Eilenberg-Kelly Reloaded

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Closed categories

- Closed categories [Eilenberg & Kelly 1966] are categories with a unit object I and an internal hom $A \multimap B$ for all objects A and B.
- Examples:
  - Categories of structured sets, e.g. normal bands, posets
  - Categories underlying deductive systems, e.g. STLC
- In many cases, the internal hom is determined by an adjunction with the tensor product of a monoidal category, but monoidal structure was not required in Eilenberg & Kelly's original definition.

A theorem by Eilenberg & Kelly

- **Theorem:** Given a category $\mathbb{C}$ equipped with a unit I and two functors

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \qquad \multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$$

related by an adjunction

$$- \otimes B \dashv B \multimap -$$

natural in B, then

$$(\mathbb{C}, \mathsf{I}, \otimes) \text{ is monoidal}$$

iff

$(\mathbb{C}, \mathsf{I}, \multimap)$ is closed and the adjunction holds *internally*.

A closer look at the theorem

- Internal adjunction: the natural transformation

$$\mathsf{p}_{A,B,C} : (A \otimes B) \multimap C \to A \multimap (B \multimap C)$$

  has to be invertible.
- Needed to invert associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
- Invertibility of α not matched by anything in defn. of closed category.

Recovering a perfect match

- [Street 2013] proposes a way to fix the mismatch: consider weak variants of monoidal and closed categories:
  - Left-skew monoidal categories [Szlachányi 2012]
  - Left-skew closed categories [Street 2013]

- **Theorem:** Given a category $\mathbb{C}$ equipped with a unit I and two functors

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  related by an adjunction

$$- \otimes B \dashv B \multimap -$$

  natural in B, then

  $(\mathbb{C}, \mathsf{I}, \otimes)$ is *left-skew* monoidal
  iff
  $(\mathbb{C}, \mathsf{I}, \multimap)$ is *left-skew* closed.

- No internal adjunction requirement!

Left-skew monoidal categories

- A *left-skew monoidal category* is a category $\mathbb{C}$ together with an object I, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and three natural transformations λ, ρ, α typed

$$\begin{aligned} \lambda_A &: \mathsf{I} \otimes A \to A \\ \rho_A &: A \to A \otimes \mathsf{I} \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \end{aligned}$$

  satisfying the 5 Mac Lane equations.

- N.B. λ, ρ, α are not required to be invertible.

Left-skew closed categories

- A *left-skew closed category* is a category $\mathbb{C}$ together with an object I, a functor $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$, and three (extra)natural transformations j, i, and L typed

$$\begin{gathered} j_A : \mathsf{I} \to A \multimap A \\ i_A : \mathsf{I} \multimap A \to A \\ L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C) \end{gathered}$$

  satisfying 5 equations.

- In the original definition of closed category, i and

$$\begin{gathered} \widehat{\jmath}_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}(\mathsf{I}, A \multimap B) \\ \widehat{\jmath}_{A,B}(f) = A \multimap f \circ j_A \end{gathered}$$

  are required to be invertible.

Contribution 1: Normality conditions

- In a left-skew monoidal category, the invertibility of a structural law (λ, ρ or α) is called a *normality condition*.
- We identify analogous normality conditions in a left-skew closed category and prove a refined version of Street's left-skew variant of Eilenbeg-Kelly theorem.
- **Theorem:** In the presence of an adjunction $- \otimes B \dashv B \multimap -$, not only there exists an isomorphism between left-skew monoidal $(\mathsf{I}, \otimes)$ and left-skew closed $(\mathsf{I}, \multimap)$ structures, but the skew-monoidal and skew-closed normality conditions are in one-to-one correspondence.

Normality conditions (ctd.)

- The precise correspondence:

ρ nat. iso.	iff	i nat. iso.		
λ nat. iso.	iff	$\widehat{\jmath}$ nat. iso.		
α nat. iso.	iff	$\widehat{L}$ nat. iso.	iff	p nat. iso.

  with

$$\widehat{\jmath}_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}(\mathrm{I}, A \multimap B)$$
$$\widehat{L}_{A,B,C,D} : \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X) \to \mathbb{C}(A, B \multimap (C \multimap D))$$

  interdefinable with j and L respectively.

