Polynomial Solutions of Recurrence Relations

O. Shkaravska M. van Eekelen A. Tamalet

Digital Security, ICIS
Radboud Universiteit Nijmegen

Tallinn, 9 June 2009

Sponsored by the Netherlands Organisation for Scientific Research (NWO), project Amortized Heap Space Usage Analysis (AHA), grantnr. 612.063.511.

Outline

1. Motivation: recurrences in program analysis and math.
2. Our Contribution: multi-step quadratic recurrences for 1-variable polynomials

Size/Resource Recurrences

tails : $L_n(\alpha) \rightarrow L_{f(n)}(\alpha)$

tails l = match l with
Nil ⇒ Nil
Cons(hd, tl) ⇒ l ++ tails(tl)

$$\begin{array}{rl} & \vdash f(0) = 0 \\ n \geq 1 & \vdash f(n) = n + f(n-1) \end{array}$$

Linear Recurrences for 1-variable Functions

$$\begin{array}{rl} & \vdash f(0) = 0 \\ n \geq 1 & \vdash f(n) = n + f(n-1) \end{array}$$

Homogenisation by symbolic differentiation:

$f'(n) := f(n) - f(n-1)$,
$f'(n) = 1 + f'(n-1)$, $f'(1) = 1 - 0 = 1$

$f''(n) := f'(n) - f'(n-1)$
$f''(n) = f''(n-1)$, $f''(2) = f'(2) - f'(1) = 1$,
$f''(n) = 1$

$f'''(n) = 0$. If the solution is a polynomial, then the degree is 2:
$f(n) = an^2 + bn + c$

$$f(0) = 0, f(1) = 1, f(2) = 3 \Longrightarrow c = 0, a = b = \frac{1}{2}$$

Non-linear recurrences: math. challenge

No general theory, as for linear recurrences. We consider polynomial solutions for such recurrences.

$$\begin{cases} p(n_1, 0) & = 4n_1^2 \\ p(0, n_2) & = 4n_2^2 \\ p(n_1, n_2) & = (p(n_1 - 1, n_2) + n_1 - (p(n_1, n_2 - 1) + n_2))^2 \\ & +17n_1 n_2 \end{cases}$$

We want to know such D, that either $degree(p) := z \leq D$ or D is not a polynomial at all.

If such D is known then we can use MUC or, as above (better!), fit a polynomial by solving SLE and check then if it suits the recurrence.

In the example we take $D = 2$ and obtain $p(n_1, n_2) = 4n_1^2 + 4n_2^2 + 9n_1 n_2$

Multi-step Quadratic Recurrence

t-step Quadratic Recurrence

$$
\begin{aligned}
p(n) = \quad & \alpha_{11}\, p^2(n - r_1) + \\
& \alpha_{12}\, p(n - r_1)\, p(n - r_2) + \alpha_{22}\, p^2(n - r_2) + \\
& \alpha_{13}\, p(n - r_1)\, p(n - r_3) + \alpha_{33}\, p^2(n - r_3) + \\
& \ldots + \\
& \alpha_{t-1,\, t}\, p(n - r_{t-1})\, p(n - r_t) + \alpha_{tt}\, p^2(n - r_t) + \\
& L\Big(p(n - r_1),\, \ldots,\, p(n - r_t) \Big)
\end{aligned}
$$

Our Aim

Find D such that $deg(p) \leq D$

Technicalities: gather coefficients at n^t in the r.h.s.

$$
\begin{array}{ll}
p(n) & = a_z n^z + \ldots + a_1 n + a_0 \\
p(n-r) & = a_z (n-r)^z + \ldots + a_1 (n-r) + a_0 \\
p(n-r_k)p(n-r_l) & = \sum_{0 \le i,j \le z} a_i a_j (n-r_k)^i (n-r_l)^j
\end{array}
$$

$$
p(n) = \sum_{1 \le k \le l \le t} \alpha_{kl} \sum_{0 \le i,j \le z} a_i a_j
\left(K_{k,l}^{i,j,\,-0} n^{i+j} + K_{k,l}^{i,j,\,-1} n^{i+j-1} + \ldots + K_{k,l}^{i,j,\,-(i+j)} \right)
$$

where

$$K_{k,l}^{i,j,\,-0} = 1$$

$\ldots$

$$K_{k,l}^{i,j,\,-m} = \sum_{\gamma=0}^{m} C_i^{\gamma} C_j^{m-\gamma} (-r_k)^{\gamma} (-r_l)^{m-\gamma}$$

Cancellation equations for multi-step recurrence

t	The coefficient at n^t	Cancellation
$2z$	$v_0 = \quad a_z a_z \Sigma_{1\leq k\leq l\leq t} K_{k,\, l}^{z,\, z,\, -0} \alpha_{kl}$	$2z > z \Rightarrow$ $v_0 = 0$
$2z-1$	$v_1 = \quad a_z a_z \Sigma_{1\leq k\leq l\leq t} K_{k,\, l}^{z,\, z,\, -1} \alpha_{kl} +$ $a_{z-1} a_z \Sigma_{1\leq k\leq l\leq t} K_{k,\, l}^{z-1,\, z,\, -0} \alpha_{kl} +$ $a_z a_{z-1} \Sigma_{1\leq k\leq l\leq t} K_{k,\, l}^{z,\, z-1,\, -0} \alpha_{kl}$	$2z-1 > z \Rightarrow$ $v_1 = 0$
	$\ldots$	
$2z-m$	$v_m = \sum_{i,j\ 0\leq i+j\leq m}$ $a_{z-i} a_{z-j} \Sigma_{1\leq k\leq l\leq t} K_{k,\, l}^{z-i,\, z-j,\, -(m-(i+j))} \alpha_{kl}$	$2z-m > z \Rightarrow$ $v_m = 0$

Cancellation conditions form a homogeneous linear system w.r.t α_{kl}

A homogeneous linear system: $A\bar{x} = 0$

- Folklore: if the amount of equations is equal to the amount of variables then the only solution is zero: $\bar{x} = \bar{0}$,
- in fact: if $rank(A) =$"the amount of variables", then $\bar{x} = \bar{0}$.

