Eilenberg-Kelly Reloaded

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Closed categories

Closed categories [Eilenberg & Kelly 1966] are categories with a unit object $I$ and an internal hom $A \multimap B$ for all objects $A$ and $B$.

Examples:
- Categories of structured sets, e.g. normal bands, posets
- Categories underlying deductive systems, e.g. STLC

In many cases, the internal hom is determined by an adjunction with the tensor product of a monoidal category, but monoidal structure was not required in Eilenberg & Kelly's original definition.
Theorem: Given a category \( C \) equipped with a unit \( I \) and two functors

\[ \otimes : C \times C \to C \quad \rightarrow : C^{\text{op}} \times C \to C \]

related by an adjunction

\[ - \otimes B \vdash B \rightarrow - \]

natural in \( B \), then

\( (C, I, \otimes) \) is monoidal

iff

\( (C, I, \rightarrow) \) is closed and the adjunction holds *internally*. 
A closer look at the theorem

- Internal adjunction: the natural transformation

\[ p_{A,B,C} : (A \otimes B) \rightarrow C \rightarrow A \rightarrow (B \rightarrow C) \]

has to be invertible.
- Needed to invert associator \( \alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C). \)
- Invertibility of \( \alpha \) not matched by anything in defn. of closed category.
Recovering a perfect match

- [Street 2013] proposes a way to fix the mismatch: consider weak variants of monoidal and closed categories:
  - Left-skew monoidal categories [Szalachányi 2012]
  - Left-skew closed categories [Street 2013]

- **Theorem:** Given a category $\mathcal{C}$ equipped with a unit $I$ and two functors
  $$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \dashv \quad - \circ : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$$
  related by an adjunction
  $$- \otimes B \vdash B \circ -$$
  natural in $B$, then
  $$(\mathcal{C}, I, \otimes)$$ is left-skew monoidal
  iff
  $$(\mathcal{C}, I, \circ)$$ is left-skew closed.

- No internal adjunction requirement!
A left-skew monoidal category is a category $\mathbb{C}$ together with an object $I$, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and three natural transformations $\lambda, \rho, \alpha$ typed

\[
\begin{align*}
\lambda_A : I \otimes A &\to A \\
\rho_A : A &\to A \otimes I \\
\alpha_{A,B,C} : (A \otimes B) \otimes C &\to A \otimes (B \otimes C)
\end{align*}
\]

satisfying the 5 Mac Lane equations.

N.B. $\lambda, \rho, \alpha$ are not required to be invertible.
Left-skew closed categories

- A *left-skew closed category* is a category $\mathcal{C}$ together with an object $I$, a functor $\circ : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$, and three (extra)natural transformations $j$, $i$, and $L$ typed

  \[
  j_A : I \to A \to A \\
  i_A : I \to A \to A \\
  L_{A,B,C} : B \to C \to (A \to B) \to (A \to C)
  \]

  satisfying 5 equations.

- In the original definition of closed category, $i$ and

  \[
  \hat{j}_{A,B} : \mathcal{C}(A, B) \to \mathcal{C}(I, A \to B) \\
  \hat{j}_{A,B}(f) = A \to f \circ j_A
  \]

  are required to be invertible.
Contribution 1: Normality conditions

- In a left-skew monoidal category, the invertibility of a structural law \((\lambda, \rho \text{ or } \alpha)\) is called a *normality condition*.

- We identify analogous normality conditions in a left-skew closed category and prove a refined version of Street’s left-skew variant of Eilenbeg-Kelly theorem.

- **Theorem:** In the presence of an adjunction \(- \otimes B \vdash B \dashv -\), not only there exists an isomorphism between left-skew monoidal \((I, \otimes)\) and left-skew closed \((I, \dashv)\) structures, but the skew-monoidal and skew-closed normality conditions are in one-to-one correspondence.
Normality conditions (ctd.)

