Introduction to Symbolic Dynamics

Part 2: Shifts of finite type

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April 14, 2010

Revised: April 14, 2010

Overview

- Shifts of finite type.
- Graphs and their shifts.
- Graphs as representations of shifts of finite type.
- Shifts of finite type and data storage.

Shifts of finite type

Definition

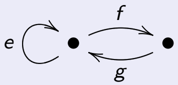
Let X be a subshift over A. There is a collection $\mathcal{F}$ of blocks over A s.t.

$$X = \mathsf{X}_{\mathcal{F}} = \{x \in A^{\mathbb{Z}} \mid x_{[i,j]} \neq u \, \forall i, j \in \mathbb{Z}, u \in \mathcal{F}\}$$

X is a shift of finite type (SFT) if $\mathcal{F}$ can be chosen finite.

Examples

Shifts of finite type

- The full shift.
- The golden mean shift.
- The set of labelings of bi-infinite paths on the graph 

- The (d, k)-run length limited shift.

A shift not of finite type

The even shift.

Memory

Definition

A SFT $X = \mathsf{X}_{\mathcal{F}}$ has memory M, or is a M-step SFT, if $\mathcal{F}$ can be chosen so that $|u| = M + 1 \, \forall u \in \mathcal{F}$.

Meaning

A SFT X has memory M when a machine with a memory size of M characters can decide whether $w \in A^{>M}$ belongs to $\mathcal{B}(X)$.

Examples

- 0-step SFT are full shifts (on smaller alphabets).
- 1-step SFT are Markov chains (minus probabilities).
- The (d, k)-run length limited shift has memory $M = k + 1$.

Characterization of memory for SFT

Theorem

Let X be a subshift over A. TFAE.

1. X is a SFT with memory M.
2. For every $w \in A^{\geq M}$, if $uw, wv \in \mathcal{B}(X)$, then $uwv \in \mathcal{B}(X)$.

Corollary: the charge constrained shift is not a SFT

- Let $A = \{+1, -1\}$.
- Define $x \in X$ iff $\sum_{i=j}^{j+p} x_i \in [-c, c]$ for every $j \in \mathbb{Z}$, $p \geq 0$.
- Fix $M \geq 0$.
- Take $w \in A^*$ s.t. $|w| \geq M$ and $\sum_{i=1}^{|w|} w_i = c - 1$.
- Then $1w, w1 \in \mathcal{B}(X)$ but $1w1 \notin \mathcal{B}(X)$.

Proof

If X is a M-step SFT

- Suppose $|w| \geq M$, $uw, wv \in \mathcal{B}(X)$.
- Let $x, y \in X$ s.t. $x_{[1,|w|]} = y_{[1,|w|]} = w, x_{[1-|u|,0]} = u, y_{|w|+1,|w|+|v|} = v$.
- Then $z = x_{(-\infty,0]}wy_{[|w|+1,\infty)} = x_{(-\infty,-|u|]}uwvy_{[|w|+|v|+1,\infty)} \in X$.

If X satisfies property 2

- Let $\mathcal{F} = A^{M+1} \setminus \mathcal{B}_{M+1}(X)$.
- Then clearly $X \subseteq \mathsf{X}_{\mathcal{F}}$.
- But if $x \in \mathsf{X}_{\mathcal{F}}$, then $x_{[0,M]}$ and $x_{[1,M+1]}$ are in $\mathcal{B}(X)$, so that $x_{[0,M+1]} \in \mathcal{B}(X)\ldots$
- $\ldots$and iterating the procedure, $x_{[i,j]} \in \mathcal{B}(X)$ for every $i \leq j$.

Finiteness of type is a shift invariant

Theorem

Let X be a SFT over A, Y a subshift over $\mathfrak{A}$.
Suppose there exists a conjugacy $\phi : X \to Y$.
Then Y is a SFT.

Reason why

- Suppose X is M-step.
- Suppose ϕ and ϕ^{-1} have memory and anticipation r.
- Then Y is $(M+4r)$-step.

Graphs

Definition

A graph G is made of:

1. A finite set $\mathcal{V}$ of vertices or states.
2. A finite set $\mathcal{E}$ of edges.
3. Two maps $\mathrm{i}, \mathrm{t} : \mathcal{E} \to \mathcal{V}$, where $\mathrm{i}(e)$ is the initial state and $\mathrm{t}(e)$ is the terminal state of edge e.

Graph homomorphisms

A graph homomorphism is made of two maps $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$, $\partial\Phi : \mathcal{V}_1 \to \mathcal{V}_2$ s.t.

$$\mathrm{i}(\Phi(e)) = \partial\Phi(\mathrm{i}(e)) \text{ and } \mathrm{t}(\Phi(e)) = \partial\Phi(\mathrm{t}(e)) \ \ \forall e \in \mathcal{E}_1 \,.$$

An embedding has Φ and $\partial\Phi$ injective.
An isomorphism has Φ and $\partial\Phi$ bijective.

Graphs and matrices

Adjacency matrix of a graph

Given an enumeration $\mathcal{V} = \{v_1, \dots, v_r\}$, the adjacency matrix of G is defined by

$$(A(G))_{I,J} = |\{e \in \mathcal{E} \mid i(e) = v_I, t(e) = v_J\}|$$

Graph of a nonnegative matrix

Given a $r \times r$ matrix A with nonnegative entries, the graph of A is defined by:

- $\mathcal{V}(G(A)) = \{v_1, \dots, v_r\}$
- $\mathcal{E}(G(A))$ has exactly $A_{I,J}$ elements s.t. $\mathrm{i}(e) = v_I$ and $\mathrm{t}(e) = v_J$.

Almost inverses

$A(G(A)) = A$ and $G(A(G)) \cong G$.

Edge shifts

Theorem

Let G be a graph and A its adjacency matrix. Then the edge shift

$$\mathsf{X}_G = \mathsf{X}_A = \{\xi : \mathbb{Z} \to \mathcal{E} \mid \mathrm{t}(\xi_i) = \mathrm{i}(\xi_{i+1}) \, \forall i \in \mathbb{Z}\}$$

is a 1-step SFT.

Essential graphs

Definition

A vertex is stranded if it has no incoming, or no outgoing, edges.
A graph is essential if it has no stranded vertices.

Theorem

For every graph G there exists exactly one essential subgraph H s.t. $\mathsf{X}_H = \mathsf{X}_G$.

Reason why

H is the maximal essential subgraph of G.

How to construct the maximal essential subgraph

Start with a graph G.

1. Remove all the vertices that are stranded.
2. Remove all the edges that have a loose end.
3. If no vertices have been remove at point 1: terminate.
4. Else: resume from point 1.

The resulting graph H is the maximal essential subgraph of G.

Not all SFT are edge shifts!

If the golden mean shift was an edge shift...

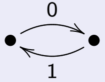
- ...then we could choose an essential graph G s.t X_G is the golden mean shift.
- This graph would have two edges, labeled 0 and 1.

But what are the essential graphs with two edges?

- One is

which is the graph of the full shift.

- The other one is



which is the graph of $\{\ldots 010101\ldots, \ldots 101010\ldots\}$.

Paths

Definition

- A path on a graph G is a finite sequence $\pi = \pi_1 \dots \pi_m$ on $\mathcal{E}$ s.t. $\mathrm{t}(\pi_i) = \mathrm{i}(\pi_{i+1})$ for every $i < m$.
- A path π is a cycle if $\mathrm{t}(\pi_m) = \mathrm{i}(\pi_1)$.
- A path π is simple if the $\mathrm{i}(\pi_i)$'s are all distinct.

The paths on G are precisely the blocks in $\mathcal{B}(\mathsf{X}_G)$.

Facts

Let G be a graph, A its adjacency matrix.

- The number of paths of length m from I to J is $(A^m)_{I,J}$.
- The number of cycles of length m is $\mathrm{tr}(A^m)$.

Irreducible graphs

Definition

A graph is irreducible if any two nodes I, J there is a path $\pi = \pi_1 \ldots \pi_m$ s.t. $I = \mathrm{i}(\pi_1)$ and $J = \mathrm{t}(\pi_m)$.

Equivalently

Let A be the adjacency matrix of G.
Then G is irreducible iff for every I and J there exists m s.t. $(A^m)_{I,J} > 0$.

Irreducible graphs and subshifts

Theorem

Let G be a graph.

1. If G is irreducible then X_G is irreducible.
2. If X_G is irreducible and G is essential then G is irreducible.

Reason why

If G is irreducible:

- Take $u, v \in \mathcal{B}(X)$.
- Make w that links $\mathrm{t}(u_{|u|})$ to $\mathrm{i}(v_1)$.

If X_G is irreducible and G is essential:

- Suppose $I = \mathrm{t}(e)$ and $J = \mathrm{i}(f)$.
- If $e\pi f \in \mathcal{B}(\mathrm{X}_G)$ then π links I to J.

Presenting SFT as edge shifts

Theorem

- Suppose X is a M-step SFT.
- Then $X^{[M+1]}$ is an edge shift.

Proof

Consider the de Bruijn graph of order M on X:

- $\mathcal{V}(G) = \mathcal{B}_M(X)$.
- $\mathcal{E}(G) = \mathcal{B}_{M+1}(X)$ with $\mathrm{i}(e) = e_{[1,M]}$ and $\mathrm{t}(e) = e_{[2,M+1]}$.

Then $\mathsf{X}_G = X^{[M+1]}$.

Higher edge graphs

Definition

Given G, define $G^{[N]}$ as follows:

- $\mathcal{V}(G^{[N]})$ is the set of paths of length $N-1$ in G.
- $\mathcal{E}(G^{[N]})$ is the set of paths of length N in G.
- For an edge $\pi = \pi_1 \ldots \pi_N$, $\mathrm{i}(\pi) = \pi_{[1,N-1]}$ and $\mathrm{t}(\pi) = \pi_{[2,N]}$.

