F\}.$$

Proof. We have

$$\begin{aligned} F \cup I \neq X &\iff [X\backslash F]\propto \cup \propto[X\backslash I] \neq X, \text{ by definition of } iso_u \text{ and } iso_l \\ &\iff (\exists\, a \notin F,\ x \in core(X),\ b \notin I)\ a \not\propto x \not\propto b, \text{ by Lemma } 5.6.11(7), (11). \end{aligned}$$

$x \notin I$ because $x \not\propto X\backslash I$ and $x \notin F$ because $X\backslash F \not\propto x$. Therefore $x \in core(X)\backslash I$ and because $X\backslash F \propto F$, $x \propto F$. For the second claim we have

$$\begin{aligned} y \notin F &\iff (X\backslash F) \not\propto y \\ &\iff X\backslash F \not\subseteq \propto[y] \\ &\iff \propto[y] \cup F \neq X \\ &\iff (\exists\, x \in core(X)\backslash\propto[y])\ x \propto F \\ &\iff (\exists\, x \in core(X))\ x \propto F \text{ and } x \not\propto y. \end{aligned}$$

Therefore

$$(\forall y \in X)\, y \in F \iff ((\forall x \in core(X))\ x \propto F \implies y \in [x]\propto).$$

□

For the next lemma we recall the following fact which was proved in chapter 4 (Corollaries 4.2.7 and 4.2.10, and Theorems 4.2.15 and 4.5.1). For a Priestley space $\langle X, \leq, \mathcal{T}\rangle$

equipped with apartness $\propto$, $iso_u(X)$ is isomorphic to the open set lattice $\Omega(core(X))$, and $iso_l(X)$ is isomorphic to the lattice $\mathcal{K}_{core(X)}$. The isomorphisms are given by

$$I \iff core(X) \cap I, \tag{5.7.1}$$

$$F \iff core(X) \backslash F. \tag{5.7.2}$$

Lemma 5.7.2. *Let $\langle X; \leq; \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Then the sets*

$$o(V) = \{F \in iso_l(X) \mid V \subseteq F\},\ V \in \mathcal{O}^{\mathcal{T}}(X).$$

form a basis for the Scott-topology on $\langle iso_l(X), \subseteq\rangle$.

Proof. By the dual of Theorem 4.2.6, the directed suprema in $iso_l(X)$ are unions so by the compactness of V the sets $o(V)$ are Scott-open. The sets $\Uparrow x = \{y \mid x \ll y\}$ in any domain form a basis for the Scott-topology. Therefore it is enough to show that these sets are of the form $o(V)$. Let $F' \in iso_l(X)$ and pick an element $F \in \Uparrow F'$. Then by the interpolation property of domains there exists $F'' \in iso_l(X)$ such that $F' \ll F'' \ll F$. The fact $F'' \ll F$ corresponds to $K'' \ll K$ for some compact saturated subsets K'' and K. So there exists an open set O such that $K \subseteq O \subseteq K''$ by Theorem 2.6.6. From the correspondence 5.7.1 and 5.7.2 above there is $I \in iso_u(X)$ such that $core(X)\backslash F \subseteq I$ and $I \cap core(X) \subseteq core(X)\backslash F''$. So $F \cup I = core(X)$. If $F \cup I \neq X$ then by Lemma 5.7.1 there is a $x \in core(X)\backslash I$ such that $x \propto F$. This implies $x \in F$ and hence $x \propto x$ which is impossible for core points (Remark 4.2.2). Hence $F \cup I = X$ implying $X\backslash I \subseteq F$. We recall that F is the union of clopen lower sets contained in it. Therefore by compactness of $X\backslash I$ there exists $V \in \mathcal{O}^{\mathcal{T}}(X)$ such that $X\backslash I \subseteq V \subseteq F$. Hence $F \in o(V)$.

We show that $o(V) \subseteq \Uparrow F'$. We have

$$
\begin{aligned}
G \in o(V) &\iff V \subseteq \bigcap\{[x]\!\propto \mid x \in core(X),\ x \propto G\},\ \text{by Lemma } 5.7.1\\
&\iff (\forall x \in core(X),\ x \propto G)\ x \propto V\\
&\implies (\forall x \in core(X),\ x \propto G)\ x \propto X\backslash I\\
&\implies (\forall x \in core(X),\ x \propto G)\ x \in I\\
&\implies (\forall x \in core(X),\ x \propto G)\ x \notin F''\\
&\implies (\forall x \in core(X),\ x \propto G)\ F'' \subseteq X\backslash{\uparrow}x = [x]\!\propto\\
&\implies (\forall x \in core(X),\ x \propto G)\ x \propto F''\\
&\implies F'' \subseteq \bigcap\{[x]\!\propto \mid x \in core(X),\ x \propto G\} = G,\ \text{by Lemma } 5.7.1\\
&\implies F' \ll G.
\end{aligned}
$$

□

Lemma 5.7.3. *Let $\langle X_1; \leq_1; \mathfrak{T}_1\rangle$ and $\langle X_2; \leq_2, \mathfrak{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Then there is a one-to-one correspondence between weak separators from X_1 to X_2 and continuous maps from $core(X_1)$ to $\mathcal{K}_{core(X_2)} \simeq iso_l(X_2)$.*

Proof. Given a weak separator $\ltimes$ from X_1 to X_2, the map

$$f_{\ltimes} : core(X_1) \longrightarrow iso_l(X_2) : x \longmapsto [x]\ltimes$$

is well-defined by Lemma 5.6.5. Now we show that $f_{\ltimes}$ is continuous

$$
\begin{aligned}
f_{\ltimes}^{-1}(O(V)) &= \{x \in core(X_1) \mid V \subseteq [x]\ltimes\}\\
&= \{x \in core(X_1) \mid x \ltimes V\}\\
&= \{x \in core(X_1) \mid x \in U \ltimes V,\ U \in \mathcal{U}^{\mathfrak{T}}(X)\}, \text{by Lemma 3.3.32}\\
&= \bigcup\{core(X_1) \cap U \mid U \in \mathcal{U}^{\mathfrak{T}}(X),\ U \ltimes V\}.
\end{aligned}
$$

We show that the map $(\ltimes \longmapsto f_{\ltimes})$ is one-to-one. Suppose $\ltimes \neq \ltimes'$. Then by Lemma 5.6.6 there is $F \in iso_l(X_1)$ such that $[X\backslash F]\ltimes \neq [X\backslash F]\ltimes'$. Without loss of generality, we can assume that there exists $y \in X_2$ such that $(X\backslash F) \ltimes y$ and $(X\backslash F) \not\ltimes' y$. So $F \cup \ltimes'[y] \neq X_1$. Then by Lemma 5.7.1 there exists $x \in core(X_1)$ such that $x \propto_1 F$ and $x \not\ltimes' y$. Therefore $x \ltimes y$ because $X\backslash F \ltimes y$ and $x \not\ltimes' y$. Hence $f_{\ltimes}(x) \neq f_{\ltimes'}(x)$ proving that $f_{\ltimes} \neq f_{\ltimes'}$.

