Concrete Process Categories

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Functional reactive programming

- extension of functional programming
- supports description of temporal behavior
- two key concepts:
  - time-dependent type membership
  - special type constructors:
    - time-varying values
    - events
- Curry–Howard correspondence to temporal logic:
  - time-dependent trueness
  - special operators:
    - will always hold
    - will eventually hold
Categorical models of simply typed calculus

- models are cartesian closed categories with coproducts
- use of basic category structure:

\[
\begin{array}{c}
\text{type} \quad \text{operation} \quad \text{morphism} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{type} \quad \text{object} \quad \text{object}
\end{array}
\]

- use of CCCC structure:

- product type \( \tau_1 \times \tau_2 \) \( \longrightarrow \) \( A \times B \) product
- sum type \( \tau_1 + \tau_2 \) \( \longrightarrow \) \( A + B \) coproduct
- function type \( \tau_1 \rightarrow \tau_2 \) \( \longrightarrow \) \( B^A \) exponential
- unit type \( 1 \) \( \longrightarrow \) \( 1 \) terminal object
- empty type \( 0 \) \( \longrightarrow \) \( 0 \) initial object
Categorical models of FRP

- ingredients:
  - totally ordered set \((T, \leq)\) time scale
  - simple types and functions
- product category \(\mathcal{B}^T\) models FRP types and operations with indices denoting inhabitation times:
Meanings of FRP type constructors

- general picture:

  simple type constructors \[\square\] and \[\Diamond\] \[\rightarrow\] CCCC structure of \(B^T\)

- CCCC structure of \(B^T\) from CCCC structure of \(B\) with operations working pointwise

- functors \[\square\] and \[\Diamond\] defined as follows:

  \[
  (\square A)(t) = \prod_{t' \in [t, \infty)} A(t')
  \]
  \[
  (\Diamond A)(t) = \bigsqcup_{t' \in [t, \infty)} A(t')
  \]
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From “until” to processes

- more temporal operators from linear-time temporal logic:
  - ▶ strong “until”
  - ▶ weak “until”
- semantics given by functors ▶ and ▶:

\[
(A \triangleright B)(t) = \prod_{t' \in [t, \infty)} \left( \prod_{t'' \in [t, t')} A(t'') \times B(t') \right)
\]

\[
(A \triangleright\triangleright B)(t) = (A \triangleright B)(t) + \prod_{t' \in [t, \infty)} A(t')
\]

- FRP analogs of “until” proofs are processes:
  - normally finite-length time-varying value plus terminal event
  - in the case of ▶ also nontermination possible
Applications of processes

- stereo playback with different guarantees:
  \[(\mathbb{R} \times \mathbb{R}) \triangleright 1\] none
  \[(\mathbb{R} \times \mathbb{R}) \triangleright 1\] termination
  \[(\mathbb{R} \times \mathbb{R}) \triangleright 0\] nontermination

- stereo playback with additional information:
  \[(\mathbb{R} \times \mathbb{R}) \triangleright (1 + 1)\] reason of termination
    (end of track vs. abort)

- alternating stereo/mono playback with different guarantees:
  \[\nu\sigma. (\mathbb{R} \times \mathbb{R}) \triangleright \mathbb{R} \triangleright \sigma\] nontermination
  \[\nu\sigma. (\mathbb{R} \times \mathbb{R}) \triangleright \mathbb{R} \triangleright \sigma\] switch, nontermination
  \[\nu\sigma. (\mathbb{R} \times \mathbb{R}) \triangleright (1 + \mathbb{R} \triangleright (1 + \sigma))\] none
  \[\nu\sigma. (\mathbb{R} \times \mathbb{R}) \triangleright (1 + \mathbb{R} \triangleright (1 + \sigma))\] switch
  \[\mu\sigma. (\mathbb{R} \times \mathbb{R}) \triangleright (1 + \mathbb{R} \triangleright (1 + \sigma))\] termination
Processes as the core concept of FRP

- introduction of processes increases expressiveness
- processes cover time-varying values and events as special cases:

\[ \Box A \Leftrightarrow A \rightarrow 0 \]
\[ \Diamond A \Leftrightarrow 1 \rightarrow A \]
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An example program component

- looks for the next key press up to a certain timeout
- emits a value of type $\diamond (\text{Key} + 1)$ when it starts:
  
  **Case 1** key press before timeout:

  $\nu_1(K) @ t_k$

  **Case 2** no key press before timeout:

  $\nu_2(tt) @ t^*$
A noncausal operation

- hypothetical polymorphic operation $d$ from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond \tau_1 + \Diamond \tau_2$:
  
  \begin{align*}
  \nu_1(x) \circ t' & \mapsto \nu_1(x \circ t') \\
  \nu_2(y) \circ t' & \mapsto \nu_2(y \circ t')
  \end{align*}

