A Divertimento on MonadPlus and Nondeterminism

Tarmo Uustalu, Institute of Cybernetics, Tallinn

Theory Days at Jõeküla, 2–4 October 2015
Monad and MonadPlus

- In Haskell, we use monads (the Monad class) as an general abstraction of notions of effectful computation. Monad knows all about sequential composition of computations.

- MonadPlus, a subclass of Monad, are used as an abstraction to specifically deal with notions of effectful computation involving nondeterminism. MonadPlus cares about combining computations with choice.

- Monads are very well understood; they are central in category theory.
- In contrast, With MonadPlus, the situation has been quite unclear.
Sequential composition vs choice
MonadPlus

- In Haskell, MonadPlus is a subclass of Monad with two added operations:

  ```haskell
  class Monad m => MonadPlus m where
    mzero :: m a
    mplus :: m a -> m a -> m a
  ```

- But what should the laws be?

  - Typical suggestions include:

    ```haskell
    mzero `mplus` c == c
    c `mplus` mzero == c
    (c `mplus` c') `mplus` c'' == c `mplus` (c' `mplus` c'')
    ```

    ```haskell
    mzero >>= k == mzero
    (c `mplus` c') >>= k == (c >>= k) `mplus` (c' >>= k)
    ```

- But also these and others have been suggested:

  ```haskell
  c >>= \_ -> mzero == mzero
  c >>= \a -> (k a `mplus` k' a) == (c >>= k) `mplus` (c >>= k')
  ```

  (The list monad does not obey that last law.)
Associativity of (binary) choice

\[
\begin{align*}
\text{a} & \quad \text{b} & \quad \text{c} \\
\text{c} & \quad \text{a} & \quad \text{b} \\
\end{align*}
\]
This talk

- I propose some general principles for choosing the axioms, following Plotkin and Power’s algebraic theories approach.
- In particular, I will look at monads of semigroups, monoids, and special classes of semigroups.
Monads of semigroups

- Say that a *monad of semigroups* on a category $\mathcal{C}$ with products is a monad $T = (T, \eta, \mu)$ with a nat. transf. $\oplus_x : TX \times TX \to TX$ such that

\[
\begin{array}{ccc}
(TX \times TX) \times TX & \xrightarrow{\oplus_x \times TX} & TX \times TX \\
\alpha_{TX,TX,TX} & \downarrow & \downarrow \oplus_x \\
TX \times (TX \times TX) & \xrightarrow{\oplus_x} & TX \\
\oplus_x \times TX & \downarrow & \downarrow \mu_x \\
TX \times TX & \xrightarrow{\oplus_x} & TX \\
\end{array}
\]

\[
\begin{align*}
(c \oplus c') \oplus c'' &= c \oplus (c' \oplus c'') \\
\mu(c \oplus c') &= \mu c \oplus \mu c'
\end{align*}
\]

(associativity of $\oplus$, right distributivity of $\mu$ over $\oplus$)

- Trivially, a monad of semigroups delivers semigroups: for any $X$, $(TX, \oplus_X)$ is a semigroup.

- A monad of semigroups is a right near-semiring without additive unit in the right near-semiringy category $([\mathcal{C}, \mathcal{C}], 1, \times, \text{Id}, \cdot)$. 

Monads of semigroups (ctd)

- Examples on \textbf{Set}: the nonempty list monad, the nonempty finite multiset monad, the nonempty finite powerset monad etc.
- A non-example: the leaf-labelled binary tree monad (associativity fails).

- I argue that monads of semigroups are canonical in several significant ways.
- And they fit the purpose of abstracting notions of associative nondeterminism without no-option choice.
Algebras (＝ handlers)

- *Algebras* of a monad of semigroups \((T, \oplus)\) are semigroups, i.e., there is a functor
  \(F : EM(T) \to \text{Semigrp}(C)\) preserving carriers:

\[
\begin{array}{ccc}
EM(T) & \xrightarrow{F} & \text{Semigrp}(C) \\
\downarrow U & & \downarrow U \\
C & \xleftarrow{\mu} & \end{array}
\]

- \(F(A, \alpha)\) is \((A, \oplus)\) where

\[
A \times A \xrightarrow{\oplus} A = A \times A \xrightarrow{\eta_A \times \eta_A} TA \times TA \xrightarrow{\oplus_A} TA \xrightarrow{\alpha} A
\]

- A monad \(T\) whose algebras are semigroups, i.e., that comes with a functor \(F\) as above, is a monad of semigroups:
  - \(\oplus_X\) is given by \(F(TX, \mu_X)\).
  - This correspondence is a bijection.
The Kleisli category

