Relating sequent calculi for bi-intuitionistic propositional logic

Luís Pinto, Universidade do Minho, Braga
Tarmo Uustalu, Institute of Cybernetics, Tallinn

CLaC 2010, Brno, 21–22 Aug 2010
Bi-intuitionistic propositional logic $\text{BiInt}$

(a.k.a. Heyting-Brouwer logic, subtractive logic)

- A symmetric (self-dual) subsystem of classical propositional logic, believed to be of computational significance
- Extends intuitionistic propositional logic with a connective dual to implication, called exclusion (pseudo-difference, subtraction)
- Conservative extension, excluded middle, Peirce not valid
- Does not enjoy the disjunction property
- Was first studied by C Rauszer; subsequent interest by T Crolard, G Restall and others
Sequent calculi for **BiInt**

- The natural sound and complete standard-style sequent calculus (à la Dragalin) is only complete with cut
- Cut-free alternatives:
  - nested sequents (R Goré et al),
  - labelled sequents (Pinto, Uustalu)
This talk

- We explain how the three sequent calculi relate to each other, by providing translations between them.
- We discuss a complete class of cuts for the standard-style sequent calculus.

To downplay bureaucracy, we gloss over
- whether weakening and contraction should be separate rules or built into other rules,
- whether axioms and two-premise rules should be context-sharing or context-splitting.
Syntax and semantics of **BiInt**

- **Formulas:**
  As in intuitionistic propositional logic, **Int**, plus exclusion formulas $A \prec B$ (“$A$ but not $B$”), dual to implication formulas $A \supset B$.
  (In addition to intuitionistic negation as a special case of implication, $\neg A = A \supset \bot$, we get dual-intuitionistic negation, $\sim A = \top \prec A$.)

- **Kripke semantics:**
  Kripke structures are $(W, \leq, I)$ as in the Kripke semantics of **Int** (with $\leq$ reflexive-transitive and $I$ monotone), truth of $A \prec B$ is defined dually to truth of $A \supset B$:
  - $w \models A \supset B$ iff, for all $w' \geq w$, $w' \not\models A$ or $w' \models B$
  - $w \models A \prec B$ iff, for some $w' \leq w$, $w' \models A$ and $w' \not\models B$

- Instead of general Kripke structures, it is fine to confine to finite trees only
Standard-style sequent calculus \textbf{LBil}

- A multiple-hypothesis, multiple-conclusion sequent calculus, naturally extending that by Dragalin for \textbf{Int}.
- Sequents are pairs $\Gamma \vdash \Delta$, where $\Gamma$, $\Delta$ are multisets of formulas.
- The rules for exclusion are dual to those for implication:

$$
\Gamma, A \supset B \vdash A, \Delta \quad \Gamma, B \vdash \Delta \\
\Gamma, A \supset B \vdash \Delta
$$

$$
\Gamma \vdash A \supset B \vdash \Delta \\
\Gamma \vdash A \supset B, \Delta
$$

(incorporating weakening and contraction)
- A sequent $\Gamma \vdash \Delta$ is valid if, for any Kripke structure and world $w$, if all formulas in $\Gamma$ are true at $w$, then so is some formula in $\Delta$. 
Necessity of cut

- **LBiI** is incomplete with cut.
- The sequent \( p \vdash q, r \supset ((p \prec q) \land r) \) is valid and derived, with cut, by

\[
\begin{align*}
\frac{\frac{\frac{\frac{p \vdash q, p}{p, q \vdash q}}{p \vdash q, p \prec q}}{p \vdash q, p \land r}}{p \vdash q, r \supset ((p \prec q) \land r)} & \quad \text{hyp}
\frac{\frac{\frac{\frac{\frac{p \prec q, r \vdash p \prec q}{p \prec q, r \vdash (p \prec q) \land r}}{p \prec q \vdash r \supset ((p \prec q) \land r)}}{p \vdash q, r \supset ((p \prec q) \land r)}}{p \vdash q, r \supset ((p \prec q) \land r)} & \quad \text{cut}
\end{align*}
\]

- The only possible last inference of a hypothetical cut-free derivation

\[
\begin{align*}
\vdots & \quad ? \\
\frac{p, r \vdash (p \prec q) \land r}{p \vdash q, r \supset ((p \prec q) \land r)} & \quad \supset R
\end{align*}
\]

has its premise invalid.
Necessity of cut (ctd)

Reduction of cut is impossible in these two cases:

(1) is an inference by an $R$ rule with $A$ as the main formula or by an $\leftarrow L$ rule and (2) is an inference by an $L$ rule with $A$ as the main formula or by an $\supset R$ rule.
Nested sequent calculus **N-LBiI** (Goré et al.)

