Walking through infinite trees with mixed induction and coinduction

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Mixing induction and coinduction

Why? — Because it is useful in practice and interesting technically.

- Properties of interactive computation/processes, e.g. weak bisimilarity, responsiveness etc. (see our SOS 2010 paper)
- Properties from modal logic (this talk)

My talk

How to traverse binary trees with infinitely deep paths

With predicates on trees via mixed induction-coinduction

How different is our traversal from path oriented traversal á la modal logic?

Infinite binary trees

$$\frac{}{R : color} \qquad \frac{}{B : color}$$

$$\frac{c : color \qquad t_0 : tree \qquad t_1 : tree}{t_0 \; c \; t_1 : tree}$$

Extensional equality on trees defined by coinduction:

$$\frac{t_0 \approx t_0' \qquad t_1 \approx t_1'}{t_0 \; c \; t_1 \approx t_0' \; c \; t_1'}$$

$\approx$ is an equivalence. We only consider setoid predicates.
(NB: double horizontal line – coinduction, single – induction)

E(GF) & A(FG)

Two properties on red-black trees are of interest (to us).

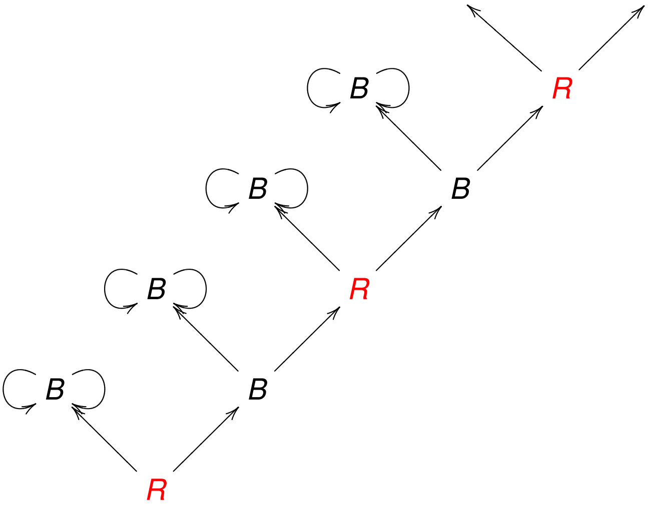
- Some path is infinitely often red.
- Every path is almost always black.

NB: we could consider symmetric ones:

- Every path is infinitely often red.
- Some path is almost always black.

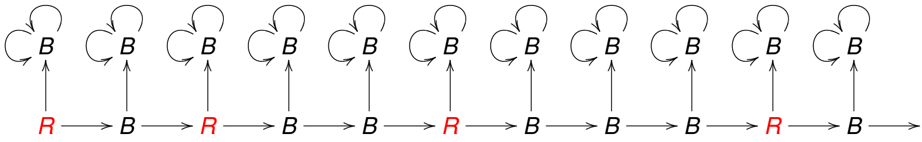
Exists, always, eventually (1)

Some path is infinitely often red.



Exists, always, eventually (2)

Some path is infinitely often red.



Interval is finite but unbounded.

Nesting induction-into-coinduction

$$\nu X.\, G(\mu Y.\, F(X, Y), X)$$

$$\frac{X\ t_0}{pop\ X\ (t_0\ \color{red}{R}\ t_1)} \qquad \frac{X\ t_1}{pop\ X\ (t_0\ \color{red}{R}\ t_1)}$$

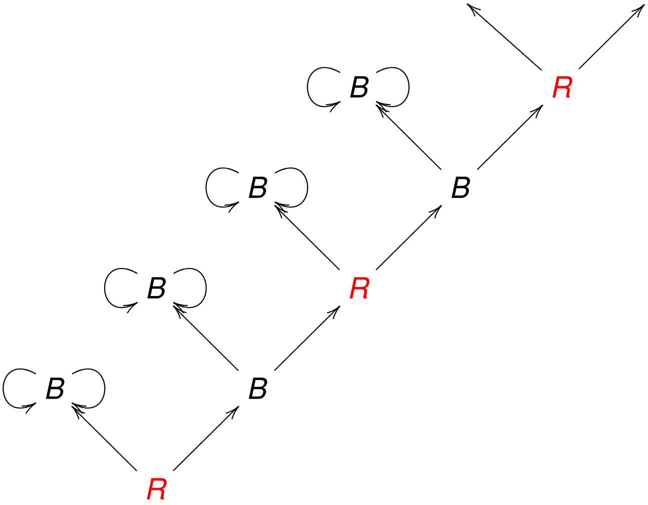
$$\frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)} \qquad \frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)}$$

