# Introduction to Symbolic Dynamics

## Part 3: Sofic shifts

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# Overview

- State splitting.
- Sofic shifts.
- Characterization of sofic shifts.
- Minimal right-resolving presentations.

# Graphs

## Definition

A graph $G$ is made of:

1. A finite set $\mathcal{V}$ of vertices or states.
2. A finite set $\mathcal{E}$ of edges.
3. Two maps $\mathrm{i}, \mathrm{t} : \mathcal{E} \to \mathcal{V}$, where $\mathrm{i}(e)$ is the initial state and $\mathrm{t}(e)$ is the terminal state of edge $e$.

## Adjacency matrix of a graph

Given an enumeration $\mathcal{V} = \{v_1, \ldots, v_r\}$, the adjacency matrix of $G$ is defined by

$$(A(G))_{I,J} = |\{e \in \mathcal{E} \mid i(e) = v_I, t(e) = v_J\}|$$

# Graph shifts

## Edge shifts

Let $G$ be a graph and $A$ its adjacency matrix. Then the edge shift

$$\mathsf{X}_G = \mathsf{X}_A = \{\xi : \mathbb{Z} \to \mathcal{E} \mid \mathrm{t}(\xi_i) = \mathrm{i}(\xi_{i+1})\ \forall i \in \mathbb{Z}\}$$

is a 1-step SFT.

## Vertex shifts

Suppose $B$ is a $r \times r$ boolean matrix.

- Put $\mathcal{F} = \left\{ IJ \in \{0, \ldots, r-1\}^2 \mid B_{I,J} = 0 \right\}$.
- Then $\widehat{\mathsf{X}}_B = \mathsf{X}_{\mathcal{F}}$ is called the vertex shift of $B$.

# State splitting

## The aim

Given a graph $G$, obtain a new graph $H$.

## Procedure

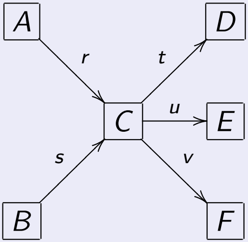
- Start with an “original” graph $G$.
- Partition the edges.
- Split each “original” state into one or more “derived” states, according to the partition of the edges.
- End with a “derived” graph $H$

## The main question

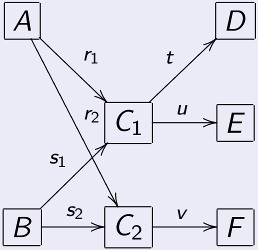
What are the properties of $H$ and $X_H$?

# Example

## Before




## After



# Out-splitting

## The basic idea

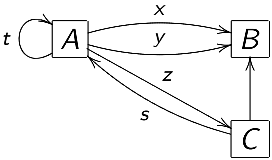
- Let $G = (\mathcal{V}, \mathcal{E})$ be a graph and let $I \in \mathcal{V}$.
- Let $\mathcal{E}_I = \{e \in \mathcal{E} \mid \mathrm{i}(e) = I\}$, $\mathcal{E}^I = \{e \in \mathcal{E} \mid \mathrm{t}(e) = I\}$.
- Partition $\mathcal{E}_I = \mathcal{E}_I^1 \sqcup \mathcal{E}_I^2 \sqcup \ldots \sqcup \mathcal{E}_I^m$.
- Put $\mathcal{V}(H) = (\mathcal{V}(G) \setminus \{I\}) \sqcup \{I^1, I^2, \ldots, I^M\}$.
- Construct $\mathcal{E}(H)$ from $\mathcal{E}$ as follows:
  - Replace every $e \in \mathcal{E}^I$ with $e^1, \ldots, e^m$ s.t. $\mathrm{i}(e^k) = \mathrm{i}(e)$ and $\mathrm{t}(e^k) = I^k$.
  - Make each $f \in \mathcal{E}_I^k$ start from $I^k$ instead of $I$.

## Out-splittings

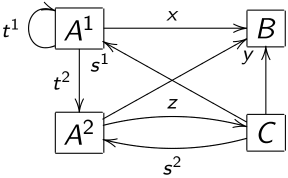
- Apply the same idea at all nodes.
- Let $\mathcal{P}$ be the partition of $\mathcal{E}$ used.
- Then $H = G^{[\mathcal{P}]}$ is an out-splitting of $G$, and $G$ is an out-amalgamation of $H$.