- The precise correspondence:

\[ \rho \text{ nat. iso. iff } i \text{ nat. iso.} \]
\[ \lambda \text{ nat. iso. iff } \hat{j} \text{ nat. iso.} \]
\[ \alpha \text{ nat. iso. iff } \hat{L} \text{ nat. iso. iff } p \text{ nat. iso.} \]

with

\[ \hat{j}_{A,B} : \mathbb{C}(A, B) \rightarrow \mathbb{C}(1, A \rightarrow B) \]
\[ \hat{L}_{A,B,C,D} : \int^X \mathbb{C}(A, X \rightarrow D) \times \mathbb{C}(B, C \rightarrow X) \rightarrow \mathbb{C}(A, B \rightarrow (C \rightarrow D)) \]

interdefinable with \( j \) and \( L \) respectively.

- In the original Eilenberg-Kelly theorem, the internal adjunction requirement can be substituted with the invertibility of \( \hat{L} \) (a condition identified first in [Day 1974; Day & Laplaza 1978]).
Contribution 2: Skewing to the right

- Changing the orientation of the structural laws $\rho, \lambda, \alpha$ of left-skew monoidal categories, we obtain \textit{right-skew monoidal categories}.

$$
\begin{align*}
\rho^R_A &: A \otimes I \to A \\
\lambda^R_A &: A \to I \otimes A \\
\alpha^R_{A,B,C} &: A \otimes (B \otimes C) \to (A \otimes B) \otimes C
\end{align*}
$$

- Similarly, changing the orientation of the structural laws $i, \hat{j}, \hat{L}$ of left-skew closed categories, we obtain the new notion of \textit{right-skew closed category}.

$$
\begin{align*}
i^R_A &: A \to I \to A \\
j^R_{A,B} &: \mathbb{C}(I, A \to B) \to \mathbb{C}(A, B) \\
L^R_{A,B,C,D} &: \mathbb{C}(A, B \to C \to D) \to \int^X \mathbb{C}(A, X \to D) \times \mathbb{C}(B, C \to X)
\end{align*}
$$
Skewing to the right (ctd.)

- We prove a right-skew variant of Street’s theorem connecting adjoint right-skew monoidal and right-skew closed structures on a category, and similar relationships between their normality conditions.

- More interestingly, we prove a new theorem connecting left-skew closed and right-skew closed structures on a category.

- **Theorem:** Let $\mathcal{C}$ be a category with an object $I$ and functors $\otimes^L, \otimes^R : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$

  together with what we call the *external Lambek condition*, viz., a bijection

  $$\sigma_{A,B,C} : \mathcal{C}(A, B \otimes^R C) \to \mathcal{C}(B, A \otimes^L C)$$

  natural in $A, B$ and $C$. Then

  $$(\mathcal{C}, I, \otimes^L) \text{ is left-skew closed}$$

  iff

  $$(\mathcal{C}, I, \otimes^R) \text{ is right-skew closed.}$$
The normality conditions on $\to^L$ and $\to^R$ also correspond:

\begin{align*}
  i \text{ nat. iso.} \iff & j^R \text{ nat. iso.} \\
  \hat{j} \text{ nat. iso.} \iff & i^R \text{ nat. iso.} \\
  \hat{L} \text{ nat. iso.} \iff & L^R \text{ nat. iso.} \iff s \text{ nat. iso.}
\end{align*}

with

$$s_{A,B,C} : A \to^L (B \to^R C) \to B \to^R (A \to^L C)$$

internal version of Lambek condition.
Contribution 3: Examples

- We discuss a large number of examples, in particular for motivating the different normality conditions and the new notion of right-skew closed category.

- In this talk we discuss:
  - Skewing a left-(right-)skew closed structure further to the left (right) using a comonad (monad).
  - Lifting left- and right-skew closed structure to a Kleisli category.
  - The non-commutative linear typed \(\lambda\)-calculus with unit type.
Ex 1: Skewing a skew closed structure further

- Let \((\mathcal{C}, I, \rightarrow)\) be a left-skew closed category with a comonad \(D\) on it.
- Suppose \(D\) lax closed, i.e., coming with a map \(e : I \to DI\) and a nat. trans. \(c_{B,C} : D(B \rightarrowtail C) \to DB \rightarrowtail DC\) cohering with \(j, i, L, \varepsilon, \delta\).
- Then \(\mathcal{C}\) has another left-skew closed structure \((I, D \rightarrowtail)\) where \(B^{D \rightarrowtail} C = DB \rightarrowtail C\) and, e.g.,

\[
D i_A = DI \rightarrowtail A \xrightarrow{e \circ A} I \rightarrowtail A \xrightarrow{i_A} A
\]

- If both \(i\) and \(e\) are invertible, then \(D i\) is invertible.