- Sequents are pairs $\Gamma \vdash \Delta$ where $\Gamma, \Delta$ are contexts. A context is a multiset of formulas and (embedded) sequents.
- The rules are those of **LBiI**, plus

  \[
  \frac{\Gamma_0 \vdash \Delta_0, \Delta}{\Gamma, (\Gamma_0 \vdash \Delta_0) \vdash \Delta} \quad \text{nestL} \quad \frac{\Gamma, \Gamma_0 \vdash \Delta_0}{\Gamma \vdash (\Gamma_0 \vdash \Delta_0), \Delta} \quad \text{nestR}
  \]

  \[
  \frac{\Gamma, \Gamma_0, (\Gamma_0 \vdash \Delta_0) \vdash \Delta_0, \Delta}{\Gamma, \Gamma_0 \vdash \Delta_0, \Delta} \quad \text{unnestL} \quad \frac{\Gamma, \Gamma_0 \vdash (\Gamma_0 \vdash \Delta_0), \Delta_0, \Delta}{\Gamma, \Gamma_0 \vdash \Delta_0, \Delta} \quad \text{unnestR}
  \]

- Intuitively, a right-hand embedded $\vdash$ is an implication, a left-hand embedded $\vdash$ is an exclusion. So **nestL** and **nestR** are structural-level versions of the $\supset L$ and $\supset R$ rules.

**unnestL** and **unnestR** are structural-level elimination rules of $\supset$ and $\supset$, corresponding to combinations of $\supset R$ and $\supset L$ with cut.
Nested sequent calculus (ctd)

- Validity is easiest defined via a translation of nested into standard sequents based on the above intuition. E.g., \( A, (B, C \vdash D) \vdash E, (F \vdash G), (H \vdash) \) translates to \( A, B \land C \vdash D \vdash E, F \supset G, H \supset \bot \).

- Our example is derived without cut (but with \textit{unnestL}) by

\[
\begin{align*}
\pmb{p \vdash q, p} & \hfill \text{hyp} \hfill \pmb{p, q \vdash q} & \hfill \text{hyp} \\
\pmb{p \vdash q, p \vdash q} & \hfill \text{\textless{}R} \\
\hfill \pmb{(p \vdash q), r \vdash p \vdash q} & \hfill \text{nestL} \hfill \pmb{(p \vdash q), r \vdash r} & \hfill \text{hyp} \\
\pmb{(p \vdash q), r \vdash (p \vdash q) \land r} & \hfill \text{\land{}R} \\
\pmb{(p \vdash q) \vdash r \supset ((p \vdash q) \land r)} & \hfill \text{\supset{}R} \\
\pmb{p \vdash q, r \supset ((p \vdash q) \land r)} & \hfill \text{unnestL}
\end{align*}
\]
Labelled sequent calculus \textbf{L-LBiI}

- Sequents are triples $\Gamma \vdash_G \Delta$ where $G$ is a tree of labels.
  $\Gamma, \Delta$ are labelled contexts, i.e., multisets of pairs $x : A$ with $x$ a label and $A$ a formula.
- The rules are labelled versions of those of \textbf{LBiI}. The implication and exclusion rules take form

\[
\begin{align*}
\Gamma, x : A \supset B &\vdash_G x : A, \Delta & \Gamma, x : B &\vdash_G \Delta & \supset L \\
\Gamma, x : A \supset B &\vdash_G \Delta & \Gamma, y : A \vdash_G \oplus_{x}(x, y) y : B, \Delta &\vdash_R \\
\Gamma, y : A &\vdash_G \oplus_{x}(x, y) y : B, \Delta & \Gamma, x : A &\vdash_G \Delta & \Gamma, x : B &\vdash_G x : A \prec B, \Delta & \prec L \\
\Gamma, x : A &\vdash_G \Delta & \Gamma, x : B &\vdash_G x : A \prec B, \Delta & \prec R
\end{align*}
\]

In addition, there are the monotonicity rules

\[
\begin{align*}
\Gamma, x : A, y : A &\vdash_G \oplus_{x}(x, y) \oplus y \Delta & \Gamma, x : A, y : A &\vdash_G \Delta & \text{monotL} \\
\Gamma, x : A &\vdash_G \oplus_{x}(x, y) \oplus y \Delta & \Gamma, y : A &\vdash_G \oplus_{y}(y, x) \oplus x \Delta & \text{monotR}
\end{align*}
\]
Labelled sequent calculus $\textsf{L-LBiI}$ (ctd)

- A sequent $\Gamma \vdash_G \Delta$ is valid, if for any Kripke structure and any assignment of worlds to labels that maps adjacent labels to accessible worlds, if all labelled formulas of $\Gamma$ are true, so is some labelled formula of $\Delta$.

