Introduction to Symbolic Dynamics

Part 2: Shifts of finite type

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Overview

- Shifts of finite type.
- Graphs and their shifts.
- Graphs as representations of shifts of finite type.
- Shifts of finite type and data storage.
Shifts of finite type

**Definition**

Let $X$ be a subshift over $A$. There is a collection $\mathcal{F}$ of blocks over $A$ s.t.

$$X = X_\mathcal{F} = \{x \in A^\mathbb{Z} | x_{[i,j]} \neq u \forall i, j \in \mathbb{Z}, u \in \mathcal{F}\}$$

$X$ is a **shift of finite type (SFT)** if $\mathcal{F}$ can be chosen finite.
Examples

Shifts of finite type

- The full shift.
- The golden mean shift.
- The set of labelings of bi-infinite paths on the graph

The $(d, k)$-run length limited shift.

A shift not of finite type

The even shift.
Memory

Definition

A sft $X = X_{\mathcal{F}}$ has memory $M$, or is a $M$-step sft, if $\mathcal{F}$ can be chosen so that $|u| = M + 1 \forall u \in \mathcal{F}$.

Meaning

A sft $X$ has memory $M$ when a machine with a memory size of $M$ characters can decide whether $w \in A^{>M}$ belongs to $B(X)$.

Examples

- 0-step sft are full shifts (on smaller alphabets).
- 1-step sft are Markov chains (minus probabilities).
- The $(d, k)$-run length limited shift has memory $M = k + 1$. 
Characterization of memory for SFT

**Theorem**

Let $X$ be a subshift over $A$. TFAE.

1. $X$ is a SFT with memory $M$.
2. For every $w \in A^{\geq M}$, if $uw, wv \in \mathcal{B}(X)$, then $uvw \in \mathcal{B}(X)$.

**Corollary: the charge constrained shift is not a SFT**

- Let $A = \{+1, -1\}$.
- Define $x \in X$ iff $\sum_{i=-j}^{j+p} x_i \in [-c, c]$ for every $j \in \mathbb{Z}$, $p \geq 0$.
- Fix $M \geq 0$.
- Take $w \in A^*$ s.t. $|w| \geq M$ and $\sum_{i=1}^{|w|} w_i = c - 1$.
- Then $1w, w1 \in \mathcal{B}(X)$ but $1w1 \notin \mathcal{B}(X)$.
Proof

If $X$ is a $M$-step SFT

- Suppose $|w| \geq M$, $uw, wv \in B(X)$.
- Let $x, y \in X$ s.t. $x[1,|w|] = y[1,|w|] = w$, $x[1-|u|,0] = u$, $y[|w|+1,|w|+|v|] = v$.
- Then $z = x(-\infty,0]wy[|w|+1,\infty) = x(-\infty,-|u|]uwvy[|w|+|v|+1,\infty) \in X$.

If $X$ satisfies property 2

- Let $\mathcal{F} = A^{M+1} \setminus B_{M+1}(X)$.
- Then clearly $X \subseteq X_{\mathcal{F}}$.
- But if $x \in X_{\mathcal{F}}$, then $x[0,M]$ and $x[1,M+1]$ are in $B(X)$, so that $x[0,M+1] \in B(X)$.
- ... and iterating the procedure, $x[i,j] \in B(X)$ for every $i \leq j$. 
Finiteness of type is a shift invariant

**Theorem**
Let $X$ be a SFT over $A$, $Y$ a subshift over $A$. Suppose there exists a conjugacy $\phi : X \rightarrow Y$. Then $Y$ is a SFT.

**Reason why**
- Suppose $X$ is $M$-step.
- Suppose $\phi$ and $\phi^{-1}$ have memory and anticipation $r$.
- Then $Y$ is $(M + 4r)$-step.
Graphs

Definition

A graph $G$ is made of:

1. A finite set $\mathcal{V}$ of vertices or states.
2. A finite set $\mathcal{E}$ of edges.
3. Two maps $i, t : \mathcal{E} \rightarrow \mathcal{V}$, where $i(e)$ is the initial state and $t(e)$ is the terminal state of edge $e$.

Graph homomorphisms

A graph homomorphism is made of two maps $\Phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$, $\partial \Phi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ s.t.

\[
i(\Phi(e)) = \partial \Phi(i(e)) \quad \text{and} \quad t(\Phi(e)) = \partial \Phi(t(e)) \quad \forall e \in \mathcal{E}_1.
\]

An embedding has $\Phi$ and $\partial \Phi$ injective.

