Notes on Restriction Categories*

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Abstract
This is a summary of some notes taken by me during the course given by Prof. Robin Cockett at the Institute of Cybernetics in February 2013.

1 Restriction categories
We assume familiarity with the basics of category theory. Given a category $\mathcal{X}$, we write $\text{Ob}(\mathcal{X})$ or $|\mathcal{X}|$ for the collection of its objects, and $\text{Hom}_{\mathcal{X}}(A,B)$ or $\mathcal{X}(A,B)$ for the collection of the maps in $\mathcal{X}$ from $A$ to $B$. We put $\text{Hom}_{\mathcal{X}} = \bigcup_{A,B \in |\mathcal{X}|} \mathcal{X}(A,B)$. If $\mathcal{X}$ is clear from the context, we may omit it from the Hom-notation.

Definition 1.1 (Restriction category). A restriction category is a category $\mathcal{X}$ together with a restriction operator $(\cdot)$ which associates to every $f \in \mathcal{X}(A,B)$ an $\overline{f} \in \mathcal{X}(A,A)$ so that the following diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{\overline{f}} & A
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{g \circ f} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{g \circ \overline{f}} & A
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
B & \xrightarrow{\overline{f}} & B
\end{array}
\]

i.e., the following properties are satisfied:

1. $g \circ \overline{f} = \overline{g \circ f}$ for every $f : A \rightarrow B$.
2. $\overline{g \circ f} = g \circ \overline{f}$ for every $f : A \rightarrow B$, $g : A \rightarrow C$.
3. $\overline{g \circ f} = g \circ \overline{f}$ for every $f : A \rightarrow B$, $g : A \rightarrow C$.
4. $\overline{g \circ f} = f \circ g \circ \overline{f}$ for every $f : A \rightarrow B$, $g : B \rightarrow C$.

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Let us interpret the rules. Rule 1 says that the result of $f$ does not change if its restriction $\overline{f}$ is applied first. Rule 2 says that restricted maps with same domain commute with each other. Rule 3 says that composing two restrictions is the same as applying a restricted map first, then an ordinary one, and finally restrict the composition. Rule 4 says that a restriction can be moved from “after” to “before” an ordinary map, provided the originally restricted map is pre-composed with the other one.

**Example 1.2.** The category $\text{Par}$ whose objects are sets and whose maps are partial functions, is a restriction category: the restriction $\overline{f}: A \to A$ of $f: A \to B$ satisfies $\overline{f}(x) = x$ if $f(x)$ is defined, and undefined otherwise.

**Lemma 1.3.** Let $X$ be a restriction category. For every $f: A \to B$, $g: B \to C$ the following hold:

1. $\overline{f} \circ \overline{f} = \overline{f}$. That is: the restriction operator on $X$ is idempotent.
2. $\overline{f} \circ \overline{g} = \overline{g} \circ \overline{f}$. That is: restricted maps are idempotents in $X$.
3. If $f$ is monic then $\overline{f} = \text{id}_A$. In particular, $\overline{\text{id}_A} = \text{id}_A$.
4. $g \circ \overline{f} = \overline{g} \circ f$ for every $f: A \to B$, $g: B \to C$.

**Proof.** We first prove point 2. By putting $g = f$ in rule 3 of restriction categories we get $\overline{f} \circ \overline{f} = \overline{f} \circ \overline{f}$, which is $\overline{f}$ by rule 1.

We now prove point 3. By rule 1, $\overline{g} \circ \overline{f} = \overline{g} \circ f$; as $f$ is monic, $\overline{f} = \text{id}_A$.

To prove point 4 we use all the rules of restriction categories:

$$\overline{g} \circ \overline{f} = \overline{f} \circ \overline{g} \circ f = \overline{g} \circ \overline{f} = \overline{g} \circ \overline{f} = \overline{g} \circ \overline{f} = \overline{g} \circ \overline{f}.$$

Finally,

$$\overline{f} = \overline{\text{id}_A} \circ \overline{f} = \overline{\text{id}_A} \circ \overline{f} = \text{id}_A \circ \overline{f} = \overline{f} ;$$

which proves point 1. □

**Definition 1.4** (Total map). Let $X$ be a restriction category. A map $f: A \to B$ in $X$ is total if $\overline{f} = \text{id}_A$.

Monic maps are total. In addition, we have the following

**Lemma 1.5.** Let $X$ be a restriction category, $f \in X(A, B)$, $g \in X(B, C)$. If $g \circ f$ is total, then so is $f$.

**Proof.** If $\overline{g} \circ \overline{f} = \text{id}_A$, then

$$\overline{f} = \text{id}_A \circ \overline{f} = \overline{g} \circ \overline{f} = \overline{g} \circ f \circ \overline{f} = \overline{g} \circ \overline{f} = \overline{g} \circ \overline{f} = \text{id}_A .$$

□

**Lemma 1.6.** For a restriction category $X$, the objects of $X$ together with the total maps of $X$ form a subcategory $\text{Total}(X)$.

**Proof.** By point 3 of Lemma 1.3, identities are total maps. By point 4 if $f: A \to B$ and $g: B \to C$ are total maps, then

$$\overline{g} \circ \overline{f} = \overline{g} \circ \overline{f} = \overline{\text{id}_B} \circ \overline{f} = \overline{f} = \text{id}_A ,$$

and $g \circ f: A \to C$ is a total map. □
Definition 1.7. Let $\mathcal{X}$ be a restriction category and let $f, g : A \to B$ maps in $\mathcal{X}$. We write $f \leq g$ if $g \circ f = f$, i.e., if the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{f} \\
A & \xrightarrow{k} & B
\end{array}
\]

In the case $A = B$ observe that $\overline{f} \leq \text{id}_A$ for every $f : A \to A$.

Example 1.8. Let $\mathcal{X} = \text{Par}$. Then $f \leq g$ if and only if $g$ is an extension of $f$.

Lemma 1.9. The relation introduced in Definition 1.7 is a partial order. Moreover, restriction is monotone with respect to such partial order, i.e., if $f \leq g$ then $f \leq g$.

Proof. Reflexivity follows by putting $g = f$ and applying the first rule of restriction categories. For transitivity, if $f \leq g$ and $g \leq h$, then

\[
h \circ f = h \circ g \circ f = h \circ \overline{g} \circ f = g \circ f = f,
\]

so that $f \leq h$ as well. For antisymmetry, if $f \leq g$ and $g \leq f$, then

\[
f = g \circ f = f \circ \overline{g} \circ f = f \circ \overline{g} = f \circ g = g.
\]

Finally, if $g \circ f = f$ then $\overline{g} \circ f = \overline{g} \circ f = g \circ f$.

Recall that a binary relation $\mathcal{R}$ defined on a partial monoid $M$ is enriching if, for every $f, g, h, k \in M$ such that $hfk$ and $hgk$ are defined, if $f \mathcal{R} g$ then $hfk \mathcal{R} hgk$.

Lemma 1.10. The ordering from Definition 1.7 is enriching.

Proof. A relation on maps is enriching if $f \mathcal{R} g$ implies $(k \circ f \circ h) \mathcal{R} (k \circ g \circ h)$ whenever $f, g, h \in \mathcal{X}(A, B)$, $h \in \mathcal{X}(Z, A)$, and $k \in \mathcal{X}(B, C)$ as $Z, A, B, C \in |\mathcal{X}|$. We will prove separately the two cases $h = \text{id}_A$ and $k = \text{id}_B$, from which the general case follows easily.

First, suppose $C = B$ and $k = \text{id}_B$: if $g \circ \overline{f} = f$, then by rule 4 of restriction categories $g \circ h \circ \overline{f} \circ h = g \circ f \circ h = f \circ h$. Thus, if $f \leq g$ then $f \circ h \leq g \circ h$.

Next, suppose $Z = A$ and $h = \text{id}_A$: if $g \circ \overline{f} = f$, then

\[
k \circ g \circ k \circ f \circ \overline{f} = k \circ g \circ k \circ f \circ \overline{f} \quad \text{by Rule 1}
= k \circ g \circ k \circ f \circ \overline{f} \quad \text{by Rule 1}
= k \circ g \circ f \circ k \circ \overline{f} \quad \text{by Rule 2}
= k \circ f \circ k \circ \overline{f}
= k \circ f \quad \text{by Rule 1}.
\]

Thus, if $f \leq g$ then $k \circ f \leq k \circ g$.

Corollary 1.11. Let $f : A \to B$, $h : A \to C$, $k : B \to D$. Then $k \circ f \circ k \leq f$.

Proof. Follows from Lemma 1.10 and $h \leq \text{id}_A$, $k \leq \text{id}_B$. 

3
Definition 1.12 (Compatible maps). Two maps \( f, g : A \to B \) in a restriction category \( \mathcal{X} \) are compatible, written \( f \sim g \), if \( g \circ \overline{f} = f \circ \overline{g} \).

Example 1.13. Two maps in \( \text{Par} \) are compatible if and only if they are equal on the intersection of their domains.

Lemma 1.14. Let \( f, g : A \to B, h : Z \to A, k : B \to C \) be maps in a restriction category \( \mathcal{X} \).

1. \( f \sim g \) if and only if \( g \circ \overline{f} \leq f \) and \( f \circ \overline{g} \leq g \). In fact, if either inequality holds, then \( f \sim g \), and the other one holds too.

2. If \( f \sim g \) then \( k \circ f \circ h \sim k \circ g \circ h \).

Proof. If \( f \sim g \), then
\[
f \circ \overline{g} \circ \overline{f} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{g} = g \circ \overline{f},
\]
that is, \( g \circ \overline{f} \leq f \); similarly, \( f \circ \overline{g} \leq g \). On the other hand, if \( g \circ \overline{f} \leq f \), then
\[
g \circ \overline{f} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{g} = g \circ \overline{f},
\]
that is, \( f \sim g \); similarly if \( f \circ \overline{g} \leq g \). This proves point [1].