We show that the map $(\ltimes \longmapsto f_{\ltimes})$ is surjective. Let f be a continuous function from $core(X_1)$ to $iso_l(X_2)$. Set

$$\ltimes_f = \{U \times V \mid U \in \mathcal{U}^{\mathcal{T}}(X_1),\ V \in \mathcal{O}^{\mathcal{T}}(X_2), \text{ and } f(core(X_1) \cap U) \subseteq o(V)\}.$$

We prove that $\propto_1 \circ \ltimes_f \circ \propto_2$ is a weak separator from X_1 to X_2. First notice that

$$\propto_1 \circ (\ltimes_f \circ \propto_2) = (\propto_1 \circ \ltimes_f) \circ \propto_2$$

as follows.

$$\begin{aligned}
x\ (\propto_1 \circ \ltimes_f) \circ \propto_2 y \iff & (\forall r \in X_2)\ x \propto_1 \circ \ltimes_f r \text{ or } r \propto_2 y \\
\iff & (\forall r \in X_2)(\forall s \in X_1)\ x \propto_1 s,\ s \ltimes_f r \text{ or } r \propto_2 y \\
\iff & (\forall s \in X_1)\ x \propto_1 s \text{ or } s \ltimes_f \circ \propto_2 y \\
\iff & x \propto_1 \circ (\ltimes_f \circ \propto_2)\ y.
\end{aligned}$$

Condition $(\uparrow_1 \ltimes \downarrow_2)$ is proved for $\propto_1 \circ \ltimes_f \circ \propto_2$ as follows.

$$\begin{aligned}
t \geq x \propto_1 \circ \ltimes_f \circ \propto_2 y \geq z \implies & (\forall r \in X_2)\ x \propto_1 \circ \ltimes_f r \text{ or } r \propto_2 y \geq z \\
\implies & (\forall r \in X_2)\ x \propto_1 \circ \ltimes_f r \text{ or } r \propto_2 z \\
\implies & t \geq x\ (\propto_1 \circ \ltimes_f) \circ \propto_2 z \\
\implies & (\forall s \in X_1)\ t \geq x \propto_1 s, \text{ or } s \ltimes_f \circ \propto_2 z \\
\implies & (\forall s \in X_1)\ t \propto_1 s \text{ or } s \ltimes_f \circ \propto_2 z \\
\implies & t \propto_1 \circ \ltimes_f \circ \propto_2 z.
\end{aligned}$$

Condition $(\forall\ltimes)$ is proved for $\propto_1 \circ \ltimes_f \circ \propto_2$ as follows.

$$\begin{aligned} x \propto_1 \circ \ltimes_f \circ \propto_2 y &\iff (\forall s \in X_1)\, x \propto_1 s \text{ or } s \ltimes_f \circ \propto_2 y \\ &\iff (\forall s, t \in X_1)\, x \propto_1 t,\ t \propto_1 s \text{ or } s \ltimes_f \circ \propto_2 y \\ &\iff (\forall t \in X_1)\, x \propto_1 t,\ t \propto_1 \circ \ltimes_f \circ \propto_2 y. \end{aligned}$$

Dually condition $(\ltimes\forall)$ is proved for $\propto_1 \circ \ltimes_f \circ \propto_2$.

We prove that $f_{\ltimes_f} = f$.

$$\begin{aligned} f_{\ltimes_f} x &= \bigcup\{V \in \mathcal{O}^{\mathcal{T}}(X_2) \mid (\exists U \in \mathcal{U}^{\mathcal{T}}(X))\, x \in \mathit{core}(X_1) \cap U \subseteq f^{-1}(o(V))\} \\ &= \bigcup\{V \in \mathcal{O}^{\mathcal{T}}(X_2) \mid x \in f^{-1}(o(V))\} \\ &= \bigcup\{V \in \mathcal{O}^{\mathcal{T}}(X_2) \mid f(x) \in o(V)\} \\ &= \bigcup\{V \in \mathcal{O}^{\mathcal{T}}(X_2) \mid V \subseteq f(x)\} \\ &= f(x). \end{aligned}$$

The second equality is proved as follows. Suppose $x \in f^{-1}(o(V))$. By continuity of f there exists an open upper set I such that $I \cap \mathit{core}(X_1) = f^{-1}(o(V))$. Therefore $x \in I \cap \mathit{core}(X_1)$. Because I is the union of clopen upper sets contained in it, there exists a clopen upper set $U \subseteq \mathcal{U}^{\mathcal{T}}(X_1)$ such that $x \in U \subseteq I$ and hence $x \in U \cap \mathit{core}(X_1) \subseteq f^{-1}(o(V))$. The last equality is again true because in a Priestley space every open lower set is the union of clopen lower sets contained in it. $\square$

In the following we would like to study the Kleisli category $\mathbf{SCS}_{\mathcal{K}}$ of the Smyth power monad $\langle \mathcal{K}, \uparrow, \bigcup \rangle$. We first study the composition in this category. Let $\langle X_1; \leq_1, \mathcal{T}_1 \rangle$, $\langle X_2; \leq_2, \mathcal{T}_2 \rangle$, and $\langle X_3; \leq_3, \mathcal{T}_3 \rangle$ be Priestley spaces equipped with apartness relations $\propto_1$, $\propto_2$ and $\propto_3$, respectively. Let $f : \mathit{core}(X_1) \longrightarrow \mathit{iso}_l(X_2)$ and $g : \mathit{core}(X_2) \longrightarrow \mathit{iso}_l(X_3)$ be continuous functions. The composition of f and g is $\bigcup \circ \mathcal{K}(g) \circ f : \mathit{core}(X_1) \longrightarrow \mathit{iso}_l(X_3)$. For $x \in \mathit{core}(X_1)$ we calculate in detail the image of x under this composition. First x is mapped to $f(x)$ which is a lower isolated subsets of X_2. The compact saturated subset of $core(X_2)$ corresponding to $f(x)$ is $core(X_2)\backslash f(x)$ (Theorem 4.5.1). Then this set is mapped to the saturation of its image under g which is equivalent to $\uparrow\{g(y) \mid y \notin f(x)\}$. Notice that this upwards closure is taken in the lattice $\langle iso_l(X_3); \subseteq\rangle$. The last step in constructing the composition is to take the union which is an infimum in the lattice $\mathcal{K}_{core(X_3)}$. Hence the last step is equivalent to taking the infimum of $\{g(y) \mid y \notin f(x)\}$ in the lattice $iso_l(X_3)$ which is $\overline{[X_3\backslash \bigcap\{g(y) \mid y \notin f(x)\}]} \propto_3$. Therefore

$$x \longmapsto \overline{[X_3\backslash \bigcap\{g(y) \mid y \notin f(x)\}]} \propto_3 .$$

Theorem 5.7.4. *The category* **PSws** *is equivalent to the Kleisli category* $\mathbf{SCS}_{\mathcal{K}}$ *of the Smyth power monad* $\langle \mathcal{K}, \uparrow, \bigcup\rangle$ *on* **SCS**.

Proof. By Lemma 4.3.4, all objects are reachable up to an isomorphism. Most of the facts needed about the morphisms have already been proved in the prove of Lemma 5.7.3. Therefore we just need to show that the mapping introduced in the prove of Lemma 5.7.3 is a functor.

For a Priestley spaces $\langle X; \leq; \mathcal{T}\rangle$ equipped with apartness relations $\propto$, the image of $\propto$ is

$$f_{\propto} : core(X) \longrightarrow iso_l(X) : x \longmapsto [x]\propto = X\backslash{\uparrow}x.$$

Note that $X\backslash{\uparrow}x$ as a lower isolated set is corresponding to $core(X)\backslash(X\backslash{\uparrow}x)$ as a compact saturated set which is the same as $core(X) \cap {\uparrow}x$, the upper closure of x in $core(X)$ (with respect to the specialisation order). Therefore the identity $\propto$ is mapped to the unit of the monad which is the identity in the Kleisli category.

Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$, $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ and $\langle X_3; \leq_3, \mathcal{T}_3\rangle$be Priestley spaces equipped with apartness relations $\propto_1$, $\propto_2$ and $\propto_3$, respectively. Suppose $\ltimes_1 \subseteq X_1 \times X_2$ and $\ltimes_2 \subseteq X_2 \times X_3$ are weak separators. To show that the composition is preserved we need to show that for every $x \in core(X_1)$:

$$f_{\ltimes_1 \circ \ltimes_2}(x) = f_{\ltimes_1} \circ f_{\ltimes_2}(x).$$

This is the equivalent to proving

$$[x]{\ltimes_1} \circ {\ltimes_2} = \overline{[X_3 \backslash \bigcap\{[y]{\ltimes_2} \mid x \not\ltimes_1 y\}]}\,{\propto_3}.$$

This is proved as follows:

$$\begin{aligned} x \ltimes_1 \circ \ltimes_2 t &\iff (\forall y \notin [x]{\ltimes_1})\; y \ltimes_2 t \\ &\iff t \in \bigcap\{[y]{\ltimes_2} \mid x \not\ltimes_1 y\} \end{aligned}$$

Therefore

$$[x]{\ltimes_1} \circ {\ltimes_2} = \bigcap\{[y]{\ltimes_2} \mid x \not\ltimes_1 y\}$$

which completes the prove since $[x]{\ltimes_1} \circ {\ltimes_2} \in iso_l(X_3)$. □

Let $\langle X; \leq; \mathcal{T}\rangle$ be a Priestley space equipped with apartness $\propto$. Clearly the dual of $\propto$ is an apartness on X^∂ and $iso_u(X^\partial) \simeq iso_l(X)$. The topology on $core(X^\partial)$ is specified by $iso_u(X^\partial)$ and hence by $iso_l(X)$. Now we notice that

$$(\forall x \in core(X))(\forall F \in iso_l(X))\; x \in F \iff \min(X \backslash {\propto}[x]) \in F.$$

This notice together with Equations 5.7.1 and 5.7.2 and Lemma 4.5.2 is enough to show that $core(X^\partial) \simeq core(X)_c$ as follows. Every open set of $core(X^\partial)$ is of the form $core(X^\partial) \cap F$ for some $F \in iso_u(X^\partial) = iso_l(X)$. But $core(X^\partial) \cap F = core(X) \cap F$ and the latter set is the complement of a compact saturated subset of $core(X)$ and hence open in $core(X)_c$. This argument is true in the other direction as well. Therefore the co-compact topology X_c of a stably compact space X is a stably compact space and $(X_c)_c = X$.

Definition 5.7.5. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Suppose $\ltimes \subseteq X_1 \times X_2$ is a weak separator. We define a binary relation $R_\ltimes \subseteq core(X_1) \times core(X_2)$ as follows:

$$x\; R_\ltimes\; y \overset{\text{def}}{\iff} x \not\ltimes y.$$

Remark 5.7.6. Let $f_{\ltimes} : core(X_1) \longrightarrow core(X_2)_c$ be the continuous function corresponding to $\ltimes$. Then it is straightforward to check that

$$x \, R_{\ltimes} \, y \Longleftrightarrow y \notin f_{\ltimes}(x).$$

Lemma 5.7.7. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively and $\ltimes \subseteq X_1 \times X_2$ a weak separator. Then $R_{\ltimes}$ is upper that is closed in $core(X_1) \times core(X_2)_c$.*

Proof. Suppose $\langle x, y\rangle$ be in $core(X_1) \times core(X_2)$ but not in $R_{\ltimes}$. Then $x \ltimes y$. Therefore there exist $U \in \mathcal{U}^{\mathcal{T}}(X_1)$ such that $x \in U$ and $U \ltimes y$. Set $A = U \cap core(X_1)$ and $B = [U]\ltimes \cap core(X_2)$. Then A is open in $core(X_1)$ and $core(X_2)\backslash[U]\ltimes$ is compact saturated in $core(X_2)$ whose complement is B and hence B is open in $core(X_2)_c$. Clearly $A \ltimes B$ and hence $(A \times B) \cap R_{\ltimes} = \emptyset$. □

Lemma 5.7.8. *Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ and $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively and $R \subseteq core(X_1) \times core(X_2)_c$ an upper relation. Then the binary relation $\propto_1 \circ \ltimes_R \circ \propto_2$, where*

$$\ltimes_R = \bigcup\{U \times V \mid U \in \mathcal{U}^{\mathcal{T}}(X_1), V \in \mathcal{O}^{\mathcal{T}}(X_2), \textit{ and } (U \times V) \cap R = \emptyset\},$$

is a weak separator.

Proof. By Lemma 2.6.13 the map

$$f_R : core(X_1) \longrightarrow \mathcal{K}_{core(X_2)}; x \longmapsto \{y \in core(X_2) \mid x \, R \, y\}$$

is continuous. By Theorem 4.5.1, the map f_R can be rewritten as

$$f'_R : core(X_1) \longrightarrow iso_l(X_2); x \longmapsto X_2\backslash\uparrow\{y \in core(X_2) \mid x \, R \, y\}.$$

Now we calculate the weak separator corresponding to f'_R. By Lemma 5.7.3, $\propto_1 \circ \ltimes_R \circ \propto_2$, where

$$\ltimes_R = \{U \times V \mid U \in \mathcal{U}^{\mathcal{T}}(X_1), \ V \in \mathcal{O}^{\mathcal{T}}(X_2), \text{ and } f'_R(core(X_1) \cap U) \subseteq o(V)\},$$

is the weak separator corresponding to f_R. We notice that

$$\begin{aligned} f'_R(core(X_1) \cap U) \subseteq o(V) &\iff (\forall x \in core(X_1) \cap U))\ f'_R(x) \in o(V) \\ &\iff (\forall x \in core(X_1) \cap U))\ V \subseteq f'_R(x) \\ &\iff (\forall x \in core(X_1) \cap U))\ X_2 \backslash f'_R(x) \subseteq X_2 \backslash V \\ &\iff (\forall x \in core(X_1) \cap U))\ \uparrow\{y \in core(X_2) \mid x\ R\ y\} \subseteq X_2 \backslash V \\ &\iff (\forall x \in core(X_1) \cap U))\ \{y \in core(X_2) \mid x\ R\ y\} \subseteq X_2 \backslash V \\ &\iff (U \times V) \cap R = \emptyset. \end{aligned}$$

□

Theorem 5.7.9. *The category* **PSws** *is equivalent to the category of stably compact spaces as objects and upper relations of the form $R \subseteq X \times Y_c$ as morphisms. The composition in the latter category is the usual relation composition.*

Proof. We prove that the translations between upper relations and continuous maps in Lemmas 5.7.7 and 5.7.8 are inverses of each other. Let $\langle X_1; \leq_1; \mathcal{T}_1\rangle$, $\langle X_2; \leq_2; \mathcal{T}_2\rangle$ and $\langle X_3; \leq_3 , \mathcal{T}_3\rangle$ be Priestley spaces equipped with apartness relations $\propto_1$, $\propto_2$ and $\propto_3$, respectively and $\ltimes_1 \subseteq X_1 \times X_2$ and $\ltimes_2 \subseteq X_2 \times X_3$ weak separators.

Let $R \subseteq core(X_1) \times core(X_2)_c$ be an upper relation. Then

$$\begin{aligned} x \ltimes_1 y &\iff (\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ x \in U, y \in V \text{ and } U \ltimes_1 V, \text{ by Lemma 3.3.32} \\ &\iff (\exists U \in \mathcal{U}^{\mathcal{T}}(X_1))(\exists V \in \mathcal{O}^{\mathcal{T}}(X_2))\ x \in U, y \in V \text{ and } (U \times V) \cap R_{\ltimes_1} = \emptyset \\ &\iff x \ltimes_{R_{\ltimes_1}} y. \end{aligned}$$

Also we have

$$\begin{aligned} x\ R\ y &\iff x \not\ltimes_R y, \text{ by definition of } \ltimes_R \\ &\iff x\ R_{\ltimes_R}\ y, \text{ by definition of } R_{\ltimes}\ . \end{aligned}$$

Therefore it remains to show that the composition is preserved under these translations. Let $\ltimes_1 \subseteq X_1 \times X_2$ and $\ltimes_2 \subseteq X_2 \times X_3$ be weak separators. We show that

$$R_{\ltimes_1 \circ \ltimes_2} = R_{\ltimes_1} \circ R_{\ltimes_2}.$$

This is proved as follows:

$$\begin{aligned}
x \, R_{\ltimes_1 \circ \ltimes_2} \, z &\iff (x, z) \notin \ltimes_1 \circ \ltimes_2 \\
&\iff (\exists\, y \in X_2)\; x \not\ltimes_1 y \not\ltimes_2 z \\
&\iff [x]\ltimes_1 \cup \ltimes_2[z] \neq X_2 \\
&\iff (\exists\, y \in \mathit{core}(X_2))\; x \not\ltimes_1 y \not\ltimes_2 z \\
&\iff (\exists\, y \in \mathit{core}(X_2))\; x \, R_{\ltimes_1} \, y \, R_{\ltimes_2} z.
\end{aligned}$$

□

5.8 Logic for Stably Locally Compact Spaces

In the previous chapter we have seen that removing the requirement of having a top element from the side of strong proximity lattices is equivalent to removing the compactness requirement from the side of stably compact spaces, in the Jung-Sünderhauf representing theorem. Therefore we proved that pointed-strong proximity lattices represent stably locally compact spaces. As strong proximity lattices were the basis for coherent sequent calculi, the question now is the following:

> *How can the notion of coherent sequent calculus be extended to provide a logical description of stably locally compact spaces?*

The answer, as the reader may have already guessed, is to remove the truth unit $\top$ from the notion of coherent sequent calculus and to do all necessary changes that must be associated with this.

We arrive at the following notion.

Definition 5.8.1. Let $\langle A; \vee, \wedge, \bot\rangle$ and $\langle B; \vee, \wedge, \bot\rangle$ be two algebras of type $\langle 2, 2, 0\rangle$. A binary relation $\vdash$ between finite subsets of A and B is *pointed-consequence* if for every $\phi, \psi \in A, \Gamma, \Gamma' \subseteq_{fin} A, \phi', \psi' \in B$ and $\Delta, \Delta' \subseteq_{fin} B$,

$(L\bot)$ $(\forall\, \Theta \subseteq_{fin} B)\ \{\bot\} \vdash \Theta$.

$(L\wedge)$ $\phi, \psi, \Gamma \vdash \Delta \Longleftrightarrow \phi \wedge \psi, \Gamma \vdash \Delta$.

$(L\vee)$ $\phi, \Gamma \vdash \Delta$ and $\psi, \Gamma \vdash \Delta \Longleftrightarrow \phi \vee \psi, \Gamma \vdash \Delta$.

$(R\bot)$ $\Gamma \vdash \Delta \Longleftrightarrow \Gamma \vdash \Delta, \bot$.

$(R\wedge)$ $\Gamma \vdash \Delta, \phi'$ and $\Gamma \vdash \Delta, \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \wedge \psi'$.

$(R\vee)$ $\Gamma \vdash \Delta, \phi', \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \vee \psi'$.

(W) $\Gamma \vdash \Delta \Longrightarrow \Gamma', \Gamma \vdash \Delta, \Delta'$.

Recall that the notion of interpolant and the composition rule Cut were reviewed in Section 2.8.

Definition 5.8.2. A *pointed-coherent sequent calculus* is an algebra $\langle A; \vee, \wedge, \bot\rangle$ together with a pointed-consequent relation $\Vdash$ on A such that $\Vdash$ is closed under Cut and has interpolants.

Definition 5.8.3. A pointed-consequent relation $\vdash$ from a pointed-coherent sequent calculus $\langle A; \vee,$
$\wedge, \bot \Vdash_A\rangle$ to a pointed-coherent sequent calculus $\langle B; \vee, \wedge, \bot, \Vdash_B\rangle$ is *compatible* if

$$\Vdash_A \dagger \vdash \;=\; \vdash \;=\; \vdash \dagger \Vdash_B.$$

We let $\mathbf{MLS}_0$ stands for the category whose objects are pointed-coherent sequent calculi and whose morphisms are compatible pointed-coherent relations. Our experiments show that the whole theory (of **MLS**) still works perfectly for $\mathbf{MLS}_0$. This means that $\mathbf{MLS}_0$

provides a logical description for stably locally compact spaces including coherent domains in their Scott topologies. Although the observations made in this section and Section 4.7 are simple, they are quite powerful as they provide algebraic and logical interpretations for coherent domains.

5.9 Domain Constructions

In this section we hint at the way simple domain constructions can be done in our Priestley form.

5.9.1 Lifting

Suppose $\langle Y, \mathcal{T}_Y\rangle$ is a stably compact space and $\langle X; \mathcal{T}_X, \leq\rangle$ is a Priestley space equipped with apartness $\propto$. Suppose $S[\![\cdot]\!] : \mathit{core}(X) \longrightarrow Y$ is a homeomorphic map. The lifting of $\langle Y, \mathcal{T}_Y\rangle$ is the space $\langle Y_\perp, \mathcal{T}_{y_\perp}\rangle$, where $Y_\perp = Y \,\dot{\cup}\, \{\perp\}$ and $\mathcal{T}_{y_\perp} = \mathcal{T}_Y \cup \{Y_\perp\}$. Now we add a bottom element (with respect to $\leq$) $\perp$ to the Priestley space as follows. We set $X_\perp = X \,\dot{\cup}\, \{\perp\}$ and let $\mathcal{T}_{X_\perp}$ be the topology generated by $\mathcal{T}_X \cup \{\{\perp\}\}$. Then $\langle X_\perp; \mathcal{T}_{X_\perp}, \leq_\perp\rangle$ is a Priestley space and the binary relation

$$\propto_\perp = \propto \cup (X \times \{\perp\})$$

is an apartness on it. It is not hard to check that $\mathit{core}(X_\perp) = \mathit{core}(X) \cup \{\perp\}$. The homeomorphic map $S[\![\cdot]\!]$ can be extended to a homeomorphism between $\mathit{core}(X_\perp)$ and $Y_\perp$ by defining $S[\![\perp]\!] = \perp$.