- The *Kleisli category* of a monad of semigroups \((\mathbb{T}, \oplus)\) has a semigroup structure \(\oplus_{X,Y}\) on every homset \(\text{Kl}(\mathbb{T})(X, Y)\) given by

\[
\begin{align*}
  \left( X \xrightarrow{k} T Y \right) \oplus_{X,Y} \left( X \xrightarrow{k'} T Y \right) &= X \xrightarrow{\langle k, k' \rangle} T Y \times T Y \xrightarrow{\oplus_Y} T Y
\end{align*}
\]

- This does not quite give enrichment of \(\text{Kl}(\mathbb{T})\) over \(\text{Semigrp}(\mathbb{C})\):
  - it is the case that \(\ell \bullet (k \oplus k') = (\ell \bullet k) \oplus (\ell \bullet k')\);
  - but \((k \oplus k') \bullet \ell = (k \bullet \ell) \oplus (k' \bullet \ell)\) fails in general, we only have \((k \oplus k') \bullet J f = (k \bullet J f) \oplus (k' \bullet J f)\).
The initial monad of semigroups

- The initial monad of semigroups $(T_0, \oplus_0)$ is the free semigroup delivering monad:
  - for any $X$, $(T_0 X, \oplus_0 X)$ is the free semigroup on $X$.
- In $\mathbf{Set}$, this is the nonempty list monad $(T_0 X = X^+)$ (as normal-form leaf-labelled binary trees are exactly nonempty lists).
- The functor $F_0 : EM(T_0) \to \text{Semigrp}(C)$ is an isomorphism:
  - the $T_0$-algebra structure $\alpha : T_0 A \to A$ for a semigroup $(A, \sqcup)$ is given by the unique semigroup morphism from the free semigroup $(T_0 A, \oplus_0 A)$, sending $\eta_0 A$ to $\text{id}_A$:

```
A \xrightarrow{\eta_0 A} T_0 A \xleftarrow{\oplus_0 A} T_0 A \times T_0 A
  \alpha \downarrow \quad \sqcup \quad \alpha \times \alpha
  \downarrow \alpha
A \xleftarrow{\sqcup} A \times A
```
One can also talk about strong monads of semigroups.

A *strong monad of semigroups* is a monad of semigroups \((\mathbb{T}, \oplus)\) with a natural transformation \(\sigma_{\Gamma,X} : \Gamma \times T X \to T (\Gamma \times X)\) satisfying the laws of a monad strength and additionally

\[
\begin{align*}
\Gamma \times (T X \times T X) & \xrightarrow{\Gamma \times \oplus X} \Gamma \times T X \\
\langle \Gamma \times \text{fst}, \Gamma \times \text{snd} \rangle & \\
(\Gamma \times T X) \times (\Gamma \times T X) & \\
\sigma_{\Gamma,X} \times \sigma_{\Gamma,X} & \\
T (\Gamma \times X) \times T (\Gamma \times X) & \xrightarrow{\oplus_{\Gamma \times X}} T (\Gamma \times X)
\end{align*}
\]

On \textbf{Set}, every monad of semigroups is uniquely strong.
Monads of monoids

- A monad of monoids is a monad of semigroups \((T, \oplus)\) with an additional nat. transf. \(o : 1 \to TX\) such that

\[
\begin{array}{ccc}
1 \times TX & \overset{o_X \times T X}{\longrightarrow} & TX \times TX \\
\lambda_TX & \downarrow & \oplus_X \\
TX & \rightarrow & TX
\end{array}
\]

\[
\begin{array}{ccc}
TX \times 1 & \overset{TX \times o_X}{\longrightarrow} & TX \times TX \\
\rho_TX & \downarrow & \oplus_X \\
TX & \rightarrow & TX
\end{array}
\]

\[
\begin{array}{ccc}
1 & \overset{o_TX}{\longrightarrow} & T(TX) \\
1 & \overset{o_X}{\longrightarrow} & TX
\end{array}
\]

\[
o \oplus c = c
\]

\[
c \oplus o = c
\]

\[
\mu o = o
\]

(left and right unital laws wrt \(\oplus\), left zero law wrt \(\mu\)).

- Monads of monoids have properties similar to those of monads of semigroups.