Point [2] is so tricky that we provide two proofs. Recall that \( f \sim g \) means \( g \circ \overline{f} = f \circ \overline{g} \).

- First proof: (by James Chapman) On the one hand,
\[
k \circ g \circ h \circ k \circ f \circ h = k \circ g \circ k \circ \overline{f} \circ h
= k \circ g \circ \overline{g} \circ k \circ \overline{f} \circ h
= k \circ g \circ k \circ \overline{f} \circ \overline{g} \circ h
= k \circ g \circ k \circ \overline{g} \circ \overline{f} \circ h
= k \circ g \circ k \circ g \circ \overline{f} \circ h
= k \circ g \circ k \circ g \circ \overline{f} \circ h
= k \circ g \circ f \circ \overline{f} \circ h.
\]
on the other hand,
\[
k \circ f \circ h \circ k \circ g \circ h = k \circ f \circ k \circ g \circ h
= k \circ f \circ k \circ g \circ \overline{f} \circ h
= k \circ f \circ k \circ g \circ \overline{f} \circ h
= k \circ f \circ k \circ g \circ \overline{f} \circ h
= k \circ f \circ k \circ g \circ \overline{f} \circ h
= k \circ f \circ \overline{g} \circ h
= k \circ f \circ \overline{g} \circ \overline{f} \circ h
= k \circ f \circ \overline{g} \circ \overline{f} \circ h
= k \circ f \circ \overline{g} \circ \overline{f} \circ h,
\]
and it follows from the hypothesis that \( k \circ g \circ \overline{f} \circ h = k \circ f \circ \overline{g} \circ h \).
Second proof: (by Robin Cockett) We prove separately that, if \( f \sim g \), then \( f \circ h \sim g \circ h \) and \( k \circ f \sim k \circ g \): from which the general case follows immediately. In fact, if \( g \circ f = f \circ g \), then

\[
g \circ h \circ f = g \circ f \circ h = f \circ g \circ h = f \circ h \circ g \circ h,
\]

and

\[
k \circ g \circ k \circ f = k \circ g \circ k \circ f \circ f = k \circ g \circ k \circ f \circ g = k \circ f \circ g \circ k \circ f = k \circ f \circ g \circ k \circ f \circ g = k \circ f \circ k \circ g.
\]

In the last chain of equalities, the leftmost \( k \) is only a placeholder that never enters the transformations. We thus get

**Corollary 1.15.** Let \( \mathcal{X} \) be a restriction category. For every \( A, B, C \in |\mathcal{X}| \), \( f, g \in \mathcal{X}(A, B) \), and \( k \in \mathcal{X}(B, C) \), if \( g \circ f = f \circ g \), then \( g \circ k \circ f = f \circ k \circ g \).

## 2 Examples of restriction categories

**Example 2.1** (The trivial restriction). Let \( \mathcal{X} \) be any category and let \( f = \text{id}_A \) for every \( f \in \mathcal{X}(A, B) \).

Example 2.1 tells that being a restriction category is additional structure, not a limitation.

**Example 2.2** (The restriction monoid). Let \( \mathcal{X} \) have \( \mathbb{N} \) as its only object and the partial recursive functions as maps. Define restriction as in \( \text{Par} \), *i.e.*, if \( f \) is a partial recursive function, then \( \overline{f}(n) = f(n) \) if \( f(n) \) is defined, and undefined otherwise.

**Example 2.3** (Meet semilattices). Let \( \mathcal{X} \) be a meet semilattice, *i.e.*, a set with a binary operation \( \wedge \) satisfying \( x \wedge y = y \wedge x \), \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \), and \( x \wedge x = x \). Then \( (\mathcal{X}, \wedge) \) is easily given a monoid structure, thus making it a category: define restriction by \( \overline{\tau} = x \). The axioms of restricted category follow then directly from those of meet semilattice.
Example 2.4 (Trunked trees). Let $G = (V,E)$ be a directed graph. Consider the path category (or free category) on $G$, whose objects are the nodes of $G$, and whose maps are the finite directed paths from a node to the other, with identities being the empty paths. We can then indicate the maps in $\text{Path}(G)$ as $(A, [a_1, \ldots, a_n], B)$; the composition of $(A, [a_1, \ldots, a_n], B)$ and $(B, [b_1, \ldots, b_m], C)$ will be $(A, [a_1, \ldots, a_n, b_1, \ldots, b_m], C)$. In a similar way, the trunked tree on $G$ is defined as the category $\textbf{TrunkT}(G)$ whose objects are the nodes of $G$, and where a map from $A$ to $B$ is a tuple $(A, (p, P), B)$ where

- $p$ is a finite path in $G$, and
- $P$ is a finite, prefix-closed set of finite paths on $G$ that contains $p$.

Then $(A, (p, P), B)(B, (q, Q), C) = (A, (pq, PQ), C)$, and $\text{id}_A = (A, ([], \{[]\}), A)$. The restriction of $(A, (p, P), B)$ is defined as $(A, ([], P), A)$: thus, restriction in trunked trees is not trivial. The total maps of $\textbf{TrunkT}(G)$ are those where $P$ is reduced to $\{[]\}$, i.e., the identities.

We recall the definition of pullback.

Definition 2.5 (Pullback). Let $f : A \to B$ and $g : C \to B$ be two convergent arrows in a category $\mathcal{X}$. A pullback of $f$ along $g$ is a map $f' : D \to C$, for which a map $g' : D \to A$ exists such that the following hold:

1. $g \circ f' = f \circ g'$, i.e., the following diagram commutes:

$$
\begin{array}{c}
D \\
| f' \downarrow \\
| g' \downarrow \\
A \\
| f \downarrow \\
B
\end{array}
$$

2. For every object $E$ and maps $u : E \to C$, $v : E \to A$ such that $g \circ u = f \circ v$ there exists a unique map $\alpha : E \to D$ such that $f' \circ \alpha = u$ and $g' \circ \alpha = v$, i.e., the following diagram commutes:

$$
\begin{array}{c}
E \\
| u \downarrow \\
| v \downarrow \\
D \\
| f \downarrow \\
| g' \downarrow \\
A \\
| f \downarrow \\
B
\end{array}
$$

Pullbacks are unique, up to the following equivalence: $f' \sim f''$ if and only if there is an isomorphism $\alpha$ such that the following diagram commutes:

$$
\begin{array}{c}
D \\
| f' \downarrow \\
| g' \downarrow \\
A \\
| f'' \downarrow \\
D
\end{array}
$$
Lemma 2.6. Let \( \mathcal{X} \) be a category and let \( m \in \mathcal{X}(A,B) \).

1. If \( m \) is monic, then every pullback of \( m \) along any map is monic.

2. \( m \) is monic if and only if \( A \xrightarrow{m} B \) is a pullback diagram.

Proof. Let \( m \) be monic and let \( A \xrightarrow{f} A \xrightarrow{m} B \) be a pullback; let then \( u,v : X \rightarrow Z \) satisfy \( m' \circ u = m' \circ v \). Then \( m \circ f' \circ u = f \circ m' \circ u = f \circ m' \circ v = m \circ f' \circ v \), so \( f' \circ u = f' \circ v \) as \( m \) is monic. As \( m' \) is a pullback, for \( h = m' \circ u = m' \circ v \) and \( k = f' \circ u = f' \circ v \), there exists a unique map \( \alpha : X \rightarrow Z \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \xrightarrow{f'} A \\
\downarrow{k} & & \downarrow{m'} \\
A & \xrightarrow{m} & B
\end{array}
\]

commutes: which implies \( u = \alpha = v \).

Now, surely \( A \xrightarrow{m} B \) commutes. If \( m \) is monic and \( u,v : Z \rightarrow A \) also make \( Z \xrightarrow{v} A \) commute, then \( u = v \) by monicness of \( m \), and clearly \( A \xrightarrow{m} B \) is a pullback. On the other hand, if the latter is a pullback and \( m \circ u = m \circ v \), then \( u = \alpha \text{id}_A = v \) for a unique \( \alpha : Z \rightarrow A \) which is only possible for \( u = \alpha = v \).

\( \Box \)

Lemma 2.7. Let

\[
\begin{array}{ccc}
X & \xrightarrow{h} & D \xrightarrow{f'} A \\
\downarrow{g''} & & \downarrow{g'} \\
Z & \xrightarrow{g} & B \xrightarrow{f} C
\end{array}
\]

be a commutative diagram.

1. If the inner squares in (6) are pullback diagrams, then so is the outer rectangle.
That is: if $g'$ is the pullback of $g$ along $f$, and $g''$ is the pullback of $g'$ along $h$, then $g''$ is the pullback of $g$ along $f \circ h$.

2. If the outer rectangle and the rightmost square in (6) are pullback diagrams, then so is the leftmost square.

That is: if $g'$ is the pullback of $g$ along $f$, and $g''$ is the pullback of $g$ along $f \circ h$, then $g''$ is the pullback of $g'$ along $h$.

Point 1 can be stated as such: the pullback of a pullback is a pullback.

Proof. First, suppose that $g'$ is the pullback of $g$ along $f$, and $g''$ is the pullback of $g'$ along $h$. Suppose that there exist an object $Y$ and two maps $u : Y \to Z$, $v : Y \to A$ such that $f \circ h \circ u = g \circ v$. As $g'$ is the pullback of $g$ along $f$, there exists a unique $\beta : Y \to D$ such that $v = f \circ \beta$ and $h \circ u = g' \circ \beta$; as $g''$ is the pullback of $g'$ along $h$, there exists a unique $\alpha : Y \to X$ such that $u = g'' \circ \alpha$ and $\beta = h' \circ \alpha$, which in turn implies $v = f \circ \beta = f' \circ h' \circ \alpha$. This proves that $g''$ is the pullback of $g$ along $f \circ h$.