5.9.2 Sum

Suppose $\langle Y_1, \mathcal{T}_{Y_1}\rangle$ and $\langle Y_2, \mathcal{T}_{Y_2}\rangle$ are stably compact spaces and suppose $\langle X_1; \mathcal{T}_{X_1}, \leq_1\rangle$ and $\langle X_2; \mathcal{T}_{X_2}, \leq_2\rangle$ are Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $S_1[\![\cdot]\!] : \mathit{core}(X_1) \longrightarrow Y_1$ and $S_2[\![\cdot]\!] : \mathit{core}(X_2) \longrightarrow Y_2$ be homeomorphic maps.

The sum of Y_1 and Y_2 is the topological space $\langle Y, \mathcal{T}\rangle$, where Y is the disjoint union of Y_1 and Y_2, i.e. $Y = Y_1 \dot{\cup} Y_2$, and $\mathcal{T}$ is generated by $\mathcal{T}_{Y_1}$ and $\mathcal{T}_{Y_2}$. Let

1. $X = X_1 \dot{\cup} X_2$,

2. $\leq = \leq_1 \cup \leq_2$, and

3. $\mathcal{T}$ be the topology generated by $\mathcal{T}_1$ and $\mathcal{T}_2$ as subbases.

Then $\langle X; \mathcal{T}; \leq\rangle$ is a Priestley space and the relation

$$\propto = \propto_1 \cup \propto_2 \cup (X_1 \times X_2) \cup (X_2 \times X_1)$$

is an apartness on it. It is obvious that $core(X) = core(X_1) \dot{\cup} core(X_2)$ and that the map

$$S[\![\cdot]\!] : core(X) \longrightarrow Y; x \longmapsto S_i[\![x]\!], \text{ if } x \in core(X_i), i = 1, 2.$$

is a homeomorphism.

If the stably compact spaces represented by the X_1 and X_2 are pointed, then $core(X_1)$ and $core(X_2)$ will have bottom elements $\perp_1$ and $\perp_2$, respectively. In this case we construct the coalesced sum just like the usual sum after unifying $\perp_1$ and $\perp_2$. The apartness $\propto$, in this case, will be constructed as follows:

$$\propto = \propto_1 \cup \propto_2 \cup (X_1 \backslash \{\perp_1\} \times X_2 \backslash \{\perp_2\}) \cup (X_2 \backslash \{\perp_2\} \times X_1 \backslash \{\perp_1\}).$$

5.9.3 Product

Suppose $\langle Y_1, \mathcal{T}_{Y_1}\rangle$ and $\langle Y_2, \mathcal{T}_{Y_2}\rangle$ are stably compact spaces and suppose $\langle X_1; \mathcal{T}_{X_1}, \leq_1\rangle$ and $\langle X_2; \mathcal{T}_{X_2}, \leq_2\rangle$ are Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$, respectively. Let $S_1[\![\cdot]\!] : core(X_1) \longrightarrow Y_1$ and $S_2[\![\cdot]\!] : core(X_2) \longrightarrow Y_2$ be homeomorphic maps.

Let $\langle X; \mathcal{T}; \leq\rangle$ be the product of $\langle X_1; \mathcal{T}_{X_1}, \leq_1\rangle$ and $\langle X_2; \mathcal{T}_{X_2}, \leq_2\rangle$. Then $\langle X; \mathcal{T}; \leq\rangle$ is a Priestley space. We equip X with the following relation

$$\langle x, y\rangle \propto \langle x', y'\rangle \overset{\text{def}}{\Longleftrightarrow} x \propto_1 x' \text{ or } y \propto_2 y'.$$

It is not hard to prove that $\propto$ is an apartness on X and $core(X) = core(X_1) \times core(X_2)$. The map

$$S[\![\cdot]\!] : core(X) \longrightarrow Y_1 \times Y_2; \langle x, y\rangle \longmapsto \langle S_1[\![x]\!], S_2[\![y]\!]\rangle.$$

is a homeomorphism.

5.9.4 The Smyth Power Domain

Suppose $\langle Y, \mathcal{T}_Y\rangle$ is a stably compact space, $\langle X; \mathcal{T}_X, \leq\rangle$ is a Priestley space equipped with an apartness $\propto$ and $S[\![\cdot]\!] : core(X) \longrightarrow Y$ is a homeomorphic map.

The Smyth power domain of Y is the set of compact saturated non-empty subsets ordered by the reverse inclusion. In this section, there is no problem in excluding the empty set therefore let us not bother removing it from the collection of compact saturated sets. We have shown that $\langle \mathcal{K}_{core(X)}, \supseteq\rangle$ is isomorphic to $\langle iso_l(X), \subseteq\rangle$.

By Lemma 5.7.2, the Scott topology, call it T_1, on $\langle iso_l(X), \subseteq\rangle$ is generated by

$$\{\mathcal{O}_A \mid A \in \mathcal{O}^{\mathcal{T}}(X)\}, \text{ where } \mathcal{O}_A = \{F \in iso_l(X) \mid A \subseteq F\}.$$

By Lemma 2.3.19, the set of compact saturated sets in the Scott topology on $\langle iso_l(X), \subseteq\rangle$ is given by

$$T_2 = \{\bigcap B \mid B \subseteq \{\uparrow F_1 \cup \cdots \cup \uparrow F_n \mid F_i \in iso_l(X)\}\}.$$

We consider the strong proximity lattice

$$\mathcal{B} = \{(O, K) \mid \mathcal{O} \in T_1, K \in T_2 \text{ and } O \subseteq K\},$$

where the order is the point-wise inclusion and the strong proximity relation is given by

$$(O, K) \prec (O', K') \overset{\text{def}}{\Longleftrightarrow} K \subseteq O'.$$

Let $\langle X'; \mathcal{T}'_X, \leq' \rangle$ equipped with $\propto'$ be the Priestley dual of $\mathcal{B}$. Then by the representation theorem introduced in Chapter 3, $core(X')$ is homeomorphic to $\mathcal{K}_Y$ equipped with the Scott topology.

Appendix A

Order Theory

This appendix reviews concepts and results from order theory, that are related to and needed in the work presented in this thesis. The appendix is based on [20, 28].

Definition A.1. A binary relation $\leq$ on a set P is called an *order* if for every $x, y, z \in P$,

1. $x \leq x$,
2. $x \leq y, y \leq x \Longrightarrow x = y$, and
3. $x \leq y, y \leq z \Longrightarrow x \leq z$.

A *pre-ordered* is a set equipped with a binary relation that satisfies conditions 1 and 3 but not necessarily 2. An *ordered set*, a *poset*, or a *partially ordered set* is a set that has an order. $x \leq y$ is read as "x is less than or equal to y".

Definition A.2. Let $\langle P, \leq \rangle$ be a pre-order and $S \subseteq P$.

1. The dual of $\langle P, \leq \rangle$ is the pre-order $\langle P, \leq^{\partial} \rangle$ where $x \leq^{\partial} y$ if and only if $y \leq x$.
2. An element $x \in P$ is an *upper bound* of S if for every $s \in S$, $s \leq x$. *Lower bounds* are defined dually.

3. An element $s \in S$ is *minimal* in S if

$$(\forall s' \in S)\ s' \leq s \Longrightarrow s = s'.$$

   Maximal elements are defined dually.