An isomorphism has $\Phi$ and $\partial \Phi$ bijective.
Graphs and matrices

Adjacency matrix of a graph

Given an enumeration $\mathcal{V} = \{v_1, \ldots, v_r\}$, the adjacency matrix of $G$ is defined by

$$(A(G))_{i,j} = |\{e \in \mathcal{E} \mid i(e) = v_i, t(e) = v_j\}|$$

Graph of a nonnegative matrix

Given a $r \times r$ matrix $A$ with nonnegative entries, the graph of $A$ is defined by:

- $\mathcal{V}(G(A)) = \{v_1, \ldots, v_r\}$
- $\mathcal{E}(G(A))$ has exactly $A_{i,j}$ elements s.t. $i(e) = v_i$ and $t(e) = v_j$.

Almost inverses

$A(G(A)) = A$ and $G(A(G)) \cong G$. 
Edge shifts

**Theorem**

Let $G$ be a graph and $A$ its adjacency matrix. Then the edge shift

$$X_G = X_A = \{ \xi : \mathbb{Z} \rightarrow \mathcal{E} \ | \ t(\xi_i) = i(\xi_{i+1}) \ \forall i \in \mathbb{Z} \}$$

is a 1-step SFT.
Essential graphs

**Definition**
A vertex is **stranded** if it has no incoming, or no outgoing, edges.
A graph is **essential** if it has no stranded vertices.

**Theorem**
For every graph $G$ there exists exactly one essential subgraph $H$ s.t. $X_H = X_G$.

**Reason why**
$H$ is the maximal essential subgraph of $G$. 
How to construct the maximal essential subgraph

Start with a graph $G$.

1. Remove all the vertices that are stranded.
2. Remove all the edges that have a loose end.
3. If no vertices have been remove at point 1: terminate.
4. Else: resume from point 1.

The resulting graph $H$ is the maximal essential subgraph of $G$. 
Not all SFT are edge shifts!

If the golden mean shift was an edge shift...

- ...then we could choose an essential graph $G$ s.t. $X_G$ is the golden mean shift.
- This graph would have two edges, labeled 0 and 1.

But what are the essential graphs with two edges?

- One is

$$
\begin{array}{c}
0 \\
\rightarrow \\
\rightarrow \\
\leftarrow \\
\leftarrow \\
1
\end{array}
$$

which is the graph of the full shift.

- The other one is

$$
\begin{array}{c}
0 \\
\rightarrow \\
\rightarrow \\
\leftarrow \\
\rightarrow \\
\leftarrow \\
1
\end{array}
$$

which is the graph of \{...010101..., ...101010...\}. 

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April 14, 2010 14 / 28
Paths

Definition

- A path on a graph $G$ is a finite sequence $\pi = \pi_1 \ldots \pi_m$ on $\mathcal{E}$ s.t. $t(\pi_i) = i(\pi_{i+1})$ for every $i < m$.
- A path $\pi$ is a cycle if $t(\pi_m) = i(\pi_1)$.
- A path $\pi$ is simple if the $i(\pi_i)$’s are all distinct.

The paths on $G$ are precisely the blocks in $\mathcal{B}(X_G)$.

Facts

Let $G$ be a graph, $A$ its adjacency matrix.

- The number of paths of length $m$ from $I$ to $J$ is $(A^m)_{I,J}$.
- The number of cycles of length $m$ is $\text{tr}(A^m)$.
Irreducible graphs

**Definition**

A graph is **irreducible** if any two nodes $I, J$ there is a path $\pi = \pi_1 \ldots \pi_m$ s.t. $I = i(\pi_1)$ and $J = t(\pi_m)$.

**Equivalently**

Let $A$ be the adjacency matrix of $G$.
Then $G$ is irreducible iff for every $I$ and $J$ there exists $m$ s.t. $(A^m)_{I,J} > 0$. 
Irreducible graphs and subshifts

Theorem

Let $G$ be a graph.

1. If $G$ is irreducible then $X_G$ is irreducible.
2. If $X_G$ is irreducible and $G$ is essential then $G$ is irreducible.

Reason why

If $G$ is irreducible:
- Take $u, v \in B(X)$.
- Make $w$ that links $t(u | u |)$ to $i(v_1)$.

If $X_G$ is irreducible and $G$ is essential:
- Suppose $I = t(e)$ and $J = i(f)$.
- If $e \pi f \in B(X_G)$ then $\pi$ links $I$ to $J$. 
Presenting SFT as edge shifts

Theorem

- Suppose $X$ is a $M$-step SFT.
- Then $X^{[M+1]}$ is an edge shift.