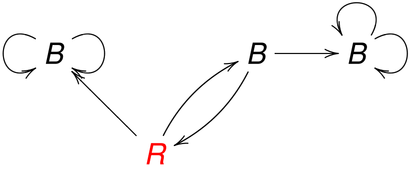
$$\frac{rep\ t_0}{rep\ (t_0\ \color{red}{R}\ t_1)} \qquad \frac{rep\ t_1}{rep\ (t_0\ \color{red}{R}\ t_1)}$$

$$\frac{pop\ rep\ t_0}{rep\ (t_0\ B\ t_1)} \qquad \frac{pop\ rep\ t_1}{rep\ (t_0\ B\ t_1)}$$

$$\left(\begin{array}{c} or,\ equivalently \\ \dfrac{X \subseteq rep \quad pop\ X\ t_0}{rep\ (t_0\ B\ t_1)} \qquad \dfrac{X \subseteq rep \quad pop\ X\ t_1}{rep\ (t_0\ B\ t_1)} \end{array} \right)$$

Some path is infinitely often red.



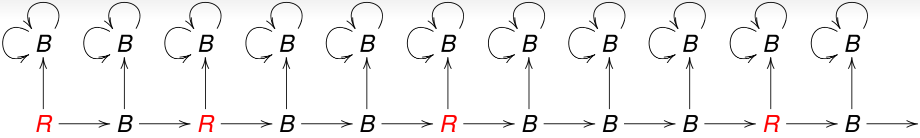


$$\frac{X\ t_0}{pop\ X\ (t_0\ \textcolor{red}{R}\ t_1)} \qquad \frac{X\ t_1}{pop\ X\ (t_0\ \textcolor{red}{R}\ t_1)}$$

$$\frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)} \qquad \frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)}$$

$$\genfrac{}{}{3pt}{}{rep\ t_0}{rep\ (t_0\ \textcolor{red}{R}\ t_1)} \qquad \genfrac{}{}{3pt}{}{rep\ t_1}{rep\ (t_0\ \textcolor{red}{R}\ t_1)}$$

$$\genfrac{}{}{3pt}{}{pop\ rep\ t_0}{rep\ (t_0\ B\ t_1)} \qquad \genfrac{}{}{3pt}{}{pop\ rep\ t_1}{rep\ (t_0\ B\ t_1)}$$



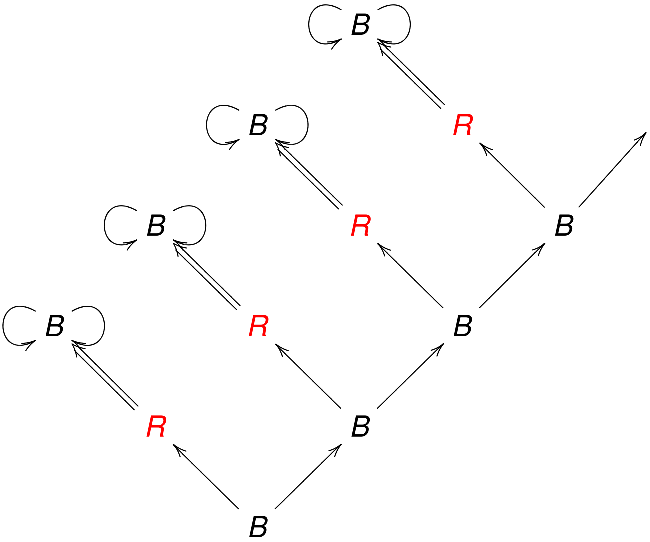
$$\frac{X\ t_0}{pop\ X\ (t_0\ R\ t_1)} \qquad \frac{X\ t_1}{pop\ X\ (t_0\ R\ t_1)}$$

$$\frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)} \qquad \frac{pop\ X\ \ t_0}{pop\ X\ (t_0\ B\ t_1)}$$

$$\frac{rep\ t_0}{rep\ (t_0\ R\ t_1)} \qquad \frac{rep\ t_1}{rep\ (t_0\ R\ t_1)}$$

$$\frac{pop\ rep\ t_0}{rep\ (t_0\ B\ t_1)} \qquad \frac{pop\ rep\ t_1}{rep\ (t_0\ B\ t_1)}$$

Every path is almost always black



Nesting coinduction-into-induction

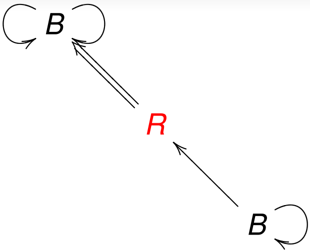
$$\mu X.\, G(\nu Y.\, F(X, Y), X)$$

$$\frac{X\ t_0 \quad X\ t_1}{\mathit{ontrack}\ X\ (t_0\ {\color{red}R}\ t_1)} \qquad \frac{\mathit{ontrack}\ X\ t_0 \quad \mathit{ontrack}\ X\ t_1}{\mathit{ontrack}\ X\ (t_0\ B\ t_1)}$$