4. An element $s \in S$ is called a *least element* of S if

$$(\forall s' \in S)\ s \leq s'.$$

   A *greatest element* of S is defined dually. If P has a least (greatest) element then this element is denoted by 0 (1) or by $\bot$ $(\top)$. P is *bounded* if it has least and greatest elements.

5. If S has a least upper bound then this element is called the *supremum* or *join* of S and is denoted by $\bigvee S$. The *infimum* or *meet* of S is defined dually to the supremum and is denoted by $\bigwedge S$. Provided that $S = \{s_1, \ldots, s_n\}$ and its supremum (infimum) exists then it is also denoted by $s_1 \vee \ldots \vee s_n$ $(s_1 \wedge \ldots \wedge s_n)$.

6. P is a *meet-semilattice* (*join-semilattice*) if it is an ordered set and for every $x, y \in P$, $x \wedge y$ $(x \vee y)$ exists. P is a *lattice* if it is an ordered set and for every $x, y \in P$, $x \vee y$ and $x \wedge y$ exist. P is a *complete lattice* if it is an ordered set and for every $A \subseteq P$, $\bigvee A$ and $\bigwedge A$ exist.

7. The *lower closure of* S (denoted by $\downarrow S$) is the set $\{x \in P \mid (\exists\, y \in S)\ x \leq y\}$. The *upper closure of* S (denoted by $\uparrow S$) is defined dually. $\uparrow x$ and $\downarrow x$ are shorthand for $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively. S is a *lower set* if $S = \downarrow S$. Dually, an *upper set* is defined. The *set of all lower (upper) subsets* of P is denoted by $\mathcal{O}(P)$ $(\mathcal{U}(P))$.

8. An element $x \in P$ is *join-prime* or $\vee$*-prime* if for every $a, b \in P$, $x \leq a \vee b$ implies $a \leq x$ or $a \leq y$. *Meet-primes* or $\wedge$*-primes* are defined dually.

Definition A.3. A filter base $\mathcal{F}$ is a non-empty collection of nonempty sets such that

$$F_1, F_2 \in \mathcal{F} \Longrightarrow (\exists F \in \mathcal{F})\ F \subseteq F_1 \cap F_2.$$

Lemma A.4. *Let P be a pre-order, $x, y \in P$, and $O \in \mathcal{O}(P)$. Then*

$$x \leq y \Longleftrightarrow \downarrow\{x\} \subseteq \downarrow\{y\} \Longleftrightarrow (y \in O \Longrightarrow x \in O).$$

Lemma A.5. *Let P be a pre-order and $x, y \in P$. Then*

$$x \leq y \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x.$$

Lemma A.6. *Let P be a pre-order, $x \in P$, and $A, B \subseteq P$. Suppose $\bigvee A, \bigwedge A, \bigvee B$, and $\bigwedge B$ exist in P. Then*

1. $(\forall a \in A)\ a \leq \bigvee A$ *and* $\bigwedge A \leq a$.

2. $x \leq \bigwedge A \Longleftrightarrow (\forall a \in A)\ x \leq a$.

3. $\bigvee A \leq x \Longleftrightarrow (\forall a \in A)\ a \leq x$.

4. $(\forall a \in A)(\forall b \in B)\ a \leq b \Longrightarrow \bigvee A \leq \bigwedge B$.

5. $A \subseteq B \Longrightarrow \bigvee A \leq \bigvee B$ *and* $\bigwedge A \geq \bigwedge B$.

Definition A.7. Let L be a lattice and $x \in L$. The element x is *join-irreducible* if $x \neq 0$ (provided that L has 0) and for every $a, b \in L$,

$$x = a \vee b \Longrightarrow x = a \text{ or } x = b.$$

Meet-irreducible elements are defined dually. *The set of all join-irreducible (meet-irreducible) elements* in L is denoted by $\mathcal{J}(L)$ ($\mathcal{M}(L)$).

Lemma A.8. *Let L be a distributive lattice and $x, y \in L$ such that $x \neq 0$ and $y \neq 1$, in case that L has 0 and/or 1. Then x is join-irreducible if and only if*

$$(\forall a_1, \ldots, a_k \in L)\, x \leq a_1 \vee \ldots \vee a_k \Longrightarrow (\exists i \in \{1, \ldots, k\})\, x \leq a_i.$$

Dually, y is meet-irreducible

$$(\forall a_1, \ldots, a_k \in L)\, x \geq a_1 \wedge \ldots \wedge a_k \Longrightarrow (\exists i \in \{1, \ldots, k\})\, x \geq a_i.$$

Definition A.9. Let P be a poset and S be a non-empty subset of P. The set S is *directed* if

$$x, y \in S \Longrightarrow (\exists z \in S)\, x, y \leq z.$$

Dually, S is *filtered* if

$$x, y \in S \Longrightarrow (\exists z \in S)\, z \leq x, y.$$

The supremum of S in P, if it exists, is denoted by $\bigvee^{\uparrow} S$. P is *directed-complete* if for every directed subset $S \subseteq P$, $\bigvee^{\uparrow} S$ exists in P.

Definition A.10. Let L be a lattice and I be a non-empty subset of L.

1. The set I is an *ideal* if it is lower and closed under binary suprema. *Filters* of L are defined dually. The set of all ideals (filters) of L is denoted by $idl(L)$ ($filt(L)$).

2. An ideal I of L is called *maximal* if

$$(\forall J \in idl(L))\, I \subseteq J \subseteq L \Longrightarrow I = J.$$

   Maximal filters are defined dually.

3. An ideal of L is *proper* if $I \neq L$. A proper ideal I of L is *prime* if for every $x, y \in L$, if $x \wedge y \in I$ then $x \in I$ or $y \in I$. Prime filters are defined dually. The set of all prime ideals (filters) of L is denoted by $idl_p(L)$ ($filt_p(L)$).

4. If L is complete then an ideal I of L is *completely prime* if for every $S \subseteq L$,

$$\bigwedge S \in I \Longrightarrow S \cap I \neq \emptyset.$$

5. If L is complete then a filter F of L is *Scott-open* if for every directed subset $S \subseteq L$, $\bigvee^{\uparrow} S \in F$ implies $S \cap F \neq \emptyset$. The set of all Scott-open filters of L is denoted by $\mathit{Sfilt}(L)$.

Definition A.11. A lattice L is *distributive* if for every $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Definition A.12. A distributive lattice $\langle B; \vee, \wedge \rangle$ is a *Boolean algebra* if it has least and greatest elements 0 and 1, respectively, and is equipped with a unary operation $'$ such that for every $b \in B$, $b \wedge b' = 0$ and $b \vee b' = 1$.

Definition A.13. A map f from an ordered set P to an ordered set Q is

1. *order-preserving (monotone)* if for every $a, b \in P$,

$$a \leq b \Longrightarrow f(a) \leq f(b).$$

2. *order-reflecting* if for every $a, b \in P$,

$$f(a) \leq f(b) \Longrightarrow a \leq b.$$

3. *order-embedding (embedding)* if for every $a, b \in P$,

$$a \leq b \Longleftrightarrow f(a) \leq f(b).$$

4. *order-isomorphism* if it is onto and an order-embedding.

Definition A.14. A map $f : A \longrightarrow B$ between lattices A and B is a *lattice homomorphism* if for every $a, b \in A$,

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

If A and B are Boolean algebras then f is a *Boolean (Boolean algebra) homomorphism* if it additionally preserves $0, 1$, and the unary operation $'$. An *embedding* is a one-to-one lattice homomorphism. A *lattice isomorphism* is a bijective lattice homomorphism. A *Boolean (Boolean algebra) isomorphism* is a bijective Boolean homomorphism.