- In the original Eilenberg-Kelly theorem, the internal adjunction requirement can be substituted with the invertibility of $\widehat{L}$ (a condition identified first in [Day 1974; Day & Laplaza 1978]).

Contribution 2: Skewing to the right

- Changing the orientation of the structural laws ρ, λ, α of left-skew monoidal categories, we obtain *right-skew monoidal categories.*

$$\rho^{\mathsf{R}}_A : A \otimes \mathsf{I} \to A$$
$$\lambda^{\mathsf{R}}_A : A \to \mathsf{I} \otimes A$$
$$\alpha^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

- Similarly, changing the orientation of the structural laws $i, \widehat{\jmath}, \widehat{L}$ of left-skew closed categories, we obtain the new notion of *right-skew closed category*.

$$i^{\mathsf{R}}{}_A : A \to \mathsf{I} \multimap A$$
$$j^{\mathsf{R}}{}_{A,B} : \mathbb{C}(\mathsf{I}, A \multimap B) \to \mathbb{C}(A, B)$$
$$L^{\mathsf{R}}{}_{A,B,C,D} : \mathbb{C}(A, B \multimap C \multimap D) \to \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)$$

Skewing to the right (ctd.)

- We prove a right-skew variant of Street's theorem connecting adjoint right-skew monoidal and right-skew closed structures on a category, and similar relationships between their normality conditions.

- More interestingly, we prove a new theorem connecting left-skew closed and right-skew closed structures on a category.
- **Theorem:** Let $\mathbb{C}$ be a category with an object I and functors

$$\multimap^{\mathsf{L}}, \multimap^{\mathsf{R}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$$

  together with what we call the *external Lambek condition*, viz., a bijection

$$\sigma_{A,B,C} : \mathbb{C}(A, B \multimap^{\mathsf{R}} C) \to \mathbb{C}(B, A \multimap^{\mathsf{L}} C)$$

  natural in A, B and C. Then

$$(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}}) \text{ is left-skew closed}$$
$$\text{iff}$$
$$(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}}) \text{ is right-skew closed.}$$

Skewing to the right (ctd.)

- The normality conditions on $\multimap^{\mathsf{L}}$ and $\multimap^{\mathsf{R}}$ also correspond:

$$
\begin{array}{lllll}
i \text{ nat. iso.} & \text{iff} & j^{\mathsf{R}} \text{ nat. iso.} & & \\
\widehat{\jmath} \text{ nat. iso.} & \text{iff} & i^{\mathsf{R}} \text{ nat. iso.} & & \\
\widehat{L} \text{ nat. iso.} & \text{iff} & L^{\mathsf{R}} \text{ nat. iso.} & \text{iff} & \mathsf{s} \text{ nat. iso.}
\end{array}
$$

with

$$\mathsf{s}_{A,B,C} : A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{R}} C) \to B \multimap^{\mathsf{R}} (A \multimap^{\mathsf{L}} C)$$

internal version of Lambek condition.

Contribution 3: Examples

- We discuss a large number of examples, in particular for motivating the different normality conditions and the new notion of right-skew closed category.

- In this talk we discuss:
  - Skewing a left-(right-)skew closed structure further to the left (right) using a comonad (monad).
  - Lifting left- and right-skew closed structure to a Kleisli category.
  - The non-commutative linear typed λ-calculus with unit type.

Ex 1: Skewing a skew closed structure further

- Let $(\mathbb{C}, \mathsf{I}, \multimap)$ be a left-skew closed category with a comonad D on it.
- Suppose D *lax closed*, i.e., coming with a map $\mathsf{e} : \mathsf{I} \to D\,\mathsf{I}$ and a nat. trans. $\mathsf{c}_{B,C} : D\,(B \multimap C) \to D\,B \multimap D\,C$ cohering with $j, i, L, \varepsilon, \delta$.
- Then $\mathbb{C}$ has another left-skew closed structure $(\mathsf{I}, {}^{D}\!\multimap)$ where $B\ {}^{D}\!\multimap C = D\,B \multimap C$ and, e.g.,

$$
{}^{D}i_A = \ D\,\mathsf{I} \multimap A \xrightarrow{\mathsf{e} \multimap A} \mathsf{I} \multimap A \xrightarrow{i_A} A
$$

- If both i and e are invertible, then ${}^{D}i$ is invertible.

