Introduction to Linear Logic

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Outline

Introduction
  Classical Logic

Linear Logic
  Background
  Syntax
  Informal Semantics
  Common Variants

Some Random Applications
  Synthesis of Web Services
  Theorem Provers
  Quantum Programming
Structural rules of weakening and contraction:

- $\Gamma, \Delta \vdash \Delta$
- $\Gamma, \Delta, A \vdash \Delta$
- $\Gamma, A, A \vdash \Delta$
- $\Gamma, A, A, \Delta \vdash \Delta$

- A fact can be used freely as many times as needed.
- What can be concluded from $A, A \Rightarrow B, A \Rightarrow C$?
Capitalistic Point of View

“A implies B” should be read as “give me as many A as I might need and I get you B”.
Linear Logic

- Proposed by Jean-Yves Girard in 1987
- Denies the structural rules of weakening and contraction
- Assumptions as consumable resources

There are other resource-oriented logics:
- Relevance logic
- Lambek calculus
Classical Linear Logic Sequent Calculus
Multiplicative conjunction

- Operator: $\otimes$
- Denotes simultaneous occurrence of resources
- Unit: 1 ($1 \otimes A = A = A \otimes 1$)

Multiplicative disjunction

- Operator: $\otimes$
- Represents simultaneous goals that must be reached
- Unit: ⊥
Linear Connective Soup

Additive conjunction

- Operator: $\&$
- Internal choice, represents alternative occurrence of resources
- Unit: $\top$

Additive disjunction

- Operator: $\oplus$
- External choice, represents a choice over which one has no control
- Unit: 0
The proposition $A \rightarrow B$ consumes resource $A$ to reach resource $B$.

- Reuse is allowed for propositions using “of course” operator: $. (contraction)$
- A fact can be weakened by additional conclusion $?A$ (“why not” operator).
Negation

Atomic formula:
- Negation of $A$ is $A^\perp$
- Negation of $A^\perp$ is $A$

Negation of non-atomic formulae is defined using the De Morgan rule:
- $(A \otimes B)^\perp = A^\perp \otimes B^\perp$
- $(A \otimes B)^\perp = A^\perp \otimes B^\perp$
- $(A \& B)^\perp = A^\perp \oplus B^\perp$
- $(A \oplus B)^\perp = A^\perp \& B^\perp$

Linear implication $A \rightarrow B$ is defined as a shorthand for $A^\perp \otimes B$
Entree:
▶ quiche lorraine ou
▶ saumon fume

et Plat:
▶ pot-au-feu ou
▶ filet de canard

et
▶ Fruit selon saison (banane ou raisin ou oranges ou ananas) ou
▶ Dessert au choix (mister, glace, tarte aux pommes)

75FF ⊗ (Q∩S) ⊗ (P∩F) ⊗ ((B ⊕ R ⊕ O ⊕ A) & (M & G & T))
Common Variants of Linear Logic

- **MLL - Multiplicative LL**
  - Only $\otimes$ and $\otimes$ are allowed
  - Decidable, NP-complete (Max I. Kanovich)

- **MALL - Multiplicative Additive LL**
  - Adds additive connectives ($\oplus$, $\&$) to MLL
  - Decidable, PSPACE-complete (P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar)

- **MELL - Multiplicative Exponential LL**
  - Adds exponential operators to MLL
  - The decision problem is open

- **MAELL - Multiplicative Additive Exponential LL**
  - Undecidable

There are also first- and higher-order extensions of LL.
Synthesis of Web Services

- How to find solutions effectively?
- General description of a service:

\[
\text{resources} \otimes \text{constraints} \otimes \text{preconditions} \otimes \text{inputs} \\
\quad \rightarrow (\text{effects} \otimes \text{outputs}) \oplus \text{exception}
\]

- Example:

\[
\text{have\_processing\_time} \otimes \text{x\_is\_known} \otimes \text{y\_is\_known} \\
\quad \rightarrow \text{z\_is\_known} \oplus \text{exception}
\]
The extralogical axiom describing a service:

\[ a \otimes i \rightarrow (f \otimes o) \oplus e \]

Where:
- \( a \) - multiplicative conjunction of resources, constraints and preconditions
- \( i \) - multiplicative conjunction of inputs
- \( f \) - multiplicative conjunction of effects
- \( o \) - multiplicative conjunction of outputs
- \( e \) - exception
Admissible derivation rule:

\[
\begin{array}{c}
\vdash a \otimes !i \rightarrow f \otimes !o \\
\Gamma \vdash a \\
\Sigma \vdash !i \\
\hline
\Gamma, \Sigma \vdash f \otimes !u
\end{array}
\]

Where \( u \) consists of something from \( i, o \).

What about the complexity of the proof-search?
Theorem Provers: linprove

- Searches a cut-free proof of the given two-sided sequent of first-order linear logic
- Written in SICStus Prolog (≈ 1400 LOC)
- Online demo: http://bach.istc.kobe-u.ac.jp/llprover/
- Author: Naoyuki Tamura

Example proof:

```
-------  Ax
a --> a
----------  L/\1
a/\b --> a
```
Linear Logic Theorem Provers

- linprove - prover for propositional linear logic
  - Written in Scheme (≈ 4000 LOC)
  - Proof Strategies in Linear Logic. 1994
  - Author: Tanel Tammet

- Forum
  - Based on intuitionistic linear logic
  - Designed by Dale Miller

- RAPS - Resource-Aware Planning System
  - LL planner to support reasoning over Web service composition problems in propositional and first-order LL
  - Written in Java
  - Peep Küngas
QML - Quantum Meta Language

QML is a functional language for quantum computations developed by T. Altenkirch and J. Grattage.

- Based on strict linear logic
  - SLL is an extension of LL with structural rule of contraction.
- Quantum control and quantum data
- Important issue: control of decoherence
Summary

- “I’m not a linear logician.” – Girard
- Linear Logic provides useful tools for different applications
- Expressive power vs complexity