Lemma A.15. *Let L be a lattice and $I \subseteq L$. Then*

$$I \in idl_p(L) \Longleftrightarrow (L \setminus I) \in \mathit{filt}_p(L).$$

Lemma A.16. *Let L be a distributive lattice and J and G be an ideal and a filter respectively of L. If $J \cap G = \emptyset$ then there exist $I \in idl_p(L)$ and $F = L \setminus I \in \mathit{filt}_p(L)$ such that $J \subseteq I$ and $G \subseteq F$.*

Lemma A.17. *Let L be a distributive lattice and J be a proper ideal of L. Then there exists $I \in idl_p(L)$ such that $J \subseteq I$.*

Lemma A.18. *In a finite distributive lattice, every element is the join (meet) of join-irreducible (meet-irreducible) elements below (above) it.*

Appendix B

Topology

This appendix reviews topological concepts and results, that are related to and needed in the work presented in this thesis. The appendix is based on [15, 22, 19].

Definition B.1. A *topological space* is a pair $\langle X, \mathcal{T} \rangle$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:

1. The sets $\emptyset$ and X belong to $\mathcal{T}$.

2. The collection $\mathcal{T}$ is closed under finite intersections and arbitrary unions.

The collection $\mathcal{T}$ is called a *topology* on the set (*space*) X. The elements of $\mathcal{T}$ are called *open sets* and their complements are called *closed sets*. A subset $A \subseteq X$ is *clopen* if it is closed and open.

Definition B.2. Let $\langle X; \mathcal{T} \rangle$ be a topological space.

1. A closed set in $\mathcal{T}$ is *irreducible* if it is non-empty and for every closed sets A' and A'',

$$A = A' \cup A'' \Longrightarrow A = A' \text{ or } A = A''.$$

2. The interior of a subset S of X is the union of all open sets contained in S and is denoted by S°.

3. A set $S \subseteq X$ is *saturated* if it is an intersection of open sets.

4. $\langle X; \mathcal{T} \rangle$ is T_0 if for every pair of distinct points $x, y \in X$, either there exists an open set that contains x but not y or there exists an open set that contains y but not x.

5. $\langle X; \mathcal{T} \rangle$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist two disjoint open sets $V, W \in \mathcal{T}$ such that $x \in V$ and $y \in W$.

6. A subset $S \subseteq \mathcal{T}$ is a *basis* for the topology $\mathcal{T}$ if every element in $\mathcal{T}$ is a union of elements of S. The set S is a *subbasis* for the topology $\mathcal{T}$ if the set of finite intersections of elements in S is a basis for $\mathcal{T}$.

7. A set $O \subseteq \mathcal{T}$ is an *open cover* for a set $S \subseteq X$ if $S \subseteq \bigcup O$.

8. A set $S \subseteq X$ is *compact* if every open cover to S has a finite subcover.

9. $\langle X, \mathcal{T} \rangle$ is *locally compact* if for every $x \in X$ and $O \in \mathcal{T}$ such that $x \in O$, there exists $V \in \mathcal{T}$ and a compact set $K \subseteq X$ such that $x \in V \subseteq K \subseteq O$.

10. $\langle X, \mathcal{T} \rangle$ is *totally disconnected* if for any two distinct points $x, y \in X$, there exists a clopen set containing x but not y.

11. The *closure of a set* $S \subseteq X$ is denoted by $\overline{S}$ and defined by

$$\overline{S} = \bigcap \{C \mid C \text{ is a closed set containing } S\}.$$

12. $\langle X, \mathcal{T} \rangle$ is *sober* if for every closed irreducible subset $C \subseteq X$, there exists a unique point $x \in X$ such that $\overline{\{x\}} = C$.

13. The binary relation $\leq_{\mathcal{T}}$ defined on X by

$$x \leq_{\mathcal{T}} y \overset{\text{def}}{\iff} ((\forall O \in \mathcal{T})\ x \in O \Longrightarrow y \in O),$$

is a pre-order and is known as the *specialisation order* of $\langle X, \mathcal{T} \rangle$. This pre-order defines an order iff $\langle X, \mathcal{T} \rangle$ is T_0.

Definition B.3. Suppose $f : X_1 \longrightarrow X_2$ is a map between two topological spaces $\langle X_1, \mathcal{T}_1\rangle$ and $\langle X_2, \mathcal{T}_2\rangle$. Then

1. f is *continuous* if it satisfies any of the following equivalent conditions.

   (a) $(\forall O \in \mathcal{T}_2)\ f^{-1}(O) \in \mathcal{T}_1$.

   (b) For every closed set C of X_2, $f^{-1}(C)$ is a closed subset of X_1.

   (c) If $\mathcal{S}$ is a basis or subbasis for $\mathcal{T}_2$ then

   $$(\forall O \in \mathcal{S})\ f^{-1}(O) \in \mathcal{T}_1.$$

2. f is a *homeomorphism* if it is bijective and continuous, and its inverse f^{-1} is continuous.

3. f is *open* if

   $$(\forall O \in \mathcal{T}_1)\ f(O) \in \mathcal{T}_2.$$

Definition B.4. An *ordered space* is a topological space $\langle X, \mathcal{T}\rangle$ equipped with an order $\leq$ that is closed in $\langle X \times X, \mathcal{T} \times \mathcal{T}\rangle$, where $\mathcal{T} \times \mathcal{T}$ is the topology on $X \times X$ generated by the basis $\{O \times O' \mid O, O' \in \mathcal{T}\}$. A *compact ordered space* is an ordered space $\langle X, \mathcal{T}, \leq\rangle$ whose topology $\mathcal{T}$ is compact.

Definition B.5. Let $\langle X_1; \mathcal{T}_1, \leq_1\rangle$ and $\langle X_1; \mathcal{T}_1, \leq_1\rangle$ be ordered spaces and $f : X_1 \longrightarrow X_2$ be a map. Then f is an *order-homeomorphism* if it is an order-isomorphism and a homeomorphism.

Lemma B.6. *Suppose $f : X_1 \longrightarrow X_2$ is a map between two topological spaces $\langle X_1, \mathcal{T}_1\rangle$ and $\langle X_2, \mathcal{T}_2\rangle$.*

1. *If $\langle X_1, \mathcal{T}_1\rangle$ is compact and f is continuous then $f(X_1)$ is compact.*

2. *If $\langle X_1, \mathcal{T}_1\rangle$ is compact Hausdorff, $\langle X_2, \mathcal{T}_2\rangle$ is Hausdorff, and f is continuous and bijective then f is a homeomorphism.*

The following lemma is known as *Alexander's subbasis lemma*.

Lemma B.7. *Suppose* $\langle X, \mathcal{T} \rangle$ *is a topological space and* $\mathcal{S}$ *is a subbasis for* $\mathcal{T}$. $\langle X, \mathcal{T} \rangle$ *is compact if and only if every open cover for* X *contained in* $\mathcal{S}$ *has a finite subcover.*

Lemma B.8. *Suppose* $\langle X, \mathcal{T} \rangle$ *is a compact Hausdorff topological space and* $C \subseteq X$. *Then* C *is closed if and only if it is compact.*

Appendix C

Category Theory

This appendix, which is based on [70, 96], reviews concepts and results from category theory, that are essential for the work presented in this thesis.