Theorem

For every graph G, $\mathrm{X}_{G^{[N]}} = \mathrm{X}_G^{[N]}$

Vertex shifts

Definition

Suppose B is a $r \times r$ boolean matrix.

- Put $\mathcal{F} = \left\{ IJ \in \{0, \ldots, r-1\}^2 \mid B_{I,J} = 0 \right\}$.
- Then $\widehat{X}_B = X_{\mathcal{F}}$ is called the vertex shift of B.

Example

The golden mean shift is a vertex shift, with

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Points of view

Theorem

1. There is a bijection between 1-step SFT and vertex shifts.
2. There is an embedding of edge shifts into vertex shifts.
3. For every M-step SFT X there exists a graph G s.t. $X^{[M]} = \widehat{\mathsf{X}}_G$ and $X^{[M+1]} = \mathsf{X}_G$.

...then why not always use vertex shifts?

- Growth in the number of states.
- Better properties of integer matrices.

Powers of a graph

Definition

Let G be a graph. Define G^N as follows:

- A vertex in G^N is a vertex in G.
- An edge from I to J in G^N is a path of length N from I to J in G.

Facts

Let G be a graph and let A be its adjacency matrix.

- Then A^N is the adjacency matrix of G^N.
- Furthermore, if $X = \mathrm{X}_G$ then $X^N = \mathrm{X}_{G^N}$.

An application to data storage

In an ideal world

- Our data is encoded in a sequence of bits.
- The device reads and writes the data *verbatim*.
- N bits require N memory allocation units.

The main issue

The world we live in, is not ideal.

Hard disk drives 101

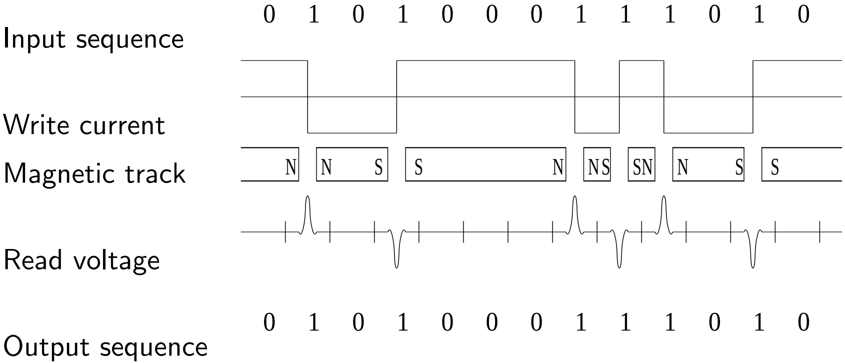
The physics

- The unit contains several rotating platters coated in a magnetic medium, and a head moving radially across the platters' tracks.
- An electrical current through the head magnetizes a portion of the track. This creates a bar magnet on the track.
- Reversing the current creates a bar with the opposite orientation.
- A polarity change generates a voltage pulse.

The logic

- Tracks are divided into cells of equal length L.
- A 0 is written by keeping the current. A 1 is written by reversing the current.
- A pulse is read as a 1. A non-pulse is read as a 0.

The scheme


0 1 0 1 0 0 0 1 1 1 0 1 0
Input sequence
Write current
Magnetic track
N
N S
S N
N S
S N
N S
S
Read voltage
0 1 0 1 0 0 0 1 1 1 0 1 0
Output sequence

Two main problems with the naïve approach

Intersymbol interference

- If polarity changes are too close, the pulses are weaker.
- There must be a "minimum safe distance" Δ between changes.
- An encoding scheme where two 1's are separated by at least d 0's allows cells of size $L = \Delta/(d+1)$.

Clock drift

- A block of the form 10^n1 is read as two pulses separated by a time interval of length $L \cdot (n+1)$.
- If the clock is not precise, then the value for n is wrong.
- This can be corrected via a feedback loop for each pulse.
- If pulses are not "too rare", then errors won't accumulate.
- A typical requirement is: no more than k 0's between two 1's.

Frequency modulation (FM)

Idea

- Store the data adding a clock 1 between each pair of data bits.
- Recover the original message by ignoring the clock bits.

Advantages

- Stored data is a $(0, 1)$-run length limited block.
- n bits can be stored on a strip of length $2n\Delta$.

Modified frequency modulation (MFM)

Ideas

- If there are at least d 0's between two 1's, the detection window can be shrunk to $L = \Delta/(d+1)$.
- Some of the 1's in the data can be used for synchronization.

The technique

Use clock bits as follows:

- Between two 0's, insert a 1.
- Otherwise, insert a 0.

Then the stored sequence is a $(1, 3)$-run length limited block.
Consequently, n bits can be stored in a strip of length $n\Delta$.