The situation is summarized by the following commutative diagram:

Next, suppose that $g'$ is the pullback of $g$ along $f$, and that $g''$ is the pullback of $g$ along $f \circ h$. Consider an object $Y$ and two maps $u : Y \to Z$, $v : Y \to D$ such that $h \circ u = g' \circ v$: then $g \circ f' \circ v = f \circ h \circ u$ as well. As $g''$ is the pullback of $g$ along $f \circ h$, there exists a unique $\alpha : Y \to X$ such that $u = g'' \circ \alpha$ and $f' \circ v = f' \circ h' \circ \alpha$; as $g'$ is the pullback of $g$ along $f$, there exists a unique $\beta : Y \to D$ such that $h \circ u = g' \circ \beta$ and $f' \circ v = f' \circ \beta$. Then $\beta = v = h' \circ \alpha$, which proves that $g''$ is the pullback of $g'$ along $f$. 
The situation is summarized by the following commutative diagram:

We recall that \( \text{Hom}_X = \bigcup_{A,B \in |X|} X(A,B) \).

**Definition 2.8** (Stable set of monics). A family \( \mathcal{M} \subseteq \text{Hom}_X \) is a stable set of monics if it satisfies the following four properties:

1. Every isomorphism of \( X \) belongs to \( \mathcal{M} \); in particular, \( \text{id}_A \in \mathcal{M} \) for every \( A \in |X| \).
2. Every \( m \in \mathcal{M} \) is monic.
3. \( \mathcal{M} \) is closed by composition.
4. For every \( m \in \mathcal{M} \), the pullbacks of \( m \) along any \( f \) exist and belong to \( \mathcal{M} \).

**Definition 2.9** (Partial map category). Let \( X \) be a category and let \( \mathcal{M} \subseteq \text{Hom}_X \) be a stable set of monics. The partial map category on \( X \) by \( \mathcal{M} \) is the category \( \text{Par}(X, \mathcal{M}) \) defined as follows:

- The objects of \( \text{Par}(X, \mathcal{M}) \) are the objects of \( X \).
- The maps of \( \text{Par}(X, \mathcal{M}) \) from \( A \) to \( B \) are the spans

\[
\begin{array}{ccc}
A' & \xleftarrow{m} & A \\
\downarrow{f} & & \downarrow{f} \\
A & & B
\end{array}
\]

in \( X \) from \( A \) to \( B \) with \( m \in \mathcal{M} \), modulo the following equivalence relation:

\[
\begin{array}{ccc}
A' & \xleftarrow{m} & A \\
\downarrow{f} & & \downarrow{f} \\
A & & B
\end{array}
\]

is equivalent to

\[
\begin{array}{ccc}
A' & \xleftarrow{m'} & A \\
\downarrow{f} & & \downarrow{f'} \\
A & & B
\end{array}
\]
if and only if there exists an isomorphism \( \alpha : A' \to A'' \) such that \( m = m' \circ \alpha \) and \( f = f' \circ \alpha \), i.e., the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A'' \\
\downarrow{m} & & \downarrow{m''} \\
A' & \xrightarrow{f} & B \\
\end{array}
\]

commutes.

- The identities of \( \text{Par}(X, \mathcal{M}) \) are the upper left corners of the form \( (\text{id}_A, \text{id}_A) \).
- The composition of \( (m, f) : A \to B \) with \( (m', g) : B \to C \) is defined (up to equivalence) as \( (m \circ m'', g \circ h) \), where \( h \) is the pullback of \( f \) along \( m' \) and \( m'' : A'' \to A' \) is the corresponding map:

\[
\begin{array}{ccc}
A'' & \xrightarrow{m''} & A' \\
\downarrow{h} & & \downarrow{f} \\
A & \xrightarrow{m} & B \\
\end{array}
\]

That \( \text{Par}(X, \mathcal{M}) \) is indeed a category, follows from Lemma 2.7 and uniqueness of pullbacks up to equivalence.

**Example 2.10.** Let \( X = \text{Set} \) and let \( \mathcal{M} \) be the family of all injective functions. Then \( \text{Par}(X, \mathcal{M}) = \text{Par} \).

**Example 2.11.** Let \( R \) be a commutative ring; let \( \Sigma \subseteq R \) not contain the zero, and let \( R(\Sigma^{-1}) \) be the smallest commutative ring that contains \( R \) and where every element of \( \Sigma \) has a multiplicative inverse. The embedding of \( R \) into \( R(\Sigma^{-1}) \) is called localization, and indicated by \( \text{Loc} \): it is known that, if \( f : R \to S \) is a ring homomorphism such that \( f(x) \) is a unit for every \( x \in \Sigma \), then there is a unique \( \phi : R(\Sigma^{-1}) \to S \) such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\text{Loc}} & R(\Sigma^{-1}) \\
\downarrow{f} & & \downarrow{\phi} \\
S & & \\
\end{array}
\]

commutes. Moreover, \( \text{Loc} \) is an epic map. It turns out that the category \( \text{Par}(\text{CRing}^{\text{op}}, \text{Loc}) \) is the opposite category of commutative rings with rational functions.

**Example 2.12.** Let \( X = \text{Top} \) and let \( \mathcal{M} \) be the family of continuous pushouts of open sets: that is, a map \( m \in \mathcal{M} \) from \( X \) to \( Y \) is a continuous function from an open subset \( U \) of \( X \) into \( Y \). Then \( \text{Par}(X, \mathcal{M}) \) is a restriction category.

\[\text{The definition in the February 27, 2013 version of these notes incorrectly assumed the spans to be pullbacks. Thanks to James Chapman for pointing out this error.}\]
Theorem 2.13. \( \text{Par}(X, \mathcal{M}) \) is a restriction category, where the restriction of \((m, f)\) is \((m, m)\).

Proof. If \((m, f)\) is a map in \( \text{Par}(X, \mathcal{M}) \) from \( A \) to \( B \), then monicness of \( m \) allows

\[
\begin{array}{ccc}
A' & \\
\downarrow m & \nearrow m & \downarrow f \\
A & \uparrow m & \downarrow B
\end{array}
\]

so that \((m, f) \circ (m, m) = (m, f)\): thus, \( \text{Par}(X, \mathcal{M}) \) satisfies Rule 1.

If \((m, f) : A \to B\) and \((n, g) : A \to C\) in \( \text{Par}(X, \mathcal{M}) \), then

\[
\begin{array}{ccc}
A''' & \\
\downarrow n' & \nearrow m' & \downarrow f' \\
A'' & \uparrow m & \downarrow C
\end{array}
\]

can be read indifferently from left to right, or from right to left: hence, \((n, n) \circ (m, m) = (m, m) \circ (n, n)\). This proves that Rule 2 holds in \( \text{Par}(X, \mathcal{M}) \).

As pullbacks are unique modulo equivalence, the diagram for \((n, g) \circ (m, f)\) says that \((n, g) \circ (m, f)\) can be constructed as

\[
\begin{array}{ccc}
A''' & \\
\downarrow n' & \nearrow m' & \downarrow g' \\
A'' & \uparrow m & \downarrow C
\end{array}
\]

whose restriction is \((m \circ n', m \circ n') = (m \circ n', n \circ m')\), which we know to be \((n, g) \circ (m, f)\). Thus, \( \text{Par}(X, \mathcal{M}) \) satisfies Rule 3.

Finally, let \((m, f) : A \to B\) and \((n, g) : B \to C\). Then \((n, g) \circ (m, f)\) is given by

\[
\begin{array}{ccc}
A'' & \\
\downarrow f' & \nearrow n' & \downarrow f \\
A & \uparrow f & \downarrow B
\end{array}
\]
By Lemma 2.6 and Lemma 2.7, we can write \((m, f) \circ (n, g) \circ (m, f)\) as follows:

\[
\begin{array}{c}
A'' \xrightarrow{n'} A' \\
\downarrow m \qquad \downarrow m \\
A \xrightarrow{m} A
\end{array}
\]

which yields \((m, f) \circ (n, g) \circ (m, f) = (m \circ n', f \circ n')\): but \(f \circ n' = n \circ f'\) by the diagram for \((n, g) \circ (m, f)\), which proves that \(\text{Par}(\mathbb{X}, \mathcal{M})\) satisfies Rule 4.

3 Idempotents

**Definition 3.1** (Split). Let \(\mathbb{X}\) be a category and let \(e : A \to A\) be an idempotent, i.e., \(e^2 = e \circ e = e\). We say that \(e\) splits if there exist an object \(B\), a monomorphism \(s : B \to A\), and an epimorphism \(r : A \to B\) such that \(s \circ r = e\) and \(r \circ s = \text{id}_B\), i.e., the diagram

\[
\begin{array}{c}
A \xrightarrow{r} B \\
\downarrow s \quad \downarrow e \\
A \xrightarrow{s} B
\end{array}
\]

commutes. We call the map \(s\) a split of \(e\), and the equality \(e = s \circ r\) a splitting of \(e\).

Observe that \(r\) is a retraction i.e., an epimorphism with a one-side converse; dually, \(s\) is a section. Also observe that, if \(e\) has such a decomposition, then \(e\) is an idempotent.

**Lemma 3.2.** Let \(e : A \to A\) be an idempotent and let \(e = s \circ r = s' \circ r'\) be two splittings of \(e\). Then there exists a unique isomorphism \(\alpha : B \to B'\) such that \(s = s' \circ \alpha\) and \(r' = \alpha \circ r\), i.e., the diagram

\[
\begin{array}{c}
A \xrightarrow{s'} B' \\
\downarrow r \quad \downarrow s' \\
B \xrightarrow{r \alpha} A
\end{array}
\]

commutes.

**Proof.** Set \(\alpha = r' \circ s\). Then \(r \circ s' = \alpha^{-1}\), as

\[
r \circ s' \circ r' \circ s = r \circ e \circ s = r \circ s = \text{id}_B \circ \text{id}_B = \text{id}_B
\]

and similarly \(r' \circ s \circ r \circ s' = \text{id}_B\). Monicness of \(s'\) ensures uniqueness of \(\alpha\). □
Lemma 3.2 states that splittings of idempotents are unique, up to (a reasonable notion of) an isomorphism. Therefore, in the rest of these notes, we will indicate $e = s \circ r$ as "the" splitting of $e$.

Recall that the equalizer of $f, g : A \to B$ is a map $h : Z \to A$ such that $f \circ h = g \circ h$, and that, for every $k : Y \to A$ such that $f \circ k = g \circ k$, there exists a unique $\alpha : Y \to Z$ such that $k = h \circ \alpha$. Dually, the coequalizer of $f, g : A \to B$ is a map $p : B \to C$ such that $p \circ f = p \circ g$ and that, for every $q : B \to D$ such that $q \circ f = q \circ g$, there exists a unique $\beta : C \to D$ such that $q = \beta \circ p$. Equalizers and coequalizers are unique, up to isomorphisms. It follows from the definition that equalizers are monic, and coequalizers are epic.

**Corollary 3.3.** If $e = s \circ r$ is the splitting of an idempotent $e : A \to A$, then $s$ is the equalizer of $e$ and $\text{id}_A$, and $r$ is the coequalizer of $e$ and $\text{id}_A$.

**Proof.** Clearly, $e \circ s = s \circ r \circ s = s \circ \text{id}_B = s = s \circ \text{id}_A$. If $h : Z \to A$ also satisfies $e \circ h = h$, then $\alpha = r \circ h$ satisfies $s \circ \alpha = s \circ r \circ h = e \circ h = h$; also, if $h = s \circ \alpha'$, then $s \circ \alpha' = e \circ h = s \circ r \circ h$, and $\alpha' = r \circ h = \alpha$ by monicness of $s$.

Dually, $r \circ e = r \circ s \circ r = \text{id}_B \circ r = r = r \circ \text{id}_A$, and if $k : A \to C$ also satisfies $k \circ e = k \circ \text{id}_A$, then $\beta = k \circ s$ is the unique map from $B$ to $C$ such that $k = \beta \circ r$. \hfill $\square$

In arbitrary categories, idempotents need not split. However, it is always possible to split idempotents.

**Definition 3.4.** Let $\mathbb{X}$ be an arbitrary category and let $E$ be a class of idempotents of $\mathbb{X}$ such that $\text{id}_A \in E$ for every $e \in E \cap \mathbb{X}(A, A)$. The category $\text{Split}_E(\mathbb{X})$ is defined as follows:

- The objects of $\text{Split}_E(\mathbb{X})$ are the elements of $E$.
- A map in $\text{Split}_E(\mathbb{X})$ from $e : A \to A$ to $e' : A' \to A'$, is a map $f : A \to A'$ in $\mathbb{X}$ such that $e' \circ f \circ e = f$.
- Composition is defined as in $\mathbb{X}$.
- The identity $\text{id}_e$ of $e : A \to A$ in $\text{Split}_E(\mathbb{X})$ is the map $e$ of $\mathbb{X}$.

Then all the elements of $E$ are split in $\text{Split}_E(\mathbb{X})$. If $\mathbb{X}$ is a restriction category, the restriction of $f : e \to e'$ in $\text{Split}_E(\mathbb{X})$ is the map $f \circ e$ of $\mathbb{X}$.

**Example 3.5.** Let $\mathbb{X}$ be a category and let $\mathcal{M}$ be a stable set of monics for $\mathbb{X}$. Then every restriction in $\text{Par}(\mathbb{X}, \mathcal{M})$ splits.

To see this, let $(m, f) : A \to B$ be a map in $\text{Par}(\mathbb{X}, \mathcal{M})$—i.e., let $m : A' \to A$ and $f : A' \to B$ constitute the pullback of a monic map—and let $(m, m)$ be its restriction. Put $s = (\text{id}_{A'}, m)$ and $r = (m, \text{id}_{A'})$: then the pullback diagrams

```
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (A') at (1,0) {$A'$};
  \node (B) at (2,0) {$B$};
  \node (A'') at (1,1) {$A'$};
  \node (B') at (2,1) {$B'$};
  \draw[->] (A) -- (A');
  \draw[->] (A') -- (A'');
  \draw[->] (A'') -- (A');
  \draw[->] (A') -- (B');
  \draw[->] (A'') -- (B'');
  \draw[->] (A) -- (B);
  \draw[->] (A') -- (B');
  \draw[->] (A') -- (B'');
\end{tikzpicture}
```
show that indeed \( s \circ r = (m, m) \) and \( r \circ s = (\text{id}_{A'}, \text{id}_{A'}) \).

**Definition 3.6** (Split restriction category). A restriction category where every restriction map splits is called a **split restriction category**.

Let \( \mathcal{X} \) be a split restriction category. We call \( \mathcal{M}_X \) the class of splits of restrictions: that is, \( m : A \rightarrow B \) belongs to \( \mathcal{M}_X \) if and only if there exist a map \( f : B \rightarrow A \) and an epic map \( r = r_m : B \rightarrow A \) such that \( m \circ r = \overline{f} \) and \( r \circ m = \text{id}_A \). Observe that, in this case, \( \overline{m} \circ r = \overline{f} = \overline{f} = m \circ r \); but then, \( m \circ r = m \circ r = m \circ r \) as \( m \) is monic.

In words: if \( f = m \circ r \) is the splitting of a restriction, then restricting the retraction is the same as postcomposing it with the split.

Let us consider some properties of \( \mathcal{M}_X \). First of all, it is a class of monic maps of \( \mathcal{X} \)—actually, of \( \text{Total}(\mathcal{X}) \), because monic maps are total. Also, as every isomorphism is clearly the split of (the restriction of) an identity, \( \mathcal{M}_X \) contains every isomorphism. Moreover, if \( \overline{f} = m \circ r_m \) and \( \overline{g} = n \circ r_n \) are splittings of restrictions, then \( n \circ m \) is the split of a restriction idempotent and \( r_m \circ r_n \) is the corresponding retraction, as

\[
\begin{align*}
n \circ m \circ r_m \circ r_n &= n \circ \overline{m} \circ r_n \\
&= n \circ r_n \circ \overline{m} \circ r_n \\
&= r_n \circ \overline{m} \circ r_n \\
&= r_m \circ r_n \circ r_n \\
&= r_m \circ r_n \circ r_n \\
&= r_m \circ r_n \\
\end{align*}
\]

and \( r_m \circ r_n \circ n \circ m = r_m \circ \text{id}_B \circ m = r_m \circ m = \text{id}_{A'} \).

So, \( \mathcal{M}_X \) is close to being a stable set of monics. The main obstacle to this is that, in general, elements of \( \mathcal{M}_X \) might not have pullbacks along arbitrary maps of \( \mathcal{X} \). Notably, by relaxing this condition a little bit, we get

**Theorem 3.7.** If \( \mathcal{X} \) is a split restriction category, then \( \mathcal{M}_X \) is a stable set of monics in \( \text{Total}(\mathcal{X}) \).

**Proof.** We must prove that, if \( m : A \rightarrow B \) is the split of a restriction idempotent \( \overline{m} = e = e_m : B \rightarrow B \) of \( \mathcal{X} \), \( r = r_m \) is the corresponding retraction, and \( f : C \rightarrow B \) is a total map in \( \mathcal{X} \), then the pullback \( m' : D \rightarrow C \) of \( m \) along \( f \) exists, and is the split of a restriction idempotent of \( \mathcal{X} \).

We observe that \( e' = e \circ f \) is a restriction idempotent: by hypothesis, \( e' \) has a splitting \( e' = m' \circ r' \) with \( m' : D \rightarrow C \) monic. Then

\[
f' = r \circ f \circ m'
\]
is such that \( m \circ f' = f \circ m' \), as
\[
\begin{align*}
m \circ f' &= m \circ r \circ f \circ m' \\
         &= e \circ f \circ m' \\
         &= \overline{e} \circ f \circ m' \\
         &= f \circ e' \circ m' \\
         &= f \circ m' \circ r' \circ m' \\
         &= f \circ m' \circ \text{id}_D \\
         &= f \circ m'.
\end{align*}
\]

We then only need to prove:

1. that \( f' \) is a total map, and

2. that \( m' \) is a pullback in \( \text{Total}(X) \).

For point 1 by monicness of \( m \) and the fact that \( e = \text{id} \) we have:
\[
\begin{align*}
m' &= r \circ f \circ m' \\
    &= \overline{m} \circ r \circ f \circ m' \\
    &= r \circ \overline{m} \circ r \circ f \circ m' \\
    &= r \circ e \circ f \circ m' \\
    &= \overline{r} \circ e \circ f \circ m':
\end{align*}
\]
but \( r \) is the retraction in the splitting of \( e = m \circ r \), so \( r = m \circ r = e \), and
\[
\begin{align*}
m' &= e \circ e \circ f \circ m' \\
    &= e \circ f \circ m' \\
    &= m' \circ r' \circ m' \\
    &= \overline{m} \\
    &= \text{id}_D.
\end{align*}
\]
as \( m' \) is monic too.