- Instead, given $(\mathbb{C}, \mathsf{I}, \multimap)$ right-skew closed and an *oplax closed* monad T on it, then $\mathbb{C}$ has another right-skew closed structure $(\mathsf{I}, {}^{T}\!\multimap)$ where $B\ {}^{T}\!\multimap C = T\,B \multimap C$.

Ex 2: Lifting skew closed structure to Kleisli category

- Let $(\mathbb{C}, \mathsf{I}, \multimap)$ be a left-skew closed category with a monad T on it.
- Suppose T *left-strong* (or *internally functorial*), i.e., endowed with a nat. trans. $\mathsf{cst}_{A,B} : B \multimap C \to T\,B \multimap T\,C$ cohering with j, L, η, μ.
- Then $\mathbf{Kl}(T)$ has a left-skew closed structure $(\mathsf{I}, \multimap^T)$ where $B \multimap^T C = B \multimap T\,C$ and, e.g.,

$$j^T_A = J(\,\mathsf{I} \xrightarrow{j_A} A \multimap A \xrightarrow{A \multimap \eta_A} A \multimap TA\,)$$

$$i^T_A = \ \mathsf{I} \multimap T\,A \xrightarrow{i_{TA}} T\,A$$

- If, instead, $(\mathbb{C}, \mathsf{I}, \multimap)$ is right-skew closed and T is *lax closed* (so both *left-strong* and *right-strong*), then $(\mathsf{I}, \multimap^T)$ is a right-skew closed structure on $\mathbf{Kl}(T)$.

Ex 3: Non-commutative linear typed λ-calculus with unit

- Types $A, B ::= X \mid \mathsf{I} \mid A \multimap B$, where X is an atomic type.
- Contexts are lists of types.
- Well-formed terms:

$$\frac{}{x : A \vdash x : A} \qquad \frac{}{\vdash \star : \mathsf{I}} \qquad \frac{\Gamma \vdash t : \mathsf{I} \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash \mathrm{let}\ \star = t\ \mathrm{in}\ u : A}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\ t : A \multimap B} \qquad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\ u : B}$$

- Definitional equality of terms is $\beta\eta$-equality.
- It is a left-skew closed category. Derivation tree of L:

$$\frac{\dfrac{}{x : B \multimap C \vdash x : B \multimap C} \qquad \dfrac{\dfrac{}{y : A \multimap B \vdash y : A \multimap B} \quad \dfrac{}{z : A \vdash z : A}}{y : A \multimap B, z : A \vdash y\ z : B}}{\dfrac{x : B \multimap C, y : A \multimap B, z : A \vdash x\ (y\ z) : C}{x : B \multimap C \vdash L_{A,B,C} = \lambda y.\ \lambda z.\ x\ (y\ z) : (A \multimap B) \multimap (A \multimap C)}}$$

Ex 3: Non-commutative linear typed λ-calculus with unit

- This calculus is a concrete presentation of the *free* left-skew closed category generated by the set of atomic types.
- Fact: $\widehat{\jmath}$ is invertible, i.e., there is a bijection between closed terms $\vdash t : A \multimap B$ and open terms with one free variable $x : A \vdash u : B$.

- i becomes invertible if we replace the elimination rule for I with the following more permissive rule:

$$\frac{\Gamma \vdash t : \mathsf{I} \quad \Delta_0, \Delta_1 \vdash u : A}{\Delta_0, \Gamma, \Delta_1 \vdash \mathsf{let}\ \star = t\ \mathsf{in}\ u : A}$$

Conclusions

- Continuing work initiated by Street on a "cleaner" Eilenberg-Kelly thm., we proved a relation between left-skew monoidal and left-skew closed categories with partial normality conditions.
- We showed that closed categories (in the sense of the standard terminology) correspond to monoidal categories that are left-skew in regards to associativity.
- We also demonstrated that there is a well-justified notion of right-skew closed category with nontrivial examples.

- Future work:
  - Find more examples relevant to mathematical semantics of programming.
  - In continuation to our prior work [UVZ 2018], develop the proof theory (sequent calculus, natural deduction) of left-skew/partially-normal monoidal, closed, monoidal closed, bi-closed and symmetric monoidal categories.