- Instead, given \((\mathcal{C}, I, \rightarrowrightarrow)\) right-skew closed and an oplax closed monad \(T\) on it, then \(\mathcal{C}\) has another right-skew closed structure \((I, T \rightarrowrightarrow)\) where \(B^{T \rightarrowrightarrow} C = TB \rightarrowrightarrow C\).
Ex 2: Lifting skew closed structure to Kleisli category

- Let \((\mathbb{C}, I, \circ)\) be a left-skew closed category with a monad \(T\) on it.
- Suppose \(T\) left-strong (or internally functorial), i.e., endowed with a nat. trans. \(\text{cst}_{A,B} : B \circ C \to TB \circ TC\) cohering with \(j, L, \eta, \mu\).
- Then \(\text{Kl}(T)\) has a left-skew closed structure \((I, \circ^T)\) where \(B \circ^T C = B \circ TC\) and, e.g.,

\[
j^T_A = J\left(1 \xrightarrow{j_A} A \circ A \xrightarrow{A \circ \eta_A} A \circ TA\right)
\]

\[
i^T_A = 1 \circ TA \xrightarrow{i_TA} TA
\]

- If, instead, \((\mathbb{C}, I, \circ)\) is right-skew closed and \(T\) is lax closed (so both left-strong and right-strong), then \((I, \circ^T)\) is a right-skew closed closed structure on \(\text{Kl}(T)\).
Ex 3: Non-commutative linear typed λ-calculus with unit

- Types $A, B ::= X \mid I \mid A \rightarrow B$, where $X$ is an atomic type.
- Contexts are lists of types.
- Well-formed terms:

  $\Gamma \vdash t : I \quad \Delta \vdash u : A$
  
  $\Gamma, x : A \vdash t : B$

  $\Gamma \vdash \lambda x. \ t : A \rightarrow B$

  $\Gamma \vdash t : A \rightarrow B \quad \Delta \vdash u : A$

  $\Gamma, \Delta \vdash t \ u : B$

- Definitional equality of terms is $\beta\eta$-equality.
- It is a left-skew closed category. Derivation tree of $L$:

  $x : B \rightarrow C \vdash x : B \rightarrow C$

  $y : A \rightarrow B \vdash y : A \rightarrow B \quad z : A \vdash z : A$

  $x : B \rightarrow C, y : A \rightarrow B, z : A \vdash x \ (y \ z) : C$

  $x : B \rightarrow C \vdash L_{A,B,C} = \lambda y. \lambda z. \ x \ (y \ z) : (A \rightarrow B) \rightarrow (A \rightarrow C)$
Ex 3: Non-commutative linear typed $\lambda$-calculus with unit

- This calculus is a concrete presentation of the free left-skew closed category generated by the set of atomic types.

- Fact: $\hat{j}$ is invertible, i.e., there is a bijection between closed terms $\vdash t : A \rightarrow B$ and open terms with one free variable $x : A \vdash u : B$.

- $i$ becomes invertible if we replace the elimination rule for $I$ with the following more permissive rule:

$$\Gamma \vdash t : I \quad \Delta_0, \Delta_1 \vdash u : A$$
$$\Delta_0, \Gamma, \Delta_1 \vdash \text{let } \star = t \text{ in } u : A$$
Conclusions

▶ Continuing work initiated by Street on a “cleaner” Eilenberg-Kelly thm., we proved a relation between left-skew monoidal and left-skew closed categories with partial normality conditions.
▶ We showed that closed categories (in the sense of the standard terminology) correspond to monoidal categories that are left-skew in regards to associativity.
▶ We also demonstrated that there is a well-justified notion of right-skew closed category with nontrivial examples.

▶ Future work:
  ▶ Find more examples relevant to mathematical semantics of programming.
  ▶ In continuation to our prior work [UVZ 2018], develop the proof theory (sequent calculus, natural deduction) of left-skew/partially-normal monoidal, closed, monoidal closed, bi-closed and symmetric monoidal categories.