Proof

Consider the de Bruijn graph of order $M$ on $X$:

- $V(G) = B_M(X)$.
- $E(G) = B_{M+1}(X)$ with $i(e) = e_{[1,M]}$ and $t(e) = e_{[2,M+1]}$.

Then $X_G = X^{[M+1]}$. 
Higher edge graphs

Definition

Given $G$, define $G^{[N]}$ as follows:

- $\mathcal{V}(G^{[N]})$ is the set of paths of length $N - 1$ in $G$.
- $\mathcal{E}(G^{[N]})$ is the set of paths of length $N$ in $G$.
- For an edge $\pi = \pi_1 \ldots \pi_N$, $i(\pi) = \pi_{[1,N-1]}$ and $t(\pi) = \pi_{[2,N]}$.

Theorem

For every graph $G$, $X_{G^{[N]}} = X_{G}^{[N]}$
Vertex shifts

Definition
Suppose $B$ is a $r \times r$ boolean matrix.
- Put $\mathcal{F} = \{IJ \in \{0, \ldots, r-1\}^2 \mid B_{I,J} = 0\}$.
- Then $\hat{X}_B = X_\mathcal{F}$ is called the vertex shift of $B$.

Example
The golden mean shift is a vertex shift, with

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
Points of view

Theorem

1. There is a bijection between 1-step sft and vertex shifts.
2. There is an embedding of edge shifts into vertex shifts.
3. For every $M$-step sft $X$ there exists a graph $G$ s.t. $X^{[M]} = \hat{X}_G$ and $X^{[M+1]} = X_G$.

...then why not always use vertex shifts?

- Growth in the number of states.
- Better properties of integer matrices.
Powers of a graph

**Definition**

Let $G$ be a graph. Define $G^N$ as follows:

- A vertex in $G^N$ is a vertex in $G$.
- An edge from $I$ to $J$ in $G^N$ is a path of length $N$ from $I$ to $J$ in $G$.

**Facts**

Let $G$ be a graph and let $A$ be its adjacency matrix.

- Then $A^N$ is the adjacency matrix of $G^N$.
- Furthermore, if $X = X_G$ then $X^N = X_{G^N}$.
An application to data storage

In an ideal world
- Our data is encoded in a sequence of bits.
- The device reads and writes the data \textit{verbatim}.
- \( N \) bits require \( N \) memory allocation units.

The main issue
The world we live in, is not ideal.
Hard disk drives 101

The physics

- The unit contains several rotating platters coated in a magnetic medium, and a head moving radially across the platters’ tracks.
- An electrical current through the head magnetizes a portion of the track. This creates a bar magnet on the track.
- Reversing the current creates a bar with the opposite orientation.
- A polarity change generates a voltage pulse.

The logic

- Tracks are divided into cells of equal length $L$.
- A 0 is written by keeping the current. A 1 is written by reversing the current.
- A pulse is read as a 1. A non-pulse is read as a 0.
The scheme

Input sequence

Write current

Magnetic track

Read voltage

Output sequence

0 1 0 1 0 0 0 1 1 1 1 0 1 0

N N S S N NS SN N S S

0 1 0 1 0 0 0 1 1 1 1 0 1 0
Two main problems with the naïve approach

**Intersymbol interference**
- If polarity changes are too close, the pulses are weaker.
- There must be a “minimum safe distance” $\Delta$ between changes.
- An encoding scheme where two 1’s are separated by at least $d$ 0’s allows cells of size $L = \Delta/(d + 1)$.

**Clock drift**
- A block of the form $10^n1$ is read as two pulses separated by a time interval of length $L \cdot (n + 1)$.
- If the clock is not precise, then the value for $n$ is wrong.
- This can be corrected via a feedback loop for each pulse.
- If pulses are not “too rare”, then errors won’t accumulate.
- A typical requirement is: no more than $k$ 0’s between two 1’s.
Frequency modulation (FM)

Idea
- Store the data adding a clock 1 between each pair of data bits.
- Recover the original message by ignoring the clock bits.

Advantages
- Stored data is a \((0, 1)\)-run length limited block.
- \(n\) bits can be stored on a strip of length \(2n\Delta\).
Modified frequency modulation (MFM)

Ideas

- If there are at least $d$ 0’s between two 1’s, the detection window can be shrunk to $L = \Delta/(d + 1)$.
- Some of the 1’s in the data can be used for synchronization.

The technique

Use clock bits as follows:

- Between two 0’s, insert a 1.
- Otherwise, insert a 0.

Then the stored sequence is a $(1, 3)$-run length limited block. Consequently, $n$ bits can be stored in a strip of length $n\Delta$. 