Categorical Semantics for Linear Logic

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Based on work by Nick Benton (1994)

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Linear logic

- useful for reasoning about resources
- each proposition must be used exactly once in a proof
- very different from the normal understanding of logic
- classical and intuitionistic variant
- in this talk, only intuitionistic linear logic

Linear logic formulas

- language:

$$F ::= F \otimes F \mid 1 \mid F \,\&\, F \mid \top \mid F \oplus F \mid 0 \mid F \multimap F \mid\, !F$$

- meanings:

$\alpha \otimes \beta$	α and β hold simultaneously
1	nothing holds
$\alpha \,\&\, \beta$	α and β hold (not necessarily simultaneously)
$\top$	tautology
$\alpha \oplus \beta$	α or β holds
0	absurdity
$\alpha \multimap \beta$	if α holds in addition, then β holds
$!\alpha$	α holds arbitrarily often

Linear logic example

- atomic propositions:

e	I have one euro.
$s/p/i$	I get a soup/a pancake/an icecream.

- derived propositions:
  - For four euros, I get a soup and a pancake:

    $$e \otimes e \otimes e \otimes e \multimap s \otimes p$$

  - For two euros, I get a soup or a pancake (my choice):

    $$e \otimes e \multimap s \mathbin{\&} p$$

  - For two euros, I get a pancake or an icecream (cafeteria's choice):

    $$e \otimes e \multimap p \oplus i$$

  - I am the central bank:

    $$!e$$

Linear λ-calculus

- the Curry–Howard analog of intuitionistic linear logic
- values have to be used exactly once:
  - a value can represent the current state of an object
  - changes to the state (destructive updates) expressible as pure functions
- some functions with destructive updates:
  - array update:

$$\iota \otimes \alpha \otimes \mathsf{Array}\ \iota\ \alpha \multimap \mathsf{Array}\ \iota\ \alpha$$

  - opening a file:

$$\mathsf{FileName} \otimes \mathsf{World} \multimap \mathsf{File} \otimes \mathsf{World}$$

  - writing to an opened file:

$$\mathsf{String} \otimes \mathsf{File} \multimap \mathsf{File}$$

  - closing a file:

$$\mathsf{File} \otimes \mathsf{World} \multimap \mathsf{World}$$

Products and coproducts

- intuitionistic (non-linear) logic:
  - finite products for $\wedge$ and $\top$
  - finite coproducts for $\vee$ and $\bot$
- intuitionistic linear logic:
  - finite products for $\&$ and $\top$
  - finite coproducts for $\oplus$ and 0
- seems strange that $\wedge/\top$ and $\&/\top$ are modeled by the same constructions, although they denote quite different things
- however, analogous statements hold for $\wedge/\top$ and $\&/\top$:

$$\alpha \vdash \alpha \wedge \alpha \qquad \alpha \vdash \alpha \,\&\, \alpha$$

$$\alpha \wedge \beta \vdash \alpha \qquad \alpha \,\&\, \beta \vdash \alpha$$

$$\alpha \vdash \top \qquad \alpha \vdash \top$$

Symmetric monoidal structure

- axioms of $\otimes$ and 1:
  - associativity of $\otimes$:

$$(\alpha \otimes \beta) \otimes \gamma \vdash \alpha \otimes (\beta \otimes \gamma)$$
$$\alpha \otimes (\beta \otimes \gamma) \vdash (\alpha \otimes \beta) \otimes \gamma$$

  - commutativity of $\otimes$:

$$\alpha \otimes \beta \vdash \beta \otimes \alpha$$

  - 1 as neutral element:

$$1 \otimes \alpha \vdash \alpha$$
$$\alpha \vdash 1 \otimes \alpha$$

- symmetric monoidal structure for $\otimes$ and 1

Adjunctions

- non-linear logic:
  - cartesian closed structure for $\wedge$ and $\rightarrow$
  - $-^B$ defined as right-adjoint of $- \times B$
  - corresponds to equivalence of

$$\alpha \wