Post-surjectivity and pre-injectivity in cellular automata: An exchange of power

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Introduction

- Cellular automata (CA) are models of parallel synchronous computation where the nodes of a regular grid take their next state according to the current state of a uniform neighborhood.
- The properties of the group underlying the grid are often linked to those of the CA defined on it.
- One of said properties is a weakening of injectivity.
- We discuss a “dual” property, which is a strengthening of surjectivity.
- We show that these two properties imply existence of a reverse CA — at least, under a very broad assumption.
Configurations and patterns over groups

Let $G = (G, \cdot, 1_G, ^{-1})$ be a group and $S$ be a finite nonempty set.

- For $E, M \subseteq G$: $EM = \{x \cdot y \mid x \in E, y \in M\}$, $E^{-1} = \{x^{-1} \mid x \in E\}$.
- A configuration is a function $c : G \rightarrow S$. We set $C = S^G$.
- $c, e \in C$ are asymptotic if $\#\{g \in G \mid c(g) \neq e(g)\} < \infty$.
- A pattern is a function $p : E \rightarrow S$ with $E \subseteq G$, $0 < \#E < \infty$.

$V \subseteq G$ generates $G$ if words over $V \cup V^{-1}$ represent all elements of $G$.

- The length of $g \in G$ is the minimum length $\|g\|$ of such a word. We set $D_n = \{g \in G \mid \|g\| \leq n\}$.
- This also induces a distance on $C$ by

$$d_V(c, e) = 2^{-N} \text{ where } N = \inf \{\|g\| \mid g \in G, c(g) \neq e(g)\}$$

In this talk we will only consider infinite, finitely generated groups.
Cellular automata over groups

A cellular automaton (CA) over a group \( G \) is a triple \( \mathcal{A} = \langle S, \mathcal{N}, f \rangle \) where:

- \( S \) is a finite set of states with two or more elements.
- The neighborhood \( \mathcal{N} = \{ \nu_1, \ldots, \nu_m \} \subseteq G \) is finite and nonempty.
- \( f : S^m \rightarrow S \) is the local update rule.

The global transition function \( F_{\mathcal{A}} : \mathcal{C} \rightarrow \mathcal{C} \) is defined by the formula

\[
F_{\mathcal{A}}(c)(g) = f(c(g \cdot \nu_1), \ldots, c(g \cdot \nu_m)) \quad \forall g \in G
\]

Note that \( F \) is continuous.

A pattern \( q : M \rightarrow S \) is a preimage of \( p : E \rightarrow S \) if \( E \mathcal{N} \subseteq M \) and

\[
f(q(x \cdot \nu_1), \ldots, q(x \cdot \nu_m))) = p(x) \quad \forall x \in E
\]

- **Fact:** if every pattern has a preimage, so does every configuration.
Pre-injectivity and the Garden of Eden theorem

Let $\mathcal{A}$ be a CA on $G$ with global function $F$. $\mathcal{A}$ is pre-injective if:

if $c, e \in C$ are asymptotic and different
then $F(c) \neq F(e)$

The Garden of Eden theorem

- Moore, 1962: Every surjective CA on $\mathbb{Z}^d$ is pre-injective.
- Myhill, 1963: Every pre-injective CA on $\mathbb{Z}^d$ is surjective.
A cellular automaton with global function $F$ is reversible if there exists a CA with global function $H$ such that $H \circ F = F \circ H = \text{id}_C$.

- **Fact**: reversibility comes for free with bijectivity.
- Every injective CA on $\mathbb{Z}^d$ is reversible. (Follows from the Garden of Eden theorem.)
- For $d = 1$ the following characterization holds:
  
  a one-dimensional cellular automaton is reversible if and only if it is injective on periodic configurations.
Surjunctive groups

The proofs by Moore and Myhill exploit that

\[ \text{in } \mathbb{Z}^d \text{ the orange grows faster than the peel} \]

This is not true for arbitrary groups!

- For the free group on two generators, \( V = \{a, b\} \), both \( \#D_n \) and \( \#(D_{n+1} \setminus D_n) \) are in \( \Theta(3^n) \).
- The class of groups where the Garden of Eden theorem holds coincides with a well-known class defined by von Neumann.

However, in all known cases, injective CA are surjective.

- We call a group \( G \) surjunctive if every injective CA on \( G \) is surjective.
- **Gottschalk’s conjecture:** every group is surjunctive.
Post-surjectivity

A cellular automaton $\mathcal{A} = \langle S, N, f \rangle$ over a group $G$ is post-surjective if:

for every $c, e \in \mathcal{C}$ with $F_\mathcal{A}(e) = c$
and every $c' \in \mathcal{C}$ asymptotic to $c$,
there exists $e' \in \mathcal{C}$ asymptotic to $e$ with $F_\mathcal{A}(e') = c'$

Post-surjective CA are surjective.

- Fix $0 \in S$ and take $0' = f(0, \ldots, 0)$.
- A preimage to any pattern $p$ can be found by pasting it on the $0'$-constant configuration.
Two examples

Reversible CA are post-surjective.
- Let $\mathcal{N}$ be a neighborhood for the reverse CA.
- Suppose $c$ and $c'$ only differ on $E$.
- Then their unique preimages $e$ and $e'$ can differ at most in $EN$.

Not all surjective CA are post-surjective.
- Let $\mathcal{A}$ be the 1D XOR with the right-hand neighbor.
- Every configuration has two preimages, uniquely determined by their value at a single point.
- However, ...00100... has no 0-finite preimage.

Is this just a case?
Post-surjective 1D CA are reversible

Step 1: characterization of non-reversible 1D CA

Step 2: swap of halves
Theorem 1: Post-surjective 1D CA are reversible (cont.)

Step 3: post-surjectivity

Step 4: contradiction of the Garden of Eden theorem
To prove Theorem 2, we have made use of two classical results:

1. a 1D CA being reversible iff it is injective on periodic configurations
2. the Garden of Eden theorem

which do not hold in the general case. However:

- pre-injectivity is a bit less than injectivity
- post-surjectivity is a bit more than surjectivity

So may it be that such “exchange of power” allows to recover reversibility?
Yes, we can! (if all goes well)

Pre-injective, post-surjective CA over surjunctive groups are reversible.

- Let $G$ be a surjunctive group.
  
  Let $F$ the global function of a pre-injective, post-surjective CA on $G$.

- **Lemma**: (by post-surjectivity and pre-injectivity combined)
  
  There exists a finite $M \subseteq G$ such that, for every $e, e' \in C$, if $c = F(e)$ and $c' = F(e')$ disagree at most on a finite $D \subseteq G$, then $e$ and $e'$ disagree at most on $DM$.

  This allows constructing a CA with neighborhood $M^{-1}$ whose global function $H$ satisfies $F \circ H = \text{id}_C$.

- But a right inverse of a surjective function is injective . . .

- . . . then $H$ is also surjective by surjunctivity of $G$, and $F$ is its inverse.
Conclusions and future work

Conclusions:

- Post-surjectivity is introduced as a “dual” to pre-injectivity.
- Under mild constraints, reversibility can be obtained by weakening injectivity while suitably strengthening surjectivity.

Future work:

1. Dispose of the surjunctivity condition (which is mainly a “patch” to quickly get our theorem)
2. Are there any CA that are post-surjective, but not pre-injective?
3. Are other such transfers possible?
4. Equivalent formulations of Gottschalk’s conjecture?

Thank you for attention!

Any questions?