For point 2 suppose \( x : X \to C \) and \( y : X \to A \) satisfy \( f \circ x = m \circ y \). As \( m' \) is monic, if \( \alpha : X \to D \) exists such that \( m' \circ \alpha = x \) and \( f' \circ \alpha = y \), then it is unique. Our candidate is thus \( \alpha = r' \circ x \), for which we prove the second equation by showing that \( m \circ f' \circ r' \circ x = m \circ y \) and exploiting monicness of \( m \):
\[
\begin{align*}
m \circ f' \circ r' \circ x &= m \circ r \circ f \circ m' \circ r' \circ x \\
                     &= e \circ f \circ e \circ f \circ x \\
                     &= e \circ f \circ x \\
                     &= m \circ r \circ m \circ y \\
                     &= m \circ y.
\end{align*}
\]
\[15\]
Given this, we get

\[
m' \circ r' \circ x = e' \circ x \\
= x \circ e' \circ x \\
= x \circ e \circ f \circ x \\
= x \circ e \circ m \circ y.
\]

but \(e \circ m \circ y = m \circ r \circ m \circ y = m \circ y\), thus

\[
m' \circ r' \circ x = x \circ m \circ y \\
= x \circ f \circ x \\
= x.
\]

as \(f\) is total. \(\square\)

**Theorem 3.8** (The Completeness Theorem). Let \(X\) be a split restriction category. Then

\[
X \cong \text{Par}(\text{Total}(X), M_X) \tag{12}
\]

via the equivalence \(F\) that sends every object into itself, and every \(f \in \text{Hom}(A, B)\) into

\[
Ff = (m f, f \circ m) \in \text{Par}(\text{Total}(X), M_X)(A, B) \tag{13}
\]

where \(f = m f \circ r f\) is the splitting. Moreover,

\[
F \bar{f} = \bar{F} f \ \forall f \in \text{Hom}_X. \tag{14}
\]

That is: every split restriction category is a full subcategory of a partial map category, up to equivalence.

**Proof.** For brevity and clarity, let us put \(m = m f\) and \(r = r f\).

Observe that \(f \circ m : A' \to B\) is total, because

\[
f \circ m = \bar{f} \circ m = m \circ r \circ m = m = \text{id}_{A'}.
\]

If \(f \in \text{Hom}(A, B)\), then \(A \xrightarrow{r f} A' \xrightarrow{f \circ m} B\): but \(f \circ m \circ r = f \circ \bar{f} = f\).

To show that \(F\) is actually a functor, let \(Ff = (m, f \circ m)\) and \(Fg = (n, g \circ n)\), where \(f = m f \circ r f = n \circ s\) is the splitting: then \((Ff)(Fg) = (m \circ n', g \circ n \circ h)\), where \(n'\) is the pullback of \(n\) along \(f\) and \(h\) is the pullback of \(f\) along \(n'\):

\[
\begin{array}{ccc}
A'' & \xleftarrow{n'} & A' \\
\downarrow & & \downarrow r f \\
A' & \xrightarrow{f \circ m} & B' \\
\downarrow m & & \downarrow g \circ n \\
A & \xrightarrow{m} & B & \xleftarrow{n'} \rightarrow C
\end{array}
\]
Now, we know from the proof of Theorem 3.7 that $n'$ is the split of $\overline{g \circ f \circ m} = \overline{g \circ f \circ m} = \overline{g \circ f \circ m}$: if $s'$ is the retraction corresponding to $n'$, then
\[
m \circ n' \circ s' \circ r = m \circ g \circ f \circ m \circ r = g \circ f \circ m \circ r = g \circ f \circ f = g \circ f,
\]
so that $m \circ n'$ is indeed the split of $\overline{g \circ f}$ (and $s' \circ r$ is the corresponding retraction). But $h = s \circ f \circ m \circ n'$, so that $g \circ n \circ h = g \circ n \circ s \circ f \circ m \circ n'$,
\[
= g \circ n \circ f \circ m \circ n' = g \circ f \circ m \circ n'.
\]
thus, $(m \circ n', g \circ n \circ h) = (m \circ n', g \circ f \circ m \circ n')$, i.e., $Fg \circ Ff = F(g \circ f)$. That $Fid_A = (id_A, id_A)$ for every object $A$ follows immediately from the definitions.

The inverse functor is defined by sending $(m, g) \in \text{Par}(\text{Total}(X), M_X)(A, B)$ into $g \circ r \in X(A, B)$, where $m \circ r = \overline{f}$ is the splitting of a restriction. In fact, $(m, f \circ m)$ is sent to $f \circ m \circ r = f \circ \overline{f} = f$, while $g \circ r$ is sent to $(m, g \circ r \circ m) = (m, g)$.

Finally, for every $f \in \text{Hom}(A, B)$ we have
\[
Ff = (m \overline{f}, m \overline{f}) = (m \overline{f}, m \overline{f} \circ r \circ m \overline{f}) = (m \overline{f}, \overline{f} \circ m \overline{f}) = (m \overline{f}, \overline{f}).
\]

\[
\square
\]

4 Cartesian restriction categories

Definition 4.1 (Restriction product). Let $X$ be a restriction category and let $A$ and $B$ be two objects in $X$. The restriction product of $A$ and $B$ is an object $A \times_R B$ together with two total maps $\pi_0 : A \times_R B \rightarrow A, \pi_1 : A \times_R B \rightarrow B$ such that, for every object $Z$ and pair of maps $f : Z \rightarrow A, g : Z \rightarrow B$ there exists a unique map $(f, g)_R : Z \rightarrow A \times_R B$ such that
\[
\pi_0 \circ (f, g)_R = f \circ \overline{g} \quad \text{and} \quad \pi_1 \circ (f, g)_R = g \circ \overline{f}.
\]

Recall that the standard product requires commutativity of the rectangle:

\[
\begin{array}{c}
\begin{array}{c}
Z \\
\downarrow f \\
A \quad \pi_0 \quad \pi_1 \\
\downarrow \\
A \times B \\
\end{array}
\end{array}
\]
Instead, the restriction product demands commutativity of the following rectangle:

\[
\begin{array}{c}
Z \\
\downarrow f \\
A \\
\leftarrow \langle f, g \rangle_R \\
\end{array}
\begin{array}{c}
Z \\
\downarrow (f \cdot g)_R \\
A \times_R B \\
\downarrow \pi_0 \\
A' \\
\end{array}
\begin{array}{c}
B \\
\downarrow \pi_1 \\
B' \\
\end{array}
\]

(16)

We stress that the “projections” \(\pi_0\) and \(\pi_1\) in Definition 4.1 are total maps. On the other hand, the “pairing” \(\langle f, g \rangle_R\) is not required to be total, as this would imply \(f\) and \(g\) to be total too (cf. Lemma 4.2 later on). However, if \(f\) and \(g\) are total, then so is \(\langle f, g \rangle_R\) by Lemma 1.5 as in this case

\[
\pi_0 \circ \langle f, g \rangle_R = f \circ \overline{g} = \overline{f} = \text{id}_Z
\]

and similarly for \(\pi_1\). The noteworthy feature of restriction product is that it moves non-totality on the component which is not involved in the projection.

Observe that restriction products are unique, up to a unique isomorphism. In fact, if \((C, p_0, p_1)\) is another candidate to the restriction product of \(A\) and \(B\), then the pairings \(\langle \overline{p_0}, \overline{p_1} \rangle_R\) and \(\langle p_0, p_1 \rangle_R\) must be each other’s inverse.

**Lemma 4.2.** Let \(\mathcal{C}\) be a restriction category. Every time the compositions and restriction products are defined, the following hold:

1. \(\langle f, g \rangle_R = \overline{f} \circ \overline{g} = \overline{g} \circ \overline{f}\).

As a consequence: if \((f, g)_R\) is total, then so are \(f\) and \(g\).

2. \((f \circ h, g)_R = (f, g)_R \circ h = (f, g \circ h)_R\).

3. \((f, g)_R \circ h = (f \circ h, g \circ h)_R\).

4. Define \(f \times_R g\) as \(\langle f \circ \pi_0, g \circ \pi_1 \rangle_R\). Then \(\overline{f} \times_R \overline{g} = \overline{f} \times_R \overline{g}\). \(f \times_R g\) for restriction products is the perfect analogous of \(f \times g\) for products. Indeed, \(f \times_R g\) is the unique map \(\phi\) such that the following diagram commutes:

\[
\begin{array}{c}
A \\
\downarrow f \\
A' \\
\leftarrow \pi_0 \\
\end{array}
\begin{array}{c}
A \times_R B \\
\downarrow \phi \\
A' \times_R B' \\
\downarrow \pi'_0 \\
A' \\
\end{array}
\begin{array}{c}
B \\
\downarrow g \\
B' \\
\end{array}
\begin{array}{c}
\pi_1 \\
\end{array}
\]

which is the same as requiring that the following one does:

\[
\begin{array}{c}
A \times_R B \\
\downarrow f \circ \pi_0 \\
A' \\
\leftarrow \pi'_0 \\
\end{array}
\begin{array}{c}
A \times_R B \\
\downarrow \phi \\
A' \times_R B' \\
\downarrow \pi'_1 \\
A' \\
\end{array}
\begin{array}{c}
B \\
\downarrow g \circ \pi_1 \\
B' \\
\end{array}
\]

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Proof of Lemma 4.2. As $\pi_1$ is total,

$$\langle f, g \rangle_R = \pi_1 \circ \langle f, g \rangle_R = \pi_1 \circ \pi_1 \circ (f, g)_R = \pi_1 \circ g = g \circ \bar{f}.$$

If, in addition, $\langle f, g \rangle_R$ is total, then $\bar{f}$ and $\bar{g}$ are each other’s inverse, so they are monic, and $\bar{f} = \bar{g} = \id_Z = \bar{f} \circ \bar{g}$. Point 1 is thus proved.

