

# Monads and More: Part 3

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# Arrows (Hughes)

- Arrows are a generalization of strong monads on symmetric monoidal categories (in their Kleisli triple form).
- An *arrow* on a symmetric monoidal category  $(\mathcal{C}, I, \otimes)$  is given by
  - an object mapping  $R : |\mathcal{C}| \times |\mathcal{C}| \rightarrow |\mathbf{Set}|$ ,
  - for any objects  $A, B$  of  $\mathcal{C}$ , a map  $\text{arr} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow R(A, B)$  of  $\mathbf{Set}$ ,
  - for any objects  $A, B, C$  of  $\mathcal{C}$ , a map  $\llcorner : R(A, B) \times R(B, C) \rightarrow R(A, C)$  of  $\mathbf{Set}$ ,
  - for any objects  $A, B, C$  of  $\mathcal{C}$ , a map  $\text{second}_{\mathcal{C}} : R(A, B) \rightarrow R(C \otimes A, C \otimes B)$  of  $\mathbf{Set}$satisfying the conditions on the next slide.

- (ctd. from the previous slide)
  - if  $k \in R(A, B)$ , then  $\text{arr id}_B \lll k = k$ ,
  - if  $k \in R(A, B)$ , then  $k \lll \text{arr id}_A = k$ ,
  - if  $k \in R(A, B)$ ,  $l \in R(B, C)$ ,  $m \in R(C, D)$ , then  $(m \lll l) \lll k = m \lll (l \lll k)$ ,
  - if  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ , then  $\text{arr}(g \circ f) = \text{arr } g \lll \text{arr } f$ ,
  - if  $f : A \rightarrow B$ , then  $\text{second}_C(\text{arr } f) = \text{arr}(\text{id}_C \times f)$ ,
  - if  $k \in R(A, B)$ ,  $l \in R(B, C)$ ,  $\text{second}_D(l \lll k) = \text{second}_D l \lll \text{second}_D k$ ,
  - if  $k \in R(A, B)$ ,  $f : C \rightarrow D$ , then  $\text{arr}(f \times \text{id}_B) \lll \text{second}_C k = \text{second}_D k \lll \text{arr}(f \times \text{id}_A)$ ,
  - if  $k \in R(A, B)$ ,  $k \lll \text{arr ul}_A = \text{ul}_B \lll \text{second}_I k$ ,
  - if  $k \in R(A, B)$ ,  $\text{second}_C(\text{second}_D k) \lll a_{C,D,A} = a_{C,D,B} \lll \text{second}_{C \otimes D} k$ .

# Examples

- Arrows from strong monoidal functors:
  - $R(A, B) =_{\text{df}} \text{Hom}_{\mathcal{C}}(FA, FB)$  where  $F$  is a monoidal endofunctor on  $\mathcal{C}$  (i.e., there is a natural isomorphism  $m_{A,B} : FA \otimes FB \rightarrow F(A \otimes B)$ ),
  - if  $f : A \rightarrow B$ , then  $\text{arr } f = Ff : FA \rightarrow FB$ ,
  - if  $k : FA \rightarrow FB$ ,  $\ell : FB \rightarrow FC$ , then  $\ell \lll k =_{\text{df}} FA \xrightarrow{k} FB \xrightarrow{\ell} FC$ ,
  - if  $k : FA \rightarrow FB$ , then  $\text{second } k =_{\text{df}} F(C \otimes A) \xrightarrow{m^{-1}} FC \otimes FA \xrightarrow{\text{id} \otimes k} FC \otimes FB \xrightarrow{m} F(C \otimes B)$ .

- Kleisli maps of strong monads:

- $R(A, B) =_{\text{df}} \text{Hom}_{\mathcal{C}}(A, TB)$  where  $T$  is a strong monad,
- if  $f : A \rightarrow B$ , then  $\text{arr } f = Jf : A \rightarrow TB$  where  $J$  is the Kleisli inclusion of  $T$ ,
- if  $k : A \rightarrow TB$ ,  $\ell : B \rightarrow TC$ , then
 
$$\ell \lll k =_{\text{df}} A \xrightarrow{k} TB \xrightarrow{\ell^*} TC,$$
- if  $k : A \rightarrow TB$ , then
 
$$\text{second } k =_{\text{df}} C \otimes A \xrightarrow{\text{id} \otimes k} C \otimes TB \xrightarrow{\text{sr}} T(C \otimes B).$$

- CoKleisli maps of comonads on Cartesian categories:

- $R(A, B) =_{\text{df}} \text{Hom}_{\mathcal{C}}(DA, B)$  where  $D$  is a comonad on  $\mathcal{C}$ ,
- if  $f : A \rightarrow B$ , then  $\text{arr } f = Jf : DA \rightarrow B$  where  $J$  is the coKleisli inclusion of  $D$ ,
- if  $k : DA \rightarrow B$ ,  $\ell : DB \rightarrow C$ , then
 
$$\ell \lll k =_{\text{df}} DA \xrightarrow{k^\dagger} DB \xrightarrow{\ell} C,$$
- if  $k : DA \rightarrow B$ , then
 
$$\text{second } k =_{\text{df}} D(C \times A) \xrightarrow{\langle D\text{fst}, D\text{snd} \rangle} DC \times DA \xrightarrow{\varepsilon \times k} C \times B.$$

- Output once more:
  - $R(A, B) =_{\text{df}} E \times \text{Hom}_{\mathcal{C}}(A, B)$  where  $(E, e, m)$  is a monoid in **Set**,
  - if  $f : A \rightarrow B$ , then  $\text{arr } f = (e, f) : E \times \text{Hom}_{\mathcal{C}}(A, B)$ ,
  - if  $(x, f) : E \times \text{Hom}_{\mathcal{C}}(A, B)$ ,  $(y, g) : E \times \text{Hom}_{\mathcal{C}}(B, C)$ , then
 
$$(y, g) \lll (x, f) =_{\text{df}} (m(x, y), g \circ f) \in E \times \text{Hom}_{\mathcal{C}}(A, C),$$
  - if  $(x, f) : E \times \text{Hom}_{\mathcal{C}}(A, B)$ , then
 
$$\text{second } (x, f) =_{\text{df}} (x, C \otimes f) \in E \times \text{Hom}_{\mathcal{C}}(C \otimes A, C \otimes B).$$

# Arrows in the monoid form (Jacobs, Heunen, Hasuo)

- An alternative definition mimicks the definition of monads in the standard, i.e., monoid form.
- An *arrow* on a symmetric monoidal category  $(\mathcal{C}, I, \otimes)$  is a strong monoid in the category of endofunctors on  $(\mathcal{C}, I, \otimes)$ .
- A *profunctor* from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$ .  
The identity profunctor on  $\mathcal{C}$  is  $\text{Id} =_{\text{df}} \text{Hom}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ .  
The composition of profunctors  $R : \mathcal{C} \rightarrow \mathcal{D}$  and  $S : \mathcal{D} \rightarrow \mathcal{E}$  is  $SR(A, C) =_{\text{df}} \int^B R(A, B) \times S(B, C)$ .

- Accordingly, the data of an arrow are the following.
  - The carrier of an arrow is a profunctor  $R$  from  $\mathcal{C}$  to  $\mathcal{C}$ , i.e., a functor  $R : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ .
  - The unit is a natural transformation from  $\text{Id}$  to  $R$ , i.e., a family of maps  $\text{arr}_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow R(A, B)$  natural in  $A, B$ .

The multiplication is a nat. transf. from  $RR$  to  $R$ , i.e., a family of maps  $\lll_{A,B,C} : R(A, B) \times R(B, C) \rightarrow R(A, C)$  natural in  $A, C$  and dinatural in  $B$ .

The strength is a family of second  $_{A,B,C} :: R(A, B) \rightarrow R(C \otimes A, C \otimes B)$  natural in  $A, B$  and dinatural in  $C$ .

# Symmetric premonoidal categories (Power, Robinson)

- Intuitively, a symmetric premonoidal category is the same as a symmetric monoidal category, except that the tensor is not necessarily a bifunctor, it must only be functorial in each argument separately.
- More officially: A *symmetric premonoidal category* is given by
  - a category  $\mathcal{K}$ ,
  - an object  $I$  of  $\mathcal{K}$ ,
  - for any object  $A$ , a functor  $A \times - : \mathcal{K} \rightarrow \mathcal{K}$ ,
  - natural isomorphisms  $a$ ,  $ul$ ,  $ur$ ,  $c$  satisfying the laws of a symmetric monoidal category and have all their components central (see further).