# Example

Consider the graph




Split $\mathcal{E}_A = \{t, x\} \cup \{y, z\}$. The resulting out-splitting is



# Out-splittings and edge shifts

## There...

Define $\Psi : \mathcal{B}_1(\mathsf{X}_H) \to \mathcal{B}_1(\mathsf{X}_G)$ as

$$\Psi(e) = \begin{cases} f & \text{if } e = f^k \,, \\ e & \text{otherwise} \,. \end{cases}$$

## ...and back again

Define $\Phi : \mathcal{B}_2(\mathsf{X}_G) \to \mathcal{B}_1(\mathsf{X}_H)$ as

$$\Phi(fe) = \begin{cases} f^k & \text{if } f \in \mathcal{E}^l \text{ and } e \in \mathcal{E}^k_l \,, \\ f & \text{otherwise} \,. \end{cases}$$

## Theorem

The SBC $\psi = \Psi_\infty^{[0,0]}$ and $\phi = \Phi_\infty^{[0,1]}$ are each other's converse.

# In-splitting

## The dual idea

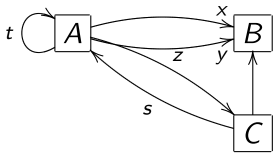
- Let $G = (\mathcal{V}, \mathcal{E})$ be a graph and let $I \in \mathcal{V}$.
- Let $\mathcal{E}^I = \{e \in \mathcal{E} \mid \mathrm{i}(e) = I\}$, $\mathcal{E}^I = \{e \in \mathcal{E} \mid \mathrm{t}(e) = I\}$.
- Partition $\mathcal{E}^I = \mathcal{E}_1^I \sqcup \ldots \sqcup \mathcal{E}_m^I$.
- Put $\mathcal{V}(H) = (\mathcal{V}(G) \setminus \{I\}) \sqcup \{I_1, \ldots, I_m\}$.
- Construct $\mathcal{E}(H)$ from $\mathcal{E}$ as follows:
  - Replace every $e \in \mathcal{E}_I$ with $e_1, \ldots, e_m$ s.t. $\mathrm{i}(e_k) = \mathrm{i}(e)$ and $\mathrm{t}(e_k) = I^k$.
  - Make each $f \in \mathcal{E}_k^I$ start from $I_k$ instead of $I$.

## In-splittings

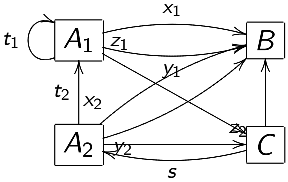
- Apply the same idea at all nodes.
- Let $\mathcal{P}$ be the partition of $\mathcal{E}$ used.
- Then $H = G_{[\mathcal{P}]}$ is an in-splitting of $G$, and $G$ is an in-amalgamation of $H$.

# Example

Consider the graph




Split $\mathcal{E}^A = \{t\} \cup \{s\}$. The resulting in-splitting is



# Conjugating it all...

## Splittings and Subshifts

- Let $G$ and $H$ be graphs.
- Suppose $H$ is a splitting of $G$.
- Then $\mathsf{X}_G$ and $\mathsf{X}_H$ are conjugate.

More in general, if

$$G = G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} G_n = H$$

and each $f_i$ is either a splitting or an amalgamation, then $\mathsf{X}_G \cong \mathsf{X}_H$.

## Advanced Splittings and Subshifts

Every conjugacy between edge shifts is a composition of splittings and amalgamations.

# Splittings and matrices

Suppose $G$ has $n$ nodes and $H = G^{[\mathcal{P}]}$ has $m$.

### The division matrix

It is the $n \times m$ boolean matrix $D$ with

$$D(I, J^k) = 1 \text{ iff } J \text{ results from the splitting of } I.$$

### The edge matrix

It is the $m \times n$ integer matrix $E$ where

$$E(I^k, J) = |\mathcal{E}_I^k \cap \mathcal{E}^J|$$

### Theorem

$DE = A(G)$ and $ED = A(H)$.

# Labeled graphs

## Definition

Let $G$ be a graph, $A$ an alphabet.

- An $A$-labeling of $G$ is a map $\mathcal{L} : \mathcal{E}(G) \to A$.
- A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$ where $G$ is a graph and $\mathcal{L}$ an $A$-labeling of $G$ (for some $A$).