For point 2 observe that

$$\pi_0 \circ \langle f, g \rangle_R \circ h = f \circ \omega \circ h = f \circ h \circ g \circ h = \pi_0 \circ \langle f \circ h, g \circ h \rangle_R,$$
and similarly,

$$\pi_1 \circ \langle f, g \rangle_R \circ h = \pi_1 \circ \langle f \circ h, g \circ h \rangle_R.$$

Point 3 then follows from uniqueness of $\langle f \circ h, g \circ h \rangle_R$.

Finally, as clearly $\langle \pi_0, \pi_1 \rangle_R$ follows from point 1, 4.3.

which proves point 4.

Definition 4.3 (Restriction functor). Let $\mathcal{X}$ and $\mathcal{Y}$ be restriction categories. A restriction functor from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $F : \mathcal{X} \to \mathcal{Y}$ such that $F(f \circ g) = F(f) \circ F(g)$ for every $f, g \in \Hom_{\mathcal{X}}$.

Point 4 of Lemma 4.2 can thus be read as such: the functor $\times_R : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ that associates to every pair of objects $(A, B)$ the restriction product $A \times_R B$ and to every pair of maps $f : A \to A', g : B \to B'$ the restriction product map $\langle f \circ \pi_0, g \circ \pi_1 \rangle_R : A \times_R B \to A' \times_R B'$, is a restriction functor.

Definition 4.4 (Restriction final object). A restriction final object in a restriction category $\mathcal{X}$ is an object $1_R$ such that every object has a unique total map $!_A : A \to 1_R$ with the following property: for every map $f : A \to 1_R$ the equality $f = !_A \circ \bar{f}$ holds, i.e., $f \leq !_A$.

If $\mathcal{X}$ has a restriction final object $1_R$, then $\mathcal{X}(A, 1_R)$ is equivalent to $\mathcal{O}(A) = \{ e : A \to A \mid \omega = e \}$, the lattice of restriction idempotents at $A$, which may be seen as a family of “open sets”.

Definition 4.5 (Cartesian restriction category). A cartesian restriction category is a restriction category with a restriction final object $1_R$ where every two objects (thus, every $n$ objects for arbitrary $n \in \mathbb{N}$) have a restriction product.
Definition 4.6 (Partial isomorphism). A map $f : A \to B$ in a restriction category $\mathcal{X}$ is a partial isomorphism if there exists a map $g = f^{-1} : B \to A$ such that $g \circ f = \overline{f}$ and $f \circ g = \overline{g}$.

Lemma 4.7. Let $\mathcal{X}$ be a restriction category and let $f : A \to B$, $g : B \to C$ be partial isomorphisms.

1. $f^{-1}$ is unique (and so is $g^{-1}$).

2. $g \circ f$ is a partial isomorphism.

Proof. If $g$ and $g'$ satisfy $g \circ f = g' \circ f = \overline{f}$, $f \circ g = \overline{g}$, $f \circ g' = \overline{g'}$, then $g = g \circ \overline{g} = g \circ f \circ g = g' \circ f \circ g = g' \circ \overline{g}$,

on the other hand, $g' \circ \overline{g} = g' \circ \overline{g} \circ \overline{g'} = g' \circ f \circ g \circ f \circ g' = g' \circ f \circ g' = g' \circ \overline{g'} = g' \circ \overline{g} = g'$.

Thus, $g = g'$, and point 1 is proved. For point 2, the partial inverse of $g \circ f$ is simply $f^{-1} \circ g^{-1}$, as

$$f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ \overline{g} \circ f$$

$$= f^{-1} \circ f \circ \overline{g} \circ \overline{f}$$

$$= \overline{f} \circ g \circ \overline{f}$$

$$= \overline{g} \circ \overline{f} \circ \overline{g}$$

$$= g \circ f \circ \overline{f}$$

$$= g \circ \overline{f}$$,

and similarly, $g \circ f \circ f^{-1} \circ g^{-1} = f^{-1} \circ g^{-1}$.

Lemma 4.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be two restriction categories and let $F : \mathcal{X} \to \mathcal{Y}$ be a restriction functor. If $f \in \text{Hom}_\mathcal{X}(A, B)$ is a partial isomorphism, then so is $Ff \in \text{Hom}_\mathcal{Y}(FA, FB)$.

Proof. If $g = f^{-1}$ is the partial inverse of $f$ in $\mathcal{X}$, then

$$Fg \circ Ff = F(g \circ f) = F(\overline{f}) = \overline{Ff}$$

and similarly $Ff \circ Fg = \overline{Fg}$, so $Fg$ is the partial inverse of $Ff$ in $\mathcal{Y}$.

Example 4.9. Let $\mathcal{X} = \text{Par}$, $A = \{a\}$, $B = \{b_1, b_2\}$, $f(a) = b_1$, $g(b_1) = a$, $g(b_2)$ undefined. Then $f$ is a partial isomorphism, $g$ is its partial inverse, $f$ is total, and $g$ is not.
Definition 4.10 (Discrete restriction category). A cartesian restriction category $\mathcal{X}$ is discrete if for every object $A$ the diagonal $\Delta_A = (\text{id}_A, \text{id}_A)_R \in \mathcal{X}(A, A \times_R A)$ is a partial isomorphism.

Observe that $\Delta_A$ is total: hence, $\Delta_A^{-1} \circ \Delta_A = \text{id}_A$. The partial inverses $\Delta_A^{-1} : A \times_R A \to A$ are called equalities.

Example 4.11. Let $\mathcal{X}$ be a discrete cartesian restriction category. Then $\text{Total}(\mathcal{X})$ has ordinary products, which coincide with restriction products in $\mathcal{X}$; and final object, which is just the restriction final object of $\mathcal{X}$. But $\text{Total}(\mathcal{X})$ has pullbacks, and the equalizer $m$ of $f, g : X \to A$ in $\text{Total}(\mathcal{X})$ can be constructed as the pullback of $\Delta_A$ along $(f, g)_R$ as follows:

From the existence theorem for limits it then follows that $\text{Total}(\mathcal{X})$ has all limits. So:

if $\mathcal{X}$ is a discrete cartesian restriction category, then $\text{Total}(\mathcal{X})$ is a “nice” category.

Definition 4.12 (Meets in a restriction category). A restriction category $\mathcal{X}$ has meets if there exists a binary operator $\wedge$ which associates to every pair of parallel maps $f, g : A \to B$ a map $f \wedge g : A \to B$ so that the following properties are satisfied:

1. $f \wedge f = f$.
2. $f \wedge g \leq f, g$. That is: $f \circ f \wedge g = g \circ f \wedge g = f \wedge g$.
3. $(f \wedge g) \circ h = (f \circ h) \wedge (g \circ h)$ for every $h : Z \to A$.

Lemma 4.13. Let $\mathcal{X}$ be a restriction category with meets and let $f, g \in \mathcal{X}(A, B)$.

1. For every $h \in \text{Hom}(A, B)$, if $h \leq f, g$, then $h \leq f \wedge g$.
   That is: the meet in a restriction category is actually a greatest lower bound.
   In particular: the meet operation is commutative.

2. For every $f' \in \text{Hom}(A, B)$, if $f \leq f'$, then $f \wedge g \leq f' \wedge g$.
   From this and the previous point: for every $g' \in \text{Hom}(A, B)$, if $g \leq g'$, then $f \wedge g \leq f \wedge g'$.

\[\text{[Version of these notes earlier that July 16, 2014 incorrectly stated: “It follows from Lemma 4.8 and point 4 of Lemma 4.2 that in a discrete cartesian restriction category $\mathcal{X}$ the diagonals have partial inverses $\Delta_A^{-1} : A \times_R A \to A$, which are called equalities.”]}\]

\[\text{[The version from July 16, 2014 contained a wrong “proof” that $\Delta_A^{-1}$ is total. On careful examination, such proof relied on the equalities $\pi_0 = \pi_1 = \Delta_A^{-1}$: which are, in general, false.]}\]
Proof. Let us prove the points one by one:

1. If \( f \circ h = g \circ h = h \), then \( (f \land g) \circ h = (f \circ h) \land (g \circ h) = h \land h = h \).

In particular, as \( f \land g \leq f, g \), we get \( f \land g \leq f \land f \); similarly, \( g \land f \leq f \land g \), and from antisymmetry we get equality.

2. If \( f \leq f' \) then \( f \land g \leq f' \land g \) too, thus \( f \land g \leq f' \land g \) by point 1.

3. By point 2, \( (f \land g) \circ h = (f \circ h) \land (g \circ h) \leq (f \circ h) \land g \). On the other hand, if \( k : A \to D \) is such that \( k \leq f \circ h, g \), then
\[
\begin{align*}
g \circ h \circ k & = g \circ (k \circ h) \\
& = k \circ h \\
& = ((f \circ h) \circ k) \circ h \\
& = f \circ h \circ k \\
& = (f \circ h) \circ k,
\end{align*}
\]
so that \( k \leq g \circ h \) too, and \( k \leq (f \circ h) \land (g \circ h) = (f \land g) \circ h \) by point 1.

Therefore, \( (f \land g) \circ h = (f \circ h) \land g \); this holds for every suitable \( f, g \), and \( h \), whence \( (f \land g) \circ h = (g \land f) \circ h = (g \circ h) \land f = f \land (g \circ h) \).

4. Suppose \( f \sim g \). Then \( g \circ f \leq g \) and \( g \circ f = f \circ g \leq f \), thus \( g \circ f \leq f \land g \).

But if \( h \leq f \), then \( g \circ f \circ h = g \circ f \circ h = g \circ h = h \), i.e., \( h \leq g \circ f \); in particular, \( f \land g \leq g \circ f \).