- In \(\textbf{Set}\), the initial monad of monoids is the list monad \((TX = X^*)\).
A *monad of commutative semigroups* is a monad of semigroups \((T, \oplus)\) such that

\[
T X \times T X \xrightarrow{\sigma_{T X, TX}} T X \times T X
\]

\[
\begin{align*}
\oplus_X & \quad \oplus_X \\
T X & \quad T X
\end{align*}
\]

\[
c \oplus c' = c' \oplus c
\]

(commutativity of \(\oplus\))

- The initial monad of commutative semigroups delivers free commutative semigroups.
- In **Set**, this is the finite nonempty multiset monad \((T_0 X = M_{\geq 0}^f X)\).
Monads of bands and semilattices

- A monad of bands [semilattices] is a monad of [commutative] semigroups \((T, \oplus)\) such that

\[
\begin{array}{c}
T X \\
\downarrow \Delta_{T X} \\
T X \times T X \\
\downarrow \Theta_X \\
T X
\end{array}
\]

\[c \oplus c = c\]

(idempotence of \(\oplus\))

- The initial monad of bands [semilattices] delivers free bands [resp. semilattices].

- In \textbf{Set}, this is the square-free nonempty list monad \((T_0 X = \{xs : X^+ \mid xs \text{ square-free}\})\) [resp. the nonempty finite powerset monad \((T_0 X = \mathcal{P}^{>0}_f X)\)].
More exotic designs (1)

- Many other equations make sense.
- We can consider, e.g., monads of semigroups satisfying

\[
T X \times (T X \times T X) \xrightarrow{T X \times \sigma_{T X, T X}} T X \times (T X \times T X) \xrightarrow{\oplus_X^{(3)}} T X \xrightarrow{T X \times \sigma_{T X, T X}} T X \times (T X \times T X) \xrightarrow{\oplus_X^{(3)}} T X
\]

\[
c \oplus (c' \oplus c'') = c \oplus (c'' \oplus c')
\]

(right commutativity)

- In \textbf{Set}, the initial special monad of this type is given by

\[
T_0 X = X \times M_f X
\]

(the leftmost outcome is singled out, the order of the others is not observable).
More exotic designs (2)

- Or we can consider, e.g., monads of bands (i.e., idempotent semigroups) that additionally satisfy

\[
T X \times T X \xrightarrow{\Delta_{T X \times T X}} (T X \times T X) \times T X \xrightarrow{\alpha_{T X, T X, T X}} T X \times (T X \times T X) \xrightarrow{T X \times \sigma_{T X, T X}} T X \times (T X \times T X) \\
\oplus \quad \oplus^{(3)}
\]

\[
c \oplus (c' \oplus c) = c \oplus c'
\]

(left regularity)

- In **Set**, the initial special monad of this type is

\[
T_0 X = \{xs : X^+ \mid xs \text{ duplicate-free}\}
\]

(only the leftmost occurrence of any outcome is observable).
More exotic designs (3)

- Or we can consider, e.g., monads of rectangular bands, i.e., monads of bands that additionally satisfy

\[
\begin{align*}
T X \times T X &\xrightarrow{\Delta_{T X \times T X}} T X \times (T X \times T X) \\
&\xrightarrow{T X \times (! \times T X)} T X \times (1 \times T X) \\
&\xrightarrow{T X \times \lambda_{T X}} T X \times T X
\end{align*}
\]

\[
\oplus_X^{(3)} \quad \oplus_X
\]

\[
c \oplus (c' \oplus c'') = c \oplus c''
\]

- The initial monad of rectangular bands is \(T_0 X = X \times X\) (only the leftmost and rightmost outcome of a nondeterministic computation are observable).
Recipe

- To define the class of monads to work with:
  - choose the intended algebraic operations and equations,
  - state them for the underlying functor,
  - add distributivity wrt. the multiplication.

- To work out the initial special monad,
  - see if your algebraic theory has (unique) normal forms.

- Particularly relevant for nondeterminism are special semigroups and monoids.
Conclusions

- The algebraic theories viewpoint helps see and map the spectrum of viable choices.
- Notions of associative nondeterminism (where temporal order of choices is not observable) are all about semigroups and monoids and their special cases.
Related work

- Hinze, ICFP 2000
- Martin, Gibbons, manuscript, 2002 — discussions of what the laws of MonadPlus should be
- Goncharov, Schröder, Mossakowski, CALCO 2009
- Coumans, Jacobs, Quantum Physics and Linguistics, 2013 — additive monads (Kleisli category \textbf{Semigrp}(C)-enriched)
- Rivas, Jaskelioff, Schrijvers, PPDP 2015 — MonadPlus and AlternativePlus as semirings in categories of endofunctors