If $\mathcal{P}$ is a property of graphs and $\mathcal{G} = (G, \mathcal{L})$ is a labeled graph, then $\mathcal{G}$ has property $\mathcal{P}$ if $G$ has property $\mathcal{P}$

## Labeled graph homomorphism

Let $\mathcal{G} = (G, \mathcal{L}_G)$ and $\mathcal{H} = (H, \mathcal{L}_H)$ be $A$-labeled graphs.

- A labeled graph homomorphism from $\mathcal{G}$ to $\mathcal{H}$ is a graph homomorphism $(\partial\Phi, \Phi)$ from $G$ to $H$ s.t. $\mathcal{L}_H(\Phi(e))) = \mathcal{L}_G(e)$ for every $e \in \mathcal{E}(G)$.
- A labeled graph isomorphism is a bijective labeled graph homomorphism.

# Sofic shifts

## Path labelings

Let $\mathcal{G} = (G, \mathcal{L})$ be an $A$-labeled graph.

- The labeling of a path $\pi = e_1 \ldots e_m$ on $G$ is the sequence $\mathcal{L}(\pi) = \mathcal{L}(e_1) \ldots \mathcal{L}(e_m)$.
- The labeling of a bi-infinite path $\xi \in \mathcal{E}(G)^{\mathbb{Z}}$ is the sequence $x = \mathcal{L}(\xi) \in A^{\mathbb{Z}}$ s.t. $x_i = \mathcal{L}(\xi_i)$ for every $i \in \mathbb{Z}$.
- We put

$$\mathsf{X}_{\mathcal{G}} = \left\{ x \in A^{\mathbb{Z}} \mid \exists \xi \in \mathsf{X}_G \mid x = \mathcal{L}(\xi) \right\}$$

## Definition

- $X \subseteq A^{\mathbb{Z}}$ is a sofic shift if $X = \mathsf{X}_{\mathcal{G}}$ for some $A$-labeled graph $\mathcal{G}$.
- In this case, $\mathcal{G}$ is a presentation of $X$.

# Basic facts on sofic shifts

## Sofic shifts are shift spaces

$\mathcal{L}$ provides a 1-block code $\mathcal{L}_\infty$ from $\mathsf{X}_G$ to $A^{\mathbb{Z}}$, and $\mathsf{X}_{\mathcal{G}} = \mathcal{L}_\infty(\mathsf{X}_G)$.

## Shifts of finite type are sofic

- Suppose $X$ has memory $M$.
- Construct the de Bruijn graph $G$ of order $M$. Then $\mathsf{X}_G = X^{[M+1]}$.
- Define $\mathcal{L} : \mathcal{E}(G) \to A$ by $\mathcal{L}([a_1, \ldots, a_{M+1}]) = a_1$.
- Then $\mathcal{G} = (G, \mathcal{L})$ is a presentation of $X$.

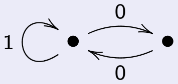
## $\mathsf{X}_{\mathcal{G}}$ is a SFT iff some $\mathcal{L}_\infty$ is a conjugacy

$\Rightarrow$ The labeling of the de Bruijn graph induces a conjugacy.

$\Leftarrow$ A conjugate of a SFT is a SFT.

# Counterexamples

## A sofic shift which is not a SFT

The even shift is presented by 


## A shift subspace which is not sofic

If the context free shift was sofic...

- Let $\mathcal{G} = (G, \mathcal{L})$. Suppose $G$ has $s$ states.
- Then any path on $G$ representing $ab^{s+1}c^{s+1}$ has a loop between the first and the last $b$.
- Let $l > 0$ be the length of the loop. Then $ab^{l+s+1}c^{s+1}a$ is a valid labeling for a path. . .

# Characterization of sofic shifts, I

**Theorem**

Let $X$ be a subshift. TFAE.

1. $X$ is a sofic shift.
2. $X$ is a factor of a SFT.

**Consequences**

- A factor of a sofic shift is sofic.
- A shift conjugate to a sofic shift is sofic.

# Proof

## Sofic shifts are factors of SFT

$\mathsf{X}_{\mathcal{G}}$ is a factor of $\mathsf{X}_G$ through $\mathcal{L}_\infty$.