- applying $d$ to the output of the key press listener gives value of type $\Diamond \text{Key} + \Diamond 1$:
  
  key press before timeout $\nu_1(K \circ t_k)$
  no key press before timeout $\nu_2(tt \circ t^*)$

- tells us immediately if the user will press a key before the timeout

- so $d$ cannot exist
Semantics allow for noncausal operations

- polymorphic operations from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond \tau_1 + \Diamond \tau_2$ modeled by natural transformations $\tau$ with

$$\tau_{A,B} : \Diamond(A + B) \rightarrow \Diamond A + \Diamond B$$

- there is such a $\tau$ (which is even an isomorphism):

$$\bigsqcup_{t' \geq t}(A(t') + B(t')) \cong \bigsqcup_{t' \geq t} A(t') + \bigsqcup_{t' \geq t} B(t')$$

- reason:

semantics do not deal with time-dependent knowledge about values
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Knowledge-aware semantics

- replace category $B^T$ by category $B^I$ where
  \[ l = \{(t, t_o) \in T \times T \mid t \leq t_o\} \]

- dealing with knowledge at $t_o$:

\[
\begin{align*}
\text{knowledge type} & \quad \rightarrow \quad A(t, t_o) \\
\text{knowledge transformation} & \quad \rightarrow \quad f(t, t_o) \\
\text{knowledge type} & \quad \rightarrow \quad B(t, t_o)
\end{align*}
\]

- $(A \triangleright B)(t, t_o)$ defined as follows:

\[
\prod_{t' \in [t, t_o]} \left( \prod_{t'' \in [t, t']} A(t'', t_o) \times B(t', t_o) \right) + \prod_{t' \in [t, t_o]} A(t', t_o)
\]
Compatibility of knowledge transformations

- Knowledge transformations may be incompatible
- Extend set $I$ to category $\mathcal{I}$ by adding morphisms
  \[ (t, t_o, t_o') : (t, t_o') \to (t, t_o) \]
  for $t \leq t_o \leq t_o'$
- Replace product category $\mathcal{B}^I$ by functor category $\mathcal{B}^\mathcal{I}$
- Objects $A(t, t_o, t_o')$ model knowledge reduction
- Morphisms of $\mathcal{B}^\mathcal{I}$ are natural transformations
- Means that knowledge transformations are compatible:

```
\begin{array}{ccc}
A(t, t_o) & \overset{A(t, t_o, t_o')}{\leftrightarrow} & A(t, t_o') \\
\downarrow & & \downarrow \\
B(t, t_o) & \overset{B(t, t_o, t_o')}{\leftrightarrow} & B(t, t_o')
\end{array}
```
Upper bounds for occurrence times

- definition of functor $\triangleright$ not directly possible
- introduction of new functor $\triangleright_\_ : \mathcal{T} \rightarrow (\mathcal{B}^\mathcal{I})^{\mathcal{B}^\mathcal{I} \times \mathcal{B}^\mathcal{I}}$
  where $\mathcal{T}$ is the category of $(T, \leq)$
- $\triangleright t_b$ models a process type constructor with upper bound $t_b$ for termination time
- $(A \triangleright t_b B)(t, t_o)$ defined as follows:

$$\begin{cases} 
0 & \text{if } t_b < t \\
\prod_{t' \in [t, t_b]} \left( \prod_{t'' \in [t, t']} A(t'', t_o) \times B(t', t_o) \right) & \text{if } t \leq t_b \leq t_o \\
(A \triangleright B)(t, t_o) & \text{if } t_o < t_b
\end{cases}$$

- $\triangleright (t_b, t'_b)$ models type conversion
Definition of the $\triangleright$-functor

- type constructor $\triangleright$ is the least upper bound of all $\triangleright_{t_b}$-constructors
- functor $\triangleright$ must be a colimit of the functor $\triangleright_{\_}$:
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The shape of the $\triangleright$-functor

**Theorem**

If $(T, \leq)$ has a maximum $t_{\text{max}}$, then $\triangleright \cong \triangleright t_{\text{max}}$.

**Theorem**

If $(T, \leq)$ has no maximum, then $\triangleright \cong \triangleright$. 
Causality ensured

**Theorem**

There are categorical models that do not contain any natural transformation $\tau$ with

$$\tau_{A,B} : \Diamond(A + B) \rightarrow \Diamond A + \Diamond B .$$
Conclusions

- Processes:
  - Result of extending the Curry–Howard correspondence between FRP and temporal logic to cover “until” operators
  - Make FRP more expressive
  - Generalize time-varying values and events nicely

- Knowledge-aware categorical models:
  - Express causality of FRP operations
  - Cannot express liveness constraint of $\triangleright$ for unbounded time

- Ultimate goal is an axiomatic semantics with the following properties:
  - Expresses causality
  - Expresses liveness constraint of $\triangleright$ generally
  - Covers concrete process categories as a special case