A depth-first implementation of Geometric Logic in Prolog

(extended overview)

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GL as a fragment of FOL

- Geometric formula: $C \Rightarrow D$
- $C = A_1 \land \ldots \land A_n$ (n>0, Ai atoms)
- $D = E_1 \lor \ldots \lor E_m$ (m>0)
- $E_j = (E x_1 \ldots x_k) C_j$ (k>0, Cj like C)
- Implicit universal closure
- No function symbols (yet), only constants
Examples

- Lattices (meet is associative, Horn clause):
  \[ x \cap y = u \land u \cap z = v \land y \cap z = w \Rightarrow x \cap w = u \]

- Projective unicity (resolution clause):
  \[ p | l \land p | m \land q | l \land q | m \Rightarrow p = q \lor l = m \]

- Diamond property (geometric clause):
  \[ a \rightarrow b \land a \rightarrow c \Rightarrow (E d) (b \rightarrow d \land c \rightarrow d) \]

- In general: \[ A_1 \land \ldots \land A_n \Rightarrow ((E x) A_{11} \land \ldots \land A_{1i}) \lor \ldots \lor ((E y) A_{k1} \land \ldots \land A_{kj}) \]
Rationale

- Horn clauses: DCG and Prolog
- Resolution: ATP
- Geometric logic: ATP and ?
  - Less skolemization
  - Direct proofs
  - Constructive logic
  - Natural proof theory/objects
Inductive definition of \( X \vdash_{(T)} D \)

- (base) \( X |- D \) if \( X \downarrow D \)
- (step) \( X,C_1 |- D, \ldots, X,C_n |- D \)

\( X \) a finite set of facts (= closed atoms)
\( D \) closed geometric disjunction (parameters in \( D \) must occur in \( X \))
\( X \downarrow D \) iff \( D = \ldots \lor (E \, x) \, C \lor \ldots \) and \( X \) contains all facts in \( C[x:=a] \) for suitable parameters \( a \)

(\%) there exists a closed instance \( C_0 \Rightarrow D_0 \) of an axiom in \( T \) with
- \( C_0 \) included in \( X \) (\( X \) contains all facts in \( C_0 \)) and
- \( D_0 = \ldots \lor (E \, x) \, C_i \lor \ldots \) and each \( C_i \) a fresh instance of \( C_i \) (\( 1 \leq i \leq n \))
Examples of derivations

- \( T = \{ \text{true} \rightarrow p, \ p \rightarrow q \}, \ \emptyset \vdash q \)
- \( T = \{ p \lor q, \ p \rightarrow r, \ q \rightarrow r \}, \ \emptyset \vdash r \)
- \( T = \{ p, \ p \rightarrow q, \ q \rightarrow \text{false} \}, \ \emptyset \vdash r \)
- \( T = \{ (E \ x) p(x), \ p(x) \rightarrow q \}, \ \emptyset \vdash q \)
- \( T = \{ s(a,b), \ s(x,y) \rightarrow (E \ z) \ s(y,z) \}, \ \emptyset \vdash (E \ x \ y) (s(a,x) \land s(x,y)) \)
- Forward reasoning (cf. Prolog)!
Metaproperties

- Soundness
- Completeness
- Constructivity
- Conservativity
- Semidecidability
- Automation (SATCHMO!)
Samples of ATP

- exist.in
- or.in
- nijm.in
Case studies

• Confluence theory: induction steps in Newman’s Lemma, Hindley-Rosen, Self-lengthening Thm, ..

• Lattice theory: \( x \cap (y \cup z) \leq (x \cap y) \cup (x \cap z) \) for all \( x, y, z \) implies \( (x \cup y) \cap (x \cup z) \leq x \cup (y \cap z) \) for all \( x, y, z \)

• Projective geometry: equivalence of two versions of Pappus’ Axiom (1 minute, 1MB proof)
Semantics and completeness

- Geometric logic: no proof by contradiction
  \( (= EM, TND, A \lor \neg A, \neg \neg A \Rightarrow A) \)

- Digression: constructivism in mathematics
  - \( p \lor q \) stronger than \( \neg(\neg p \land \neg q) \)
  - \( (E x) p(x) \) stronger than \( \neg(A x) \neg p(x) \)
  - more strict on ontology of objects
  - EM only in specific cases, f.e., for integers
    \( (A x)(x=0 \lor x \neq 0) \), but not for reals
Example of non-constructivism

- Do there exist irrational real numbers $x$ and $y$ such that $x^y$ is rational?
- Greek constructivists: $\sqrt{2}$ is irrational
- Non-constructivist: take $x = y = \sqrt{2}$. If $x^y$ is rational, then I’m done. If $x^y$ is not rational, then I’m also done: $(x^y)^y = x^{y^y} = x^2 = 2$ is rational. Next problem, please.
- Constructivist: what do you mean?
Tarskian semantics

- Truth values from a complete Boolean algebra, without loss of generality ($\{0,1\}$, max, min, not($x$) = 1 - $x$), $[|p\lor q|]=\max([|p|][|q|])$ etc.
- Thus $p\lor q$ is true iff $p$ is true or $q$ is true (Girard: `what a discovery!`)
- Sound but not complete for constructive logic, not sound for some forms of constructive mathematics
- Constructive logic is more expressive ($\lor,E$) and requires a more refined semantics …
Semantics for constructivism (digr.)

- Algebraic: complete Heyting algebras (plural!)
- Topological: open sets as truth values
- Kripke semantics: tree-structured Tarski models (graph-structured for modal logic), creative subject
- Curry-Howard interpretation: $\left\{ |\varphi| \right\}$ is the set of proofs of $\varphi$
- Kleene, Beth, Joyal, …
- Different aspects, counter models, metatheory, …
Semantics for GL

- Tarskian (non-constructive completeness)
- Beth-Joyal-Coquand (fully constructive, extra information, but highly non-trivial)
- Curry-Howard (for proof objects)
- Other semantics unexplored …
Completeness wrt Tarskian models

• Given D true in all models of T, how do you find a proof? Try them all!
• Breadth-first derivability on the blackboard
• Herbrand models along the branches
• König’s Lemma to get the tree finite
• Finite tree => breadth-first proof => |- proof