## Factors of SFT are sofic

- Suppose $X = \Phi_\infty^{[-m,n]}(Y)$ for a SFT $Y$.
- Suppose $Y$ has memory $m+n$. (Can always do by increasing $m$.)
- Let $G$ be the de Bruijn graph of $Y$ of order $m+n$. Then $Y \cong \mathsf{X}_G$.
- Define $\mathcal{L} : E(G) \to A$ by $\mathcal{L}(e) = \Phi(e)$. Then

$$\begin{array}{ccc} Y & \xrightarrow{\beta_{m+n+1} \circ \sigma^{-m}} & Y^{[m+n+1]} \\ {\scriptstyle \Phi_\infty} \downarrow & \swarrow {\scriptstyle \mathcal{L}_\infty} & \\ X & & \end{array}$$

# Follower sets

**Definition**

Let $X$ be a subshift over $A$.

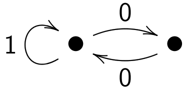
- For $w \in \mathcal{B}(X)$ let $F_X(w) = \{u \in \mathcal{B}(X) \mid wu \in \mathcal{B}(X)\}$.
- $F_X$ is called the follower set of $w$ in $X$.
- Put $C_X = \{F_X(w) \mid w \in \mathcal{B}(X)\}$.

**Examples**

- If $X = A^{\mathbb{Z}}$ then $C_X = \{A^*\}$.
- If $X = \mathrm{X}_G$ and $G$ is essential then $C_X$ has $|\mathcal{E}(G)|$ elements.
- If $X$ is the context free shift then the $F_X(ab^m)$'s are pairwise different.

# A more detailed example

Consider the even shift presented by




Then $C_X = \{C_0, C_1, C_2\}$ with

$$\begin{array}{lllll} C_0 & = & F_X(0) & = & 0^*((00)^*1)^*0^* \\ C_1 & = & F_X(1) & = & 0^* \cup ((00)^*1)^*0^* \\ C_2 & = & F_X(10) & = & 0((00)^*1)^*0^* \end{array}$$

In fact,

$$F_X(w) = \begin{cases} C_0 & \text{if } w \in 0^*, \\ C_1 & \text{if } w \in \mathcal{B}(X)1(00)^*, \\ C_2 & \text{if } w \in \mathcal{B}(X)10(00)^* \end{cases}$$

# The follower set graph

## Construction

Suppose $C_X$ is finite.

- Set $\mathcal{V}(G) = C_X$.
- For every $w \in \mathcal{B}(X)$ and $a \in A$ s.t. $wa \in \mathcal{B}(X)$, draw an edge from $F_X(w)$ to $F_X(wa)$. labeled with $a$.

The resulting labeled graph $\mathcal{G} = (G, \mathcal{L})$ is the follower set graph $\mathcal{G}_X$ of $X$.

## Why the construction works

If $F_X(w) = F_X(v) = U$ then $wa \in \mathcal{B}(X)$ iff $va \in \mathcal{B}(X)$.

- In fact, this is the same as saying that $a \in U$.

Moreover, in this case $F_X(wa) = F_X(va)$.

- In fact, $u \in F_X(wa)$ iff $au \in F_X(w)$...

# Characterization of sofic shifts, II

## Theorem

- Let $X$ be a subshift s.t. $C_X$ is finite.
- Then the follower set graph is a presentation of $X$.
- In particular, $X$ is sofic.

## Proof

- Let $\mathcal{G}$ be the follower set graph of $X$.
- If path $\pi$ with label $u$ starts from node $F_X(w)$, then $wu \in \mathcal{B}(X)$, and $u \in \mathcal{B}(X)$ as well.
- Suppose then $u \in \mathcal{B}(X)$. Take $w$ s.t. $wu \in \mathcal{B}(X)$ and $|w| > |\mathcal{V}(X)|$.
- Then $wu$ is the labeling of a path $\alpha\beta\gamma\pi$ where $\pi$ is labeled by $u$ and $\beta$ is a loop.
- Then there exists a left-infinite path terminating with $\pi$, which can be extended to a bi-infinite path.

# Characterization of sofic shifts, II (cont.)

**Theorem**

A sofic shift has finitely many follower sets.