- Our example is proved without cut by

\[
\begin{align*}
  x : p, y : r & \vdash (x, y) \quad \text{hyp} \\
  x : p, y : r, x : q, x : p & \vdash (x, y) \quad \text{hyp} \\
  x : p, y : r & \vdash (x, y) \\
  x : q, x : p & \vdash q \quad \text{hyp} \\
  x : p, y : r & \vdash (x, y) \quad \text{monotR} \\
  x : p, y : r & \vdash (x, y) \\
  x : q, y : p & \vdash q \quad \text{hyp} \\
  x : p, y : r & \vdash (x, y) \quad \text{R} \\
  x : p & \vdash \langle x \rangle \\
  x : q, x : r & \vdash ((p \prec q) \land r) \quad \text{R}
\end{align*}
\]
Translations

- In the paper, we describe six translations:

\[
\begin{array}{c}
\text{LBiI} \\
\text{(1a)} \\
\downarrow \\
\text{N-LBiI} \\
\text{(1b)} \\
\uparrow \\
\text{L-LBiI} \\
\text{(2a)} \\
\text{(2b)} \\
\text{(3)}
\end{array}
\]

- Translation (1a) is trivial (as any sequent/derivation in \text{LBiI} is also one in \text{N-LBiI}), as long as we are happy that cuts in \text{LBiI} are rendered as cuts in \text{N-LBiI}.
- Translation (1b) is based on flattening of nested sequents to standard-style sequents. nest\(R\) inferences are mapped to \(\supset R\) inferences, unnest\(R\) inferences into a combination of \(\supset L\) with cut.
- Translations (3) are best introduced as compositions of translations (1) and (2).
Translations between **N-LBiI** and **L-LBiI**

- Translating between **N-LBiI** and **L-LBiI** requires that we can convert nested sequents into labelled sequents and vice versa.

This can be done based on the equivalences:

\[
\forall x (A(x) \land \exists y (y \to x \land A_0(y) \land \neg B_0(y)) \supset B(x))
\]

\[
\Leftrightarrow \forall x \forall y (y \to x \land (A(x) \land A_0(y) \supset B_0(y) \lor B(x)))
\]

\[
\forall x (A(x) \supset \forall y (x \to y \supset \neg A_0(y) \lor B_0(y)) \lor B(x))
\]

\[
\Leftrightarrow \forall x \forall y (x \to y \land (A(x) \land A_0(y) \supset B_0(y) \lor B(x)))
\]
Translations between **N-LBil** and **L-LBil**

- E.g., $A, (B, C \vdash D) \vdash E, (F \vdash G), (H \vdash)$ translates to

  $$x : A, y : B, y : C, z : F, w : H \vdash_{(y,x),(x,z),(x,w)} x : E, y : D, z : G,$$

  which translates back to

  $$A, (B, C \vdash D) \vdash E, (F \vdash G), (H \vdash)$$

  (taking $x$ to be the root of the tree), but also to

  $$B, C \vdash D, (A \vdash E, (F \vdash G), (H \vdash))$$

  (taking $y$ to be the root)

- The translations render the $\supset$, $\prec$, *nest*, *unnest* inferences of **N-LBil** in terms of $\supset$, $\prec$, *monot* inferences of **L-LBil** alongside with inferences by some admissible rules, and vice versa.
Unnest cuts

- Translating from \( \text{L-LBiI} \) (which is cut-free) to \( \text{LBiI} \) we only need cuts to translate the \( \text{unnestL} \) and \( \text{unnestR} \) rules.

- Hence the cuts translating them, i.e., cuts of the form

\[
\begin{align*}
\Gamma_0 \vdash \land \Gamma_0 \prec \lor \Delta_0, \Delta_0 & \quad \Gamma, \Gamma_0, \land \Gamma_0 \prec \lor \Delta_0 \vdash \Delta_0, \Delta \\
\Gamma, \Gamma_0 \vdash \Delta_0, \Delta & \\
\end{align*}
\]

(\text{unnestcut}_L)

\[
\begin{align*}
\Gamma, \Gamma_0 \vdash \land \Gamma_0 \supset \lor \Delta_0, \Delta_0, \Delta & \quad \Gamma_0, \land \Gamma_0 \supset \lor \Delta_0 \vdash \Delta_0 \\
\Gamma, \Gamma_0 \vdash \Delta_0, \Delta & \\
\end{align*}
\]

(\text{unnestcut}_R)

must be complete.
Conclusion and future work

- **N-LBiI** and **L-LBiI** may look quite different on the surface, but are closely related. **N-LBiI** – fixed root that can be changed. **L-LBiI** – unrooted tree.

- Is it because of the absence of exclusion that it is difficult to give a (simple, structurally recursive) translation from the labelled sequent calculus for ***Int*** to the standard-style sequent calculus?


- Other complete classes of cuts (e.g., cuts only on atomic, implicational and exclusive subformulas of the goal formula)? Translations between restrictions to different classes of cuts?