Strong functors, strong monads
Strong functors

- A (left) strong functor on a monoidal category \((C, I, \otimes)\) is given by
  - an endofunctor \(F\) on \(C\)
  - with a nat. transf. \(\theta_{A,B} : A \otimes FB \to F(A \otimes B)\) (the (left) strength)

satisfying

\[
\begin{align*}
I \otimes FA & \xrightarrow{\theta_{I,A}} F(I \otimes A) \\
\lambda_{FA} & \downarrow \quad \downarrow F\lambda_A \\
FA & \quad \quad \quad \quad \quad FA \\
\end{align*}
\]

\[
\begin{align*}
(A \otimes B) \otimes FC & \xrightarrow{\theta_{A \otimes B,C}} F((A \otimes B) \otimes C) \\
\alpha_{A,B,FC} & \downarrow \\
A \otimes (B \otimes FC) & \xrightarrow{A \otimes \theta_{B,C}} A \otimes F(B \otimes C) \xrightarrow{\theta_{A,B \otimes C}} F(A \otimes (B \otimes C)) \xrightarrow{F\alpha_{A,B,C}}
\end{align*}
\]

- Similarly, one defines a right strong functor.
On a monoidal closed category \((\mathcal{C}, I, \otimes, \multimap)\), a strong functor can also be defined as an endofunctor on \(\mathcal{C}\) equipped with

- a natural transformation \(F(A \multimap B) \to A \multimap FB\)

or

- a natural transformation \(A \multimap B \to FA \multimap FB\)

subject to appropriate conditions.
A **bistrong functor** is a functor $F$ with a left strength $\theta$ and a right strength $\vartheta$ such that

$$(A \otimes FB) \otimes C \xrightarrow{\theta_{A,B} \otimes C} F(A \otimes B) \otimes C \xrightarrow{\vartheta_{A \otimes B, C}} F((A \otimes B) \otimes C)$$

$A \otimes (FB \otimes C) \xrightarrow{A \otimes \vartheta_{B,C}} A \otimes F(B \otimes C) \xrightarrow{\theta_{A,B} \otimes C} F(A \otimes (B \otimes C))$$

A bistrong functor on a symmetric monoidal category $(C, I, \otimes)$ category is **symmetric** bistrong, if

$A \otimes FB \xrightarrow{\theta_{A,B}} F(A \otimes B)$

$FB \otimes A \xrightarrow{\vartheta_{B,A}} F(B \otimes A)$

In this situation $\theta$ determines $\vartheta$ via

$$\vartheta_{A,B} = \text{def} \ F(A \otimes B) \xrightarrow{\sigma_{FA,B}} B \otimes FA \xrightarrow{\theta_{B,A}} F(B \otimes A) \xrightarrow{F\sigma_{B,A}} F(A \otimes B)$$
Strong natural transformations

- A (left) strong natural transformation between two (left) strong functors \((F, \theta), (G, \theta')\) is a natural transformation \(\tau : F \to G\) satisfying

\[
\begin{align*}
A \otimes FB & \xrightarrow{\theta_{A,B}} F(A \otimes B) \\
\text{id}_{A \otimes \tau_B} & \downarrow \quad \tau_{A \otimes B} \\
A \otimes GB & \xrightarrow{\theta'_{A,B}} G(A \otimes B)
\end{align*}
\]

- Right strong and bistrong natural transformations are defined analogously.
Set functors, nat. transfs. are uniquely strong

- On \((\mathbf{Set}, 1, \times)\), any functor \(F\) has a unique left strength given by
  \[
  \theta_{A,B} (a, c) = F (\lambda b. (a, b)) c
  \]
  and are therefore uniquely bistrong.

- Any natural transformation is left strong and bistrong.
Strong monads

- A (left) strong monad on a monoidal category \((\mathcal{C}, I, \otimes)\) is a monad \((T, \eta, \mu)\) with a strength \(\theta\) for \(T\) for which \(\eta\) and \(\mu\) are strong, i.e., satisfy

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{\text{id}_A \otimes \eta_B} & A \otimes B \\
\downarrow & & \downarrow \\
A \otimes TB & \xrightarrow{\theta_{A,B}} & T(A \otimes B)
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes T(TB) & \xrightarrow{\theta_{A,TB}} & T(A \otimes TB) \\
\downarrow & & \downarrow \\
A \otimes TB & \xrightarrow{T \theta_{A,B}} & T(T(A \otimes B))
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes T(TB) & \xrightarrow{\theta_{A,TB}} & T(A \otimes TB) \\
\downarrow & & \downarrow \\
A \otimes TB & \xrightarrow{\theta_{A,B}} & T(A \otimes B)
\end{array}
\]

(Id is always strong; if \(F, G\) are strong, then so is \(G \cdot F\).)

Right strong and bistrong monads are defined analogously.
An alternative: strong Kleisli triples

- A strong Kleisli triple is
  - an object mapping $T : |\mathcal{C}| \to |\mathcal{C}|$,
  - for any object $A$, a map $\eta_A : A \to TA$,
  - for any map $k : X \otimes A \to TB$, a map $k^* : X \otimes TA \to TB$
    (the strong Kleisli extension operation)

such that

- if $k : X \otimes A \to TB$, then $k^* \circ (X \otimes \eta_A) = k$,
- $(\eta_A \circ \lambda_A)^* = \lambda_{TA} : I \otimes TA \to TA$,
- if $k : X \otimes A \to TB$, $\ell : Y \otimes B \to TC$, then
  $(\ell^* \circ (Y \otimes k) \circ \alpha_{Y,X,A})^* = \ell^* \circ (Y \otimes k^*) \circ \alpha_{Y,X,TA} : (Y \otimes X) \otimes TA \to TC$.

(No explicit functoriality and naturality conditions! No explicit strength operation, no explicit laws for strength.)

- Strong monads and strong Kleisli triples are in a bijection.
Set monads are uniquely strong

- On \((\text{Set}, 1, \times)\), every monad is uniquely left strong and bistrong.
Commutative bistrong monads

- A [symmetric] bistrong monad \((T, \eta, \mu, \theta, \vartheta)\) defines two natural transformations

\[
m_{A,B}^{lr} = \text{df} \quad TA \otimes TB \xrightarrow{\vartheta_{A,TB}} T(A \otimes TB) \xrightarrow{T\theta_{A,B}} T(T(A \otimes B)) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)
\]

and

\[
m_{A,B}^{rl} = \text{df} \quad TA \otimes TB \xrightarrow{\theta_{TA,B}} T(TA \otimes B) \xrightarrow{T\vartheta_{A,B}} T(T(A \otimes B)) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)
\]

- The [symmetric] bistrong monad is called a commutative, if \(m^{lr} = m^{rl}\).