**Constructing follower sets from labeled graphs**

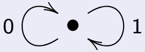
- Consider $w \in \mathcal{B}(X)$, where $X = \mathsf{X}_{\mathcal{G}}$ is a sofic shift.
- There are finitely many labeled paths on $\mathcal{G}$ that present $w$.
- Then, there are also finitely many states where a path presenting $w$ can terminate.
- But words with the same set of final states must have same followers.
- Hence, there are at most as many follower sets as subsets of the set of states of $\mathcal{G}$.

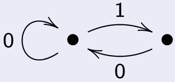
# Right-resolving presentations

## Right-resolving labelings

A labeled graph $\mathcal{G} = (G, \mathcal{E})$ is right-resolving if $\mathcal{L}$ is injective on each $\mathcal{E}_I$, *i.e.*, $\mathcal{L}$ puts different labels on different edges from same node.

## Examples

- The labeled graph 

is right-resolving.
- The labeled graph 

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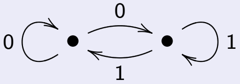
- The labeled graph 

is not right-resolving.

  (But presents the same shift as the first one.)

# The subset graph of a labeled graph

## Definition

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph.

- Let $\mathcal{V}(H)$ be the set of non-empty subsets of $\mathcal{V}(G)$.
- For $I \in \mathcal{V}(H)$ and $a \in A$, let

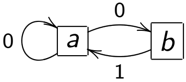
$$J = \{\mathrm{t}(e) \mid \mathrm{i}(e) \in I, \mathcal{L}(e) = a\}$$

- If $J$ is non-empty, set $e'$ from $I$ to $J$ in $\mathcal{E}(H)$, and put $\mathcal{L}'(e') = a$.

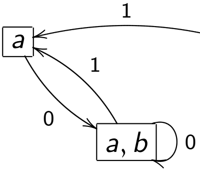
$\mathcal{H} = (H, \mathcal{L}')$ is the subset graph of $\mathcal{G}$. Observe that $\mathcal{H}$ is right-resolving.

# Example

Let $\mathcal{G}$ be the labeled graph




Then the subset graph of $\mathcal{G}$ is



# Right-resolving presentations

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph and let $\mathcal{H} = (H, \mathcal{L}')$ its subset graph.

**Theorem**

$\mathcal{H}$ is a presentation of $X_{\mathcal{G}}$.

**Reason why**

- Clearly $\mathcal{B}(X_{\mathcal{G}}) \subseteq \mathcal{B}(X_{\mathcal{H}})$.
- On the other hand, let $\pi$ be a path in $\mathcal{H}$ from $R$ to $S$ labeled $u$.
- By iterating the construction, we observe that $S$ is the set of vertices of $G$ reachable from vertices in $R$ via a path labeled $u$.
- Since $R$ is nonempty, $u \in \mathcal{B}(X_{\mathcal{G}})$.

# The merged graph of a labeled graph

## Follower sets of a labeled graph

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. The follower set of $I \in \mathcal{V}(G)$ is

$$F_{\mathcal{G}}(I) = \{\mathcal{L}(\pi) \mid \mathrm{i}(\pi) = I\}$$

$\mathcal{G}$ is follower-separated if $I \neq J$ implies $F_{\mathcal{G}}(I) \neq F_{\mathcal{G}}(J)$.

## The merged graph

Given $\mathcal{G}$, define $\mathcal{H} = (H, \mathcal{L}')$ as follows:

- A state in $H$ is a set of states in $G$ having the same follower set.
- There is an edge in $H$ from $I$ to $J$ labeled $a$ iff there is an edge in $G$ from a state in $I$ to a state in $J$ labeled $a$.

# The merged graph lemma

## Statement

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph and $\mathcal{H} = (H, \mathcal{L}')$ its merged graph.

1. $\mathcal{H}$ is follower-separated.
2. $\mathcal{H}$ is a presentation of $\mathsf{X}_{\mathcal{G}}$.
3. If $\mathcal{G}$ is irreducible then $\mathcal{H}$ is irreducible.
4. If $\mathcal{G}$ is right-resolving then $\mathcal{H}$ is right-resolving.

## Corollary

A minimal right-resolving presentation of a sofic shift is follower-separated.