Definition C.1. A *category* is a collection of three entities:

1. a class whose elements are called *objects*.

2. a class whose elements are called *morphisms*. Every morphism is assigned two objects as its *domain* and *codomain*. $f : A \longrightarrow B$ denotes the morphism f whose domain is the object A and whose codomain is the object B.

3. a *composition operation* which assigns to every pair of morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ a morphism $g \circ f : A \longrightarrow C$ such that:

   (a) given $h : C \longrightarrow D,\ (h \circ g) \circ f = h \circ (g \circ f)$.

   (b) for every object A, there an *identity morphism* $1_A : A \longrightarrow A$ such that $f \circ 1_A = f$ and $1_B \circ f = f$.

Definition C.2. Let C be a category. Then the *opposite* or *dual* C^{op} of C is the category which has the same objects as C and whose morphisms are of the form $f^{op} : B \longrightarrow A$

where $f : A \longrightarrow B$ is a morphism in C. The composition in C^{op} is defined by

$$f^{op} \circ g^{op} = (g \circ f)^{op}.$$

Note that in the previous definition the identity in C^{op} at object A is $1_{A^{op}}$.

Definition C.3. Let C and D be two categories. A *functor* $F : C \longrightarrow D$ is a pair of functions (both denoted by F):

1. F : objects of $C \longrightarrow$ objects of D, and

2. F : morphisms of $C \longrightarrow$ morphisms of D.

This pair is subject to the following conditions:

1. for every morphism $f : A \longrightarrow B$ in C, $F(f) : F(A) \longrightarrow F(B)$,

2. $F(f \circ g) = F(f) \circ F(g)$, and

3. $F(1_A) = 1_{F(A)}$.

A functor $F : C \longrightarrow D$ is

1. *full* if for all objects A, B in C and for every morphism $g : F(A) \longrightarrow F(B)$ in D there is a morphism $f : A \longrightarrow B$ such that $F(f) = g$.

2. *faithful* or an *embedding* if for all objects A, B in C and for all morphisms $f, f' : A \longrightarrow B$ in C, $F(f) = F(f')$ implies $f = f'$.

A *contravariant functor* has the same definition as functor except that the directions of morphisms get reversed under it, i.e. it maps a morphism $f : A \longrightarrow B$ to a morphism $F(f) : F(B) \longrightarrow F(A)$. Consequently, the second condition above has the following form in case of a contravariant functor:

$$F(f \circ g) = F(g) \circ F(f).$$

An *endofunctor* is a functor from a category to itself.

Definition C.4. Let $F, G : C \longrightarrow D$ be two functors between categories C and D. A *natural transformation* is a mapping η that assigns for every object A in C a morphism $\eta_A : F(A) \longrightarrow G(A)$ in D such that for every morphism $f : A \longrightarrow B$ in C

$$G(f) \circ \eta_A = \eta_B \circ F(f).$$

If both F and G are contravariant then the later equation takes the following form

$$\eta_A \circ F(f) = G(f) \circ \eta_B.$$

If η_A is an isomorphism in D, for every A in C, then η is a *natural isomorphism.*

Definition C.5. Suppose $\eta : F \longrightarrow G$ and $\mu : G \longrightarrow H$ are natural transformations between functors $F, G, H : C \longrightarrow D$. Then the composition of η and μ is the natural transformation

$$\mu \circ \eta : F \longrightarrow H,$$

which sends an object A of C to the arrow $\mu_A \circ \eta_A$. For the functor F, the *identity transformation* is the natural transformation

$$1_F : F \longrightarrow F; A \longmapsto 1_{F(A)}.$$

Definition C.6. Let C and D be two categories. An *adjunction* from C to D is a triple $\langle F, G, \eta \rangle$ such that $F : C \longrightarrow D$ and $G : D \longrightarrow C$ are functors and η is a mapping that assigns for every pair of objects $A \in C$ and $B \in D$ a bijection of sets

$$\eta_{A,B} : D(F(A), B) \longrightarrow C(A, G(B)),$$

which is natural in A and B. $D(F(A), B)$ is the set of all morphisms in D from $F(A)$ to B. Similarly $C(A, G(B))$ is defined. The functor F and G are called *left-adjoint* and *right-adjoint*, respectively. If F and G are contravariant then the bijection $\eta_{A,B}$ has the form

$$\eta_{A,B} : D(B, F(A)) \longrightarrow C(A, G(B))$$

and F and G are said to be *dual adjoints* to each other.

Lemma C.7. *Let C and D be two categories and $\langle F, G, \eta \rangle$ an adjunction from C to D. Then the adjunction is completely determined by the functors F and G and any pair of natural transformations $\eta : I_C \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow I_D$ such that the composites:*

1. *$G \xrightarrow{\eta G} G \circ F \circ G \xrightarrow{G\varepsilon} G$ and*

2. *$F \xrightarrow{F\eta} F \circ G \circ F \xrightarrow{\varepsilon F} F$*

are the identity transformations.

Definition C.8. Let C and D be two categories. Then C and D are *equivalent* to each other if there are two functors $F : C \longrightarrow D$ and $G : D \longrightarrow C$ and two natural isomorphisms $G \circ F \cong I : C \longrightarrow C$ and $F \circ G \cong I : D \longrightarrow D$. The functor F is called an equivalence of categories. The category C is *dual equivalent* or *dual* to the category D if C is equivalent to the opposite of D.

Lemma C.9. *Let C and D be two categories and $F : C \longrightarrow D$ a functor. Then the following statements are equivalent.*

1. *F is an equivalence of categories.*

2. *F is full, faithful, and for each object B in D there is an object $A \in C$ such that $F(A)$ and B are isomorphic.*

Definition C.10. Let C be a category. A collection S of objects and morphisms of C is a *subcategory* of C if

1. for every morphism $f : A \longrightarrow B$ in S, A and B belong to S,

2. for every object A in S, 1_A belongs to S, and

3. for every pair of arrows $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in S, $g \circ f$ belongs to S.

A subcategory S of a category C is *full* if the functor from S to C that sends every object to itself and every morphism to itself is full.

Definition C.11. Let C be a category. Then C is *Cartesian-closed* if it satisfies the following conditions.

1. C has an object T such that for every object A in C there exists a single morphism $f_A : A \longrightarrow T$. The object T is called *terminal*.

2. For every pair of objects A_1 and A_2 in C, there exists an object X and a pair of morphisms $\pi_i : X \longrightarrow A_i$, $i = 1, 2$, in C such that for every object B and every pair of morphisms $f_i : B \longrightarrow A_i$, $i = 1, 2$, there exists a unique morphism $f : B \longrightarrow X$ such $f_i = \pi_i \circ f, i = 1, 2$. The object X is called the product of A_1 and A_2 and is denoted by $A_1 \times A_2$.

3. Provided that condition 2 is satisfied, for every pair of objects A_1 and A_2 in C, there exists an object E and a morphism $\eta : E \times A_1 \longrightarrow A_2$ such that for any object X in C and morphism $g : X \times A_1 \longrightarrow A_2$ there is a unique morphism $\lambda_g : X \longrightarrow E$ such that

   $$\eta \circ (\lambda_g \times 1_{A_1}) = g.$$

   The object E is called an *exponential* and is denoted by $A_2^{A_1}$.

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Index