5. By exploiting point 3 and \( f \circ f \land g = g \circ f \land g = f \land g \), we find
\[
\begin{align*}
\overline{f} \circ (f \land g) & = (f \land g) \circ k \circ (f \land g) \\
& = (f \land g) \circ k \circ (f \land g) \circ \overline{f} \circ (f \land g) \\
& = (f \land g) \circ k \circ g \circ f \land g \circ k \circ f \land g \\
& = (f \land g) \circ k \circ g \circ f \land g \circ k \circ g \circ k \circ f \land g \\
& = (f \land g) \circ k \circ g \circ k \circ f \land g \\
& = (f \circ k \circ f) \land (g \circ k \circ g) \\
& = (\overline{k} \circ f) \land (\overline{k} \circ g).
\end{align*}
\]

\[\square\]

**Theorem 4.14.** A cartesian restriction category is discrete if and only if it has meets.

\[\text{4 Thanks to Prof. Cockett for pointing out this chain of equalities.}\]
Proof. Let $X$ be a cartesian restriction category. If $X$ is discrete, for any $f, g \in \text{Hom}_X(A, B)$ set
\[
f \land g = \Delta_B^{(-1)} \circ (f, g)_R .
\]
Then:

1. By uniqueness, $(f, f)_R = \Delta_B \circ f$ as the diagram
\[
\begin{array}{ccc}
A & \xleftarrow{J} & A \\
\downarrow f & & \downarrow f \\
B & \xleftarrow{\pi_0 \times_R} & B \times_R B \\
\downarrow \Delta_B & & \downarrow \pi_1 \\
B & \xleftarrow{\pi_0} & B \\
\end{array}
\]
clearly commutes: whence, $\Delta_B$ being total,
\[
f \land f = \Delta_B^{(-1)} \circ (f, f)_R = \Delta_B^{(-1)} \circ \Delta_B \circ f = \Delta_B \circ f = f .
\]
2. We have\(^5\)
\[
f \circ f \land g = f \circ \Delta_B^{(-1)} \circ (f, g)_R \\
= f \circ \Delta_B^{(-1)} \circ (f, g \circ \bar{g})_R \\
= f \circ \Delta_B^{(-1)} \circ (f, g)_R \circ \bar{g} \\
= f \circ \Delta_B^{(-1)} \circ (f, g)_R \circ g \\
= f \circ \bar{g} \circ \Delta_B^{(-1)} \circ (f, g)_R \\
= \pi_0 \circ (f, g)_R \circ \Delta_B^{(-1)} \circ (f, g)_R \\
= \pi_0 \circ \Delta_B^{(-1)} \circ (f, g)_R \\
= \pi_0 \circ \Delta_B \circ \Delta_B^{(-1)} \circ (f, g)_R \\
= \pi_0 \circ \Delta_B \circ (f, g)_R \\
= \text{id}_B \circ (f \land g) \\
= f \land g ;
\]
that is, $f \land g \leq f$. Similarly, $f \land g \leq g$.

3. In our context, $(f \land g) \circ h = (f \circ h) \land (g \circ h)$ means
\[
\Delta_B^{(-1)} \circ (f, g)_R \circ h = \Delta_B^{(-1)} \circ (f \circ h, g \circ h)_R :
\]
which follows from point 3 of Lemma 4.2

If $X$ has meets, define
\[
\Delta_A^{(-1)} = \pi_0 \land \pi_1 .
\]

\(^5\)The proof in the versions before July 17, 2014 contained an error.
(Recall that $\Delta_A \in \text{Hom}_X(A, A \times R)$, so that $\pi_0, \pi_1 \in \text{Hom}_X(A \times R, A).$) In fact,

$$
\begin{align*}
(\pi_0 \land \pi_1) \circ \Delta_A &= (\pi_0 \circ \Delta_A) \land (\pi_1 \circ \Delta_A) = \text{id}_A \land \text{id}_A = \overline{\Delta_A}
\end{align*}
$$

as $\Delta_A$ is total, while

$$
\begin{align*}
\Delta_A \circ (\pi_0 \land \pi_1) &= (\text{id}_A, \text{id}_A)_R \circ (\pi_0 \land \pi_1) \\
&= (\text{id}_0, \text{id}_1)_R \\
&= (\pi_0 \circ \overline{\pi_0} \land \pi_1 \circ \overline{\pi_0} \land \overline{\pi_1})_R \\
&= (\pi_0, \pi_1)_R \circ \overline{\pi_0} \land \overline{\pi_1} \\
&= \overline{\pi_0} \land \overline{\pi_1}.
\end{align*}
$$

# Turing categories

**Definition 5.1.** A Turing category is a cartesian restriction category $X$ that has a Turing object $T$ with an associate Turing structure, i.e., a collections of (partial) maps $\bullet_{X,Y}: T \times_R X \to Y$ satisfying the following property: for every (partial) map $f: A \times_R X \to Y$ there exists a total map $\lambda^f: A \to T$ such that

$$
\bullet_{X,Y} \circ (\lambda^f \times_R \text{id}_X) = f, \text{ i.e.,}
$$

$$
\begin{array}{c}
\xymatrix{T \times_R X \ar[rr]^\bullet_{X,Y} \ar[d]_{\lambda^f \times_R \text{id}_X} & & Y \\
A \times_R X \ar[ur]^f & & \\
\end{array}
$$

(19)

Observe that $T$ behaves as a weak exponential object for every pair of objects. Recall that, in a cartesian category (with standard products) an exponential object for $X$ and $Y$ is an object $Y^X$ together with a map $\text{eval}: Y^X \times X \to Y$, called evaluation, such that for every object $A$ and map $f: A \times R X \to Y$ there exists a unique morphism $\lambda f: A \to Y^X$ such that $\text{eval} \circ (\lambda f \times_R \text{id}_X) = f$. Then $T$ is behaving like $Y^X$, $\bullet_{X,Y}$ like eval, and $\lambda^f$ like $\lambda f$, with two important differences:

1. The single object $T$ takes the role of $Y^X$ whatever the objects $X$ and $Y$ are.

2. The map $\lambda^f$ is required not to be unique, but to be total.

In a Turing category there may be more than one Turing object, and any given Turing object may have more than one Turing structure.

**Example 5.2** (The degenerate Turing category). The category with a single object and a single morphism is a Turing category.

**Example 5.3** (Kleene’s first model). Given a Gödel enumeration of the Turing machines, consider the partial map $\bullet: \mathbb{N} \times_R \mathbb{N} \to \mathbb{N}$ given by $\bullet(n, m) = \{n\} m$, the result of the computation of the $n$th Turing machine over the input $m$. In
In this category, the objects are the powers of \( \mathbb{N} \)—i.e., the arities—and the Turing object is \( T = \mathbb{N} \), while the Turing structure is defined similarly to \( \bullet = \bullet_{\mathbb{N},\mathbb{N}} \) described above. This is a partial model, because application is defined as computation by Turing machines, and cannot be anything but partial.

**Example 5.4** (\( \lambda \)-calculus with \( \beta \)-equality). Consider the category \( \mathcal{X} \) whose objects are the natural numbers, 0 is the final object, a map \( n \to n \) has the form \( (x_1, \ldots, x_n) \mapsto (t_1, \ldots, t_m) \), where the \( t_i \)'s are \( \lambda \)-terms in the \( x_j \)'s, and composition is defined by substitution, up to \( \beta \)-equality.

This is a total model, because application between \( \lambda \)-terms (or combinators) is total, even modulo \( \beta \)-reduction. Evaluation, on the other hand, corresponds to reduction to normal form: which is only partial.

The following statement is immediate.

**Proposition 5.5.** The product of two Turing categories is a Turing category.

**Definition 5.6** (Point in a restriction category). Let \( \mathcal{X} \) be a restriction category with a restriction final object \( 1_R \) and let \( A \) be an object in \( \mathcal{X} \). A point on \( A \) is a total map \( p : 1_R \to A \).

Definition 5.6 mimics the standard definition of point in categories with a final object, with the key difference that we require \( 1_R \) to be a restriction final object.

**Proposition 5.7.** Let \( \mathcal{X} \) be a Turing category with Turing object \( T \) and restriction final object \( 1_R \). For every \( f : A \to B \) there exists a point \( \tilde{f} \) on \( T \) such that the following diagram commutes:

\[
\begin{array}{ccc}
T \times_R A & \xrightarrow{\lambda_{A,B}^{-1}} & B \\
\downarrow{f \times_R \text{id}_A} & & \\
1_R \times_R A & \xrightarrow{(1_A, \text{id}_A)_R} & A
\end{array}
\]

**Proof.** Just set \( \tilde{f} = \lambda^*(f \circ \pi_1) \). Then the upper triangle commutes, while the lower one does by construction.

Recall that an object \( A \) is a retract of an object \( B \) if there exist an epic map \( r : B \to \text{id}_A \) (the retraction) and a monic map \( s : \text{id}_A \to B \) (the section) such that \( r \circ s = \text{id}_A \). An object \( U \) is universal if every object is a retract of \( U \).

**Proposition 5.8.** Let \( \mathcal{X} \) be a Turing category.

1. The Turing object is universal.
2. Every universal object is a Turing object.

**Proof.** In the definition of Turing category, set \( X = 1_R \) and \( Y = A \). Then the following diagram commutes:

\[
\begin{array}{ccc}
T \times_R 1_R & \xrightarrow{\lambda_{1_R,A}^{-1}} & A \\
\downarrow{\lambda \times_R \text{id}_1} & & \\
A \times_R 1_R & \xleftarrow{(\text{id}_A, 1_A)_R} & A
\end{array}
\]
Setting \( r_A = \bullet_{1_R, A} \circ (\text{id}_T, !T)_R \) and \( s_A = \lambda^* \pi_0 \) proves point \(^1\).