$$\frac{\mathit{pretrack}\ t_0 \quad \mathit{pretrack}\ t_1}{\mathit{pretrack}\ (t_0\ {\color{red}R}\ t_1)}$$

$$\frac{\mathit{ontrack}\ \mathit{pretrack}\ t_0 \quad \mathit{ontrack}\ \mathit{pretrack}_1\ t_1}{\mathit{pretrack}\ (t_0\ B\ t_1)}$$

$$\left(\begin{array}{c}\textit{or equivalently} \\ \dfrac{X_0 \subseteq \mathit{pretrack} \quad \mathit{ontrack}\ X_0\ t_0 \quad X_1 \subseteq \mathit{pretrack} \quad \mathit{ontrack}\ X_1\ t_1}{\mathit{pretrack}\ (t_0\ B\ t_1)}\end{array}\right)$$



$$\frac{X\ t_0 \quad X\ t_1}{\mathit{ontrack}\ X\ (t_0\ R\ t_1)} \qquad \frac{\mathit{ontrack}\ X\ t_0 \quad \mathit{ontrack}\ X\ t_1}{\mathit{ontrack}\ X\ (t_0\ B\ t_1)}$$

$$\frac{\mathit{pretrack}\ t_0 \quad \mathit{pretrack}\ t_1}{\mathit{pretrack}\ (t_0\ R\ t_1)}$$

$$\frac{\mathit{ontrack}\ \mathit{pretrack}\ t_0 \quad \mathit{ontrack}\ \mathit{pretrack}\ t_1}{\mathit{pretrack}\ (t_0\ B\ t_1)}$$

Splitting trees with *rep* and *pretrack*

Proposition

$\forall t : \mathit{tree}, \neg(\mathit{rep}\ t \wedge \mathit{pretrack}\ t)$.

Proof.

Follows from $\forall t, \mathit{pretrack}\ t \rightarrow \neg(\mathit{rep}\ t)$ proved by induction. □

Proposition

Classically, $\forall t : \mathit{tree}, \mathit{rep}\ t \vee \mathit{pretrack}\ t$.

Proof.

Follows from $\forall t, \neg(\mathit{pretrack}\ t) \rightarrow (\mathit{rep}\ t)$ proved by coinduction classically. □

Walking along paths

Every path is almost always black

$$\dfrac{p : path}{r\ p : path} \qquad \dfrac{p : path}{l\ p : path} \qquad\qquad \dfrac{c : color : stream}{c\ s : stream}$$

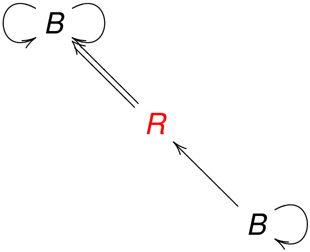
$$[\![\cdot]\!]_{\cdot} : tree \to path \to stream$$

$[\![p]\!]_t$ returns a stream, by following t along p.

$$\dfrac{\mathcal{G}\ s}{\mathcal{G}\ (B\ s)} \qquad\qquad \dfrac{\mathcal{F}^{\mathcal{G}}\ s}{\mathcal{F}^{\mathcal{G}}\ (c\ s)} \qquad \dfrac{\mathcal{G}\ s}{\mathcal{F}^{\mathcal{G}}\ s}$$

Every path is almost always black, or

$$\forall p.\, \mathcal{F}^{\mathcal{G}}\ [\![p]\!]_t$$



$$\frac{\mathcal{G}\ s}{\mathcal{G}\ (B\ s)} \qquad \frac{\mathcal{F}^{\mathcal{G}}\ s}{\mathcal{F}^{\mathcal{G}}\ (c\ s)}$$

Every path is almost always black, i.e.,

$$\forall p.\, \mathcal{F}^{\mathcal{G}}[\![p]\!]_t$$

Every path is almost always black

$$\forall p.\, \mathcal{F}^{\mathcal{G}}[\![p]\!]_t \qquad \textit{pretrack}\ t$$

where

$$\frac{\mathcal{G}\ s}{\mathcal{G}\ (B\ s)} \qquad \frac{\mathcal{F}^{\mathcal{G}}\ s}{\mathcal{F}^{\mathcal{G}}\ (c\ s)}$$

and

$$\frac{X\ t_0 \quad X\ t_1}{\textit{ontrack}\ X\ (t_0\ R\ t_1)} \qquad \frac{\textit{ontrack}\ X\ t_0 \quad \textit{ontrack}\ X\ t_1}{\textit{ontrack}\ X\ (t_0\ B\ t_1)}$$

$$\frac{\textit{pretrack}\ t_0 \quad \textit{pretrack}\ t_1}{\textit{pretrack}\ (t_0\ R\ t_1)}$$

$$\frac{\textit{ontrack pretrack}\ t_0 \quad \textit{ontrack pretrack}\ t_1}{\textit{pretrack}\ (t_0\ B\ t_1)}$$