- Symmetry yields symmetric functors  $- \times A : \mathcal{K} \rightarrow \mathcal{K}$  where  $A_0 \times A_1 = A_0 \times A_1$  (which we also denote more symmetrically by  $A_0 \otimes A_1$ ).
- A morphism  $f : A \rightarrow B$  is called *central* if, for any  $g : C \rightarrow D$ , both

$$\begin{array}{ccc}
 A \otimes C & \xrightarrow{A \times g} & A \otimes D \\
 f \times C \downarrow & & \downarrow f \times D \\
 B \otimes C & \xrightarrow{B \times g} & C \otimes D
 \end{array}
 \qquad
 \begin{array}{ccc}
 C \otimes A & \xrightarrow{g \times A} & D \otimes A \\
 C \times f \downarrow & & \downarrow D \times f \\
 C \otimes B & \xrightarrow{g \times B} & D \otimes C
 \end{array}$$

# Freyd categories

- A Freyd category on a symmetric monoidal category  $\mathcal{C}$  is given by
  - a symmetric premonoidal category  $(\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}})$  with the same objects as  $\mathcal{C}$
  - together with an identity-on-objects inclusion functor  $J : \mathcal{C} \rightarrow \mathcal{K}$  that preserves centrality and strictly preserves its the  $(I, \otimes)$  structure as premonoidal (meaning that  $I^{\mathcal{K}} = I$ ,  $A \otimes^{\mathcal{K}} B = A \otimes B$ ).

# Freyd categories vs. arrows (Jacobs, Heunen, Hasuo)

- Freyd categories are in a bijection with arrows.
- For an arrow  $R$  on a symmetric monoidal category  $(\mathcal{C}, I, \otimes)$ , the Freyd category  $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$  is defined by
  - an object is an object of  $\mathcal{C}$ ,
  - a map from  $A$  to  $B$  is an element of  $R(A, B)$ ,
  - $\text{id}_A^{\mathcal{K}} =_{\text{df}} \text{arr id}_A$ ,
  - if  $k : A \rightarrow^{\mathcal{K}} B$ ,  $\ell : B \rightarrow^{\mathcal{K}} C$ , then  $\ell \circ^{\mathcal{K}} k =_{\text{df}} \ell \lll k$ ,
  - $I^{\mathcal{K}} = I$ ,  $A \otimes^{\mathcal{K}} B =_{\text{df}} A \otimes B$ ,
  - if  $k : A \rightarrow^{\mathcal{K}} B$ , then  $C \times^{\mathcal{K}} k =_{\text{df}} \text{second } k$ ,
  - if  $f : A \rightarrow B$ , then  $Jf =_{\text{df}} \text{arr } f$ .

- Given a Freyd category  $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$  on  $\mathcal{C}$ , the corresponding arrow  $R$  is defined by
  - $R(A, B) =_{\text{df}} \text{Hom}_{\mathcal{K}}(A, B)$ ,
  - if  $f : A' \rightarrow A$ ,  $g : B \rightarrow B'$ ,  $k \in \text{Hom}_{\mathcal{K}}(A, B)$ , then  $R(f, g) k =_{\text{df}} Jg \lll k \lll Jf$ ,
  - if  $f : A \rightarrow B$ , then  $\text{arr } f =_{\text{df}} Jf \in \text{Hom}_{\mathcal{K}}(A, B)$ ,
  - if  $k \in \text{Hom}_{\mathcal{K}}(A, B)$ ,  $l \in \text{Hom}_{\mathcal{K}}(B, C)$ , then  $l \lll k =_{\text{df}} l \circ^{\mathcal{K}} k \in \text{Hom}_{\mathcal{K}}(A, C)$ ,
  - if  $k \in \text{Hom}_{\mathcal{K}}(A, B)$ , then  $\text{second } k =_{\text{df}} C \rtimes^{\mathcal{K}} k \in \text{Hom}_{\mathcal{K}}(C \otimes A, C \otimes B)$ .

## When is Freyd Kleisli? (Power)

- Given a Freyd category  $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$  on a symmetric monoidal category  $(\mathcal{C}, I, \otimes)$ , when is it the Kleisli category of a strong monad?
- A simple condition is in terms of Kleisli exponents.
- Suppose  $J(- \times A) : \mathcal{C} \rightarrow \mathcal{K}$  has a right adjoint  $A \Rightarrow^{\mathcal{K}} -$ . In this case we say the Freyd category is *closed*. Then also  $TB =_{\text{df}} I \Rightarrow^{\mathcal{K}} B$  is a strong monad with Kleisli exponents and  $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$  is its Kleisli category.