# The merged graph of a right-resolving graph is right-resolving

- Let $\mathcal{G}$ be a right-resolving graph and let $\mathcal{H}$ be its merged graph.
- Let $\eta$ be an edge in $\mathcal{H}$ from $\mathcal{I}$ to $\mathcal{J}$ labeled $a$.
- Then there exists an edge $e$ in $\mathcal{G}$ from $I \in \mathcal{I}$ to $J \in \mathcal{J}$ labeled $a$.
- Since $\mathcal{G}$ is right-resolving, $e$ is unique, and $a$ and $F_{\mathcal{G}}(I)$ determine $F_{\mathcal{G}}(J)$.
- However, $F_{\mathcal{H}}(\mathcal{I}) = F_{\mathcal{G}}(I)$ and $F_{\mathcal{H}}(\mathcal{J}) = F_{\mathcal{G}}(J)$.
- This means that $a$ and $F_{\mathcal{H}}(\mathcal{I})$ determine $F_{\mathcal{H}}(\mathcal{J})$.

# Irreducible shifts and presentations

**If $\mathcal{G}$ is irreducible then $X_{\mathcal{G}}$ is irreducible**

- Let $u, v \in \mathcal{B}(X_{\mathcal{G}})$.
- Let $\xi$ and $\eta$ be paths on $\mathcal{G}$ labeled $u$ and $v$, respectively.
- Take the labeling $w$ of a path $\pi$ from $\mathrm{t}(\xi)$ to $\mathrm{i}(\eta)$.

**It does not work the other way around!**

- Let $\mathcal{H}$ be made of two disjoint copies of $\mathcal{G}$.
- Then $X_{\mathcal{H}} = X_{\mathcal{G}}$.

# Right-resolving presentations and reducibility

**Theorem**

- Let $X$ be an irreducible sofic shift.
- Let $\mathcal{G}$ be a minimal right-resolving presentation for $X$.
- Then $\mathcal{G}$ is irreducible.

**Corollary**

A sofic shift is irreducible iff it has an irreducible presentation.

# Proof of previous theorem

For every state $I$ there exists a word $u_I \in \mathcal{B}(X_{\mathcal{G}})$ s.t. every path presenting $u_I$ contains $I$.

- Suppose otherwise.
- Form $\mathcal{H}$ from $\mathcal{G}$ by removing $I$ and the adjacent edges.
- Then $\mathcal{H}$ is a presentation of $X$—against minimality of $\mathcal{G}$.

Let then $I$ and $J$ be any two states in $\mathcal{G}$.

- Since $X$ is irreducible, $u_i w u_J \in \mathcal{B}(X)$ for some $w$.
- Let $\pi$ be a path on $\mathcal{G}$ s.t. $\mathcal{L}(\pi) = u_I w u_J$.
- Then $\pi = \tau_I \omega \tau_J$ with $\omega$ being a path from $I$ to $J$.

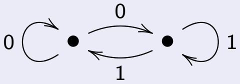
# Synchronizing words

## Definition

Let $\mathcal{G}$ be a labeled graph.

- A word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ is synchronizing if every path representing $w$ terminates in the same node.
- $w$ focuses on $I$ if every path representing $w$ terminates in $I$.

## Examples

- Every word in an edge shift is synchronizing.
- No word over 

is synchronizing.

# The importance of being right-resolving

## Theorem

Let $\mathcal{G}$ be a right-resolving labeled graph.

- If $w$ is synchronizing and $wu \in \mathcal{B}(\mathsf{X}_\mathcal{G})$ then $wu$ is synchronizing.
- Moreover, if $w$ focuses on $I$, then $F_{\mathsf{X}_\mathcal{G}}(w) = F_\mathcal{G}(I)$.
- If, in addition, $\mathcal{G}$ is follower-separated then every $u \in \mathcal{B}(\mathsf{X}_\mathcal{G})$ is a prefix of a synchronizing word.

## Reason why the third point holds

Put $T(w) = \{\mathrm{t}(\pi) \mid \mathcal{L}(\pi) = w\}$.

- Observe that $|T(w)| = 1$ iff $w$ is synchronizing.
- If $I, J \in T(u)$ then:
  - Find $v_I \in F_I(u) \setminus F_J(u)$.
  - There is at most one path labeled $v_I$ starting at each element of $T(u)$.
  - But $v_I \notin F_J(u)$ implies $|T(uv_I)| < |T(u)|$. Iterate...