Let now \( U \) be an arbitrary universal object. Then \( T \) is a retract of \( U \): let \( s : T \to U \) monic and \( r : U \to T \) epic satisfy \( r \circ s = \text{id}_T \). Let \( X \) and \( Y \) be arbitrary objects: for every object \( A \) and map \( f : A \times_R X \to Y \) there exists a total map \( \lambda^* f : A \to T \) such that the following diagram commutes:

\[
\begin{array}{c}
U \times_R X \xrightarrow{r \times_R \text{id}_X} T \times_R X \xrightarrow{\lambda^* f \times_R \text{id}_X} A \times_R X \\
\downarrow{s \times_R \text{id}_X} \quad \downarrow{\lambda^* f \times_R \text{id}_X} \\
T \times_R X \xleftarrow{A \times_R X}
\end{array}
\]

where \( \bullet = \bullet^T \) is the Turing structure relative to \( T \). Then a Turing structure \( \bullet^U \) relative to \( U \) can be defined as \( \bullet^U_{X, Y} = \bullet^T_{X, Y} \circ (r \times_R \text{id}_X) \): the total map from \( A \) to \( U \) corresponding to \( f : A \times_R X \to Y \) will be \( \lambda^U f = s \circ \lambda^* f \). \( \square \)

**Corollary 5.9.** In a Turing category with Turing object \( T \), an object \( A \) is a Turing object if and only if \( T \) is a retract of \( A \).

**Corollary 5.10.** In Kleene’s first model, every infinite recursively enumerable set is a Turing object.

**Theorem 5.11** (First characterization theorem). A cartesian restriction category \( \mathbb{X} \) is a Turing category if and only if it has a universal object \( T \) and a universal application \( \bullet : T \times_R T \to T \) (also called the Turing map) with the following weak exponential property: for every map \( f : A \times_R T \to T \) there exists a total map \( \lambda^f : A \to T \) such that \( \bullet \circ (\lambda^f \times_R \text{id}_T) = f \), i.e., the following diagram commutes:

\[
\begin{array}{c}
T \times_R T \xrightarrow{\bullet} T \\
\downarrow{\lambda^f \times_R \text{id}_T} \quad \downarrow{f} \\
A \times_R T
\end{array}
\] \hspace{1cm}\text{(21)}

Shortly: in a cartesian restriction category, being able to perform evaluation over arbitrary objects, is the same as being able to perform evaluation over a single universal object.

**Proof.** Let \( X \) and \( Y \) be arbitrary objects. We want to construct a map \( \bullet_{X, Y} : T \times_R X \to Y \) such that, for every object \( A \) and map \( f : A \times_R X \to Y \), there exists a total map \( \lambda^f : A \to T \) such that the following diagram commutes:

\[
\begin{array}{c}
T \times_R X \xrightarrow{\bullet_{X, Y}} Y \\
\downarrow{\lambda^f \times_R \text{id}_X} \quad \downarrow{f} \\
A \times_R X
\end{array}
\]

As \( T \) is universal, \( X \) and \( Y \) are retracts of \( T \): let \( s_X : X \to T, s_Y : Y \to T \) monic and \( r_X : T \to X, r_Y : T \to Y \) epic satisfy \( r_X \circ s_X = \text{id}_X \) and \( r_Y \circ s_Y = \text{id}_Y \). Set
\[ g = s_Y \circ f \circ (\text{id}_A \times_R r_X) \]. Then the following diagram commutes:

\[ \begin{array}{c}
\xymatrix{
T \times_R \lambda T \times_R s X \ar[r]^\bullet & T \ar[r]^{r_Y} & Y \\
A \times_R \lambda T \times_R s X \ar[r]^f \ar[u]_{\lambda T g \times_R \text{id} X} & A \times_R T \ar[r]^{\text{id}_A \times_R s_X} & A \times_R X \ar[u]_{\lambda T g \times_R \text{id} T}
} \end{array} \]

But \( r_X \circ s_X = \text{id}_X \), so \((\text{id}_A \times_R r_X) \circ (\text{id}_A \times_R s_X) = \text{id}_A \times_R \text{id}_X = \text{id}_A \times_R X \), and the diagram above can be shrunk into

\[ \begin{array}{c}
\xymatrix{
T \times_R \lambda T \times_R s X \ar[r]^\bullet & T \ar[r]^{r_Y} & Y \\
A \times_R \lambda T \times_R s X \ar[r]^f \ar[u]_{\lambda T g \times_R \text{id} X} & A \times_R T \ar[r]^{\text{id}_A \times_R s_X} & A \times_R X \ar[u]_{\lambda T g \times_R \text{id} T}
} \end{array} \]

We then put

\[ \bullet_{X,Y} = r_Y \circ \bullet \circ (\text{id}_T \times_R s_X), \quad (22) \]

which correctly depends only on \( X \) and \( Y \) and not on \( A \), and

\[ \lambda^* f = \lambda^T g = \lambda^T (s_Y \circ f \circ (\text{id}_A \times_R r_X)) : (23) \]

then

\[ \bullet_{X,Y} \circ (\lambda^* f \times_R \text{id}_X) = r_Y \circ \bullet \circ (\text{id}_T \times_R s_X) \circ (\lambda^T g \times_R \text{id} X) = r_Y \circ s_Y \circ f = \text{id}_Y \circ f = f. \]

An object which is the retract of its own self-power is called a powerful object: universal objects, in particular Turing objects, are powerful.

**Definition 5.12 (Reduction of maps).** Let \( e = \overrightarrow{e} : A \to A \) and \( e' = \overrightarrow{e'} : A' \to A' \). We say that \( e \) reduces to \( e' \) if there exists a total map \( f : A \to A' \) such that \( \overrightarrow{e} \circ f = e \). We say that \( e \) is complete if every \( \overrightarrow{e'} \) reduces to \( e \).

**Definition 5.13 (Halting).** Let \( X \) be a restriction category with Turing object \( T \). The halting set for \( X \) is defined as

\[ \Delta \bullet = \bullet_{T,T} \circ \Delta_T. \quad (24) \]

The restriction \( h = \Delta \bullet \) of the halting set is called the halting predicate.

**Lemma 5.14.** In every Turing category the halting predicate is complete.

**Proof.** For simplicity, we write \( \Delta \) for \( \Delta_T \), and \( \bullet \) for \( \bullet_{T,T} \). Let \( e = \overrightarrow{e} : A \to A \); let \( s = s_A : A \to T \) monic and \( r = r_A : T \to A \) epic such that \( r_A \circ s_A = \text{id}_A \). Set \( p = s_A \circ \pi \circ \pi_1 \) and \( k = \lambda^* p \). Then the following diagram commutes:

\[ \begin{array}{c}
\xymatrix{
T \ar[r]^\Delta & T \times_R T \ar[r]^\bullet & T \\
A \ar[r]^-{(\text{id}_A, s_A)} & A \times_R T \ar[r]^-{s_A \circ \pi_1} & A
} \end{array} \]
Consequently, as \( s_A \) is monic,
\[
\Delta \bullet \circ k = \Delta \bullet \circ k
\]
\[
= s_A \circ \pi_1 \circ (\text{id}_A, s_A)_R
\]
\[
= \pi_1 \circ (\text{id}_A, s_A)_R
\]
\[
= \pi_1 \circ s_A
\]
\[
= \pi.
\]

Definition 5.15 (Partial combinator algebra). Let \( X \) be a cartesian restriction category with restriction final object \( 1_R \). A partial combinator algebra on \( X \) is a quadruple \((A, \bullet, k, s)\) where \( A \in |X| \), \( \bullet \in \text{Hom}_X(A \times R A, A) \), and \( k, s \in \text{Hom}_X(1_R, A) \) are points over \( A \) satisfying the following conditions:

1. The following diagram commutes:
\[
\begin{array}{c}
1_R \times_R A \times R A \\
\langle k, \text{id}_A \times_R A \rangle_R \\
A \times_R A \times R A \times R A
\end{array}
\begin{array}{c}
\pi_1 \\
\circ \\
\circ \pi_1 \circ \text{id}_R A
\end{array}
\begin{array}{c}
A \\
\bullet \\
\bullet \times_R \text{id}_A \\
\times_R \text{id}_A \\
\times_R \text{id}_A
\end{array}
\]

2. The following diagram commutes:
\[
\begin{array}{c}
1_R \times_R A \times R A \times R A \\
\langle \langle \pi_1, \pi_3 \rangle_R, \langle \pi_2, \pi_3 \rangle_R \rangle_R \\
A \times_R A \times R A \times R A \times R A
\end{array}
\begin{array}{c}
\pi_1 \\
\circ \\
\circ \pi_1 \circ \text{id}_R A
\end{array}
\begin{array}{c}
A \times_R A \times R A \times R A \\
\bullet \\
\bullet \circ (s, \text{id}_A)_R
\end{array}
\begin{array}{c}
A \\
\circ \\
\circ (a, b, c, d)
\end{array}
\]

where \((a, b, c, d) \rightarrow ((a \bullet b) \bullet c) \bullet d.\)

3. \( \bullet \circ (s, \text{id}_A)_R \) is total.

Condition 1 in Definition 5.15 can be translated by saying that \( kxy = x \) for every \( x \) and \( y \); similarly, condition 2 can be expressed by saying that \( sxyz = xz(yz) \) for every \( x, y \), and \( z \). The points \( k \) and \( s \) thus behave as the combinators \( K \) and \( S \) of Curry’s combinatory logic.

Observe that every Turing category is a partial combinator algebra.

Theorem 5.16 (Second characterization theorem). Every Turing category is the computable functions for some partial combinator algebra.

The “is” in Theorem 5.16 means that one might need to split some idempotents.
References