We note:

- the first $m + 1$ cancellation conditions form a homogeneous system w.r.t. α_{kl}: $v_0 = 0$, $v_1 = 0$, ... v_m;
- $z > m$ implies $2z - m > z$ then all the $m + 1$ cancellation conditions must hold simultaneously, i.e. they form this system of $m + 1$ equations;
  our coefficients α_{kl} form exactly its solution;

Cancellation conditions form a homogeneous linear system w.r.t α_{kl}

We note (continue):

- Let $z > \#\{\alpha_{kl}\} - 1$ ("*the amount of coefficients*" α_{kl} - 1). Then we have a homogeneous system where the amount of equations, $\#\{\alpha_{kl}\}$ is equal to the amount of variables. Folklore: "it implies" that the system has only zero solution, i.e. all the coefficients α_{kl} are zero and the recurrence is linear.
- The real problem: we have to show that the RANK of the matrix of the system $v_m = 0$, where $0 \leq m \leq \#\{\alpha_{kl}\} - 1$, is equal to $\#\{\alpha_{kl}\}$;
  It is difficult: its determinant after $m \geq 4$ is really weird, with the unknown coefficients a_i (at the moment I do not know if you can get rid of them).

Cancellation conditions form a homogeneous linear system w.r.t α_{kl}

What we do know:

- for $1 \leq m \leq 3$ the coefficients for m are expressible via the coefficients for $m - 1$,
- using this, we show that for $m \leq 3$ the unknown coefficients a_i may be omitted,
- the determinant for the two-step recurrence over $p(n - r_1)$ and $p(n - r_2)$ with $m = 2$ is non-zero, that is the homogeneous system over $\alpha_{11}, \alpha_{12}, \alpha_{22}$ has a solution and it is zero, i.e. the recurrence is linear.

Cancellation conditions form a homogeneous linear system w.r.t α_{kl}

Theorem 1

If a quadratic two-step recurrence has a polynomial solution then its degree $z \leq 2$

If $z > 2$ then $2z - 2 > z$ and the cancellation conditions for $m = 0, 1, 2$ must hold. Moreover, the determinant of the matrix of the corresponding linear system is non-zero.
Therefore, all the coefficients $\alpha_{11}, \alpha_{12}, \alpha_{22}$ are zero and the recurrence is linear.

Idea: coefficients at n^{2z-m} are polynomials on z

$\#\{\alpha_{kl}\} \geq 4$: reduce to a system with simpler determinants.

We want to obtain the presentation
$v_m(z) = A_{mm}z^m + \ldots + A_{m0} = 0$, from which follows:

$$\begin{cases} \textit{either} \;\; A_{mm} \neq 0 \Rightarrow z \leq |\frac{A_{0m}}{A_{mm}}| \\ \textit{or} \;\; A_{mm} = 0 \Rightarrow \text{we have a simpler equation instead of} \\ v_m = 0 \end{cases}$$

Computing A_{mi} for $v(z) = A_{mm}z^m + \ldots + A_{m0}$

Lemma 1

The coefficient at the highest degree of z in $v_m(z)$ is

$$A_{mm} = \frac{(-r_k - r_l)^m}{m!} a_z a_z$$

To our aim: find D, such that $z \leq D$

Theorem 1

$z < \#(\alpha_{kl})$ or

$z \leq \dfrac{|A_{d_0 0}|}{|A_{d_0 d_0}|}$, where $d_0 = \min_{1 \leq d \leq \#(\alpha_{kl})}\{A_{dd} \neq 0\}$.

Suppose that $z \geq \#(\alpha_{kl})$. Then all $v_m = 0$, where $0 \leq m \leq \#(\alpha_{kl})$, hold.

Suppose that d with the property $A_{dd} \neq 0$ does not exist, that is for all $1 \leq m \leq \#(\alpha_{kl})$ we have $A_{mm} = 0$.

To our aim: find D, such that $z \leq D$

From what follows that $\sum_{1 \leq k \leq l \leq t} (r_k + r_l)^m \alpha_{kl} = 0$ for $0 \leq m \leq \#(\alpha_{kl})$.

The determinant of this system is Vandermonde determinant.

If all the sums $r_k + r_l$ are different, then the determinant is non-zero. Therefore, the system has only the zero solution, which means that the recurrence is linear.

But this is often not the case: e.g. $\alpha_{13} p(n-1)p(n-3)$ and $\alpha_{22} p(n-2)p(n-2)$.

To our aim: find D, such that $z \leq D$

Def.: $\{R_1, \ldots, R_s\} = \{R | \exists\, r_k\, r_l.\, R = r_k + r_l\}$

$\sum_{i=1}^{s} R_i^m \beta_i = 0$ for $0 \leq m \leq \#(\alpha_{kl})$.

$\beta_i = \Sigma_{r_k + r_l = R_i} \alpha_{kl} = 0$
$(\#\{\alpha_{kl}\} + 1) - 1 + s$ equations over $s + \#\{\alpha_{kl}\}$ variables.

Future Work

- continue with multi-step quadratic recurrences, $t \geq 3$.
- extend to degree $d \geq 2$ recurrences
- extend to recurrences over multivariate polynomial solutions