# Fischer's theorem

## Statement of the theorem

- Let $X$ be an irreducible sofic shift.
- Let $\mathcal{G}$ and $\mathcal{H}$ be minimal right-resolving presentations of $X$.
- Then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic as labeled graphs.

## Corollary

- Let $X$ be an irreducible sofic shift.
- Let $\mathcal{G}$ be an irreducible right-resolving presentation of $X$.
- Then the merged graph of $\mathcal{G}$ is the minimal right-resolving presentation of $X$.

# Proof of Fischer's theorem

## Auxiliary lemma

If $X$ is a sofic shift and $\mathcal{G}$ and $\mathcal{H}$ are presentations of $X$ that are

- irreducible,
- right-resolving, and
- follower-separated

then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic as labeled graphs.

## Fischer's theorem follows then...

Let $\mathcal{G}$ and $\mathcal{H}$ be minimal right-resolving presentations for $X$.

- Being minimal, they are follower-separated.
- Since $X$ is irreducible and $\mathcal{G}$ and $\mathcal{H}$ are minimal right-resolving presentations of $X$, they are irreducible.
- By the lemma, they are isomorphic as labeled graphs.

# Proof of the auxiliary lemma

**$\mathcal{G}$ and $\mathcal{H}$ have a common synchronizing word**

- Let $u$ be any word.
- Since $\mathcal{G}$ is follower-separated, some $uv$ is synchronizing for $\mathcal{G}$.
- Since $\mathcal{H}$ is follower-separated, some $w = uvz$ is synchronizing for $\mathcal{H}$.

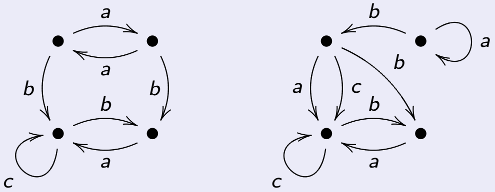
**Suppose $w$ focuses on $I \in \mathcal{V}(G)$ and $J \in \mathcal{V}(H)$**

- Put $\partial\Phi(I) = J$.
- Put $\partial\Phi(I') = J'$ if:
  - There is a word $u$ s.t. $wu$ focuses on $I'$ in $\mathcal{G}$.
  - The same word $wu$ focuses on $J'$ in $\mathcal{H}$.
- Let now $e$ be an edge from $I_1$ to $I_2$ in $\mathcal{G}$ labeled $a$.
  - If $wu$ focuses on $I_1$, then $wua$ focuses on $I_2$.
  - Put $J_k = \partial\phi(I_k)$. Then $wu$ focuses on $J_1$ and $wua$ focuses on $J_2$.
  - There is one edge $f$ from $J_1$ to $J_2$ labeled $a$. Set $\Phi(e) = f$.

# Fischer's theorem does not hold for reducible shifts

## Counterexample (Jonoska, 1996)

Let $\mathcal{G}$ and $\mathcal{H}$ be the following labeled graphs:




## $\mathcal{G}$ and $\mathcal{H}$ are not isomorphic

$\mathcal{H}$ has a self-loop labeled $a$, which $\mathcal{G}$ has not.

## $\mathcal{G}$ and $\mathcal{H}$ present the same sofic shift

Check that the language is the same.
Observe that such sofic shift $X$ is reducible.

# No right-resolving graph on three states can present $X$

## $X$ has at least three follower sets

- $aab \in F_X(aa) \setminus (F_X(c) \cup F_X(cb))$
- $c \in F_X(c) \setminus (F_X(aa) \cup F_X(cb))$
- $ac \in F_X(cb) \setminus (F_X(aa) \cup F_X(c))$

## If $\mathcal{K}$ has only three states. . .

- Then we could associate them so that:
  - $F_{\mathcal{K}}(1) \subseteq F_X(aa)$
  - $F_{\mathcal{K}}(2) \subseteq F_X(c)$
  - $F_{\mathcal{K}}(3) \subseteq F_X(cb)$
- But $F_X(aab) = F_X(c) \cup F_X(cb)$.
  - Then there must be two paths representing *aab*, one starting from 2 and one from 3. . .
- But $aab \notin (F_X(c) \cup F_X(cb))$
  - . . . so they must both end at 1.

# Soon on these screens. . .

- Constructions and algorithms with sofic shifts
- Entropy of a shift subspace
- Perron-Frobenius theory for non-negative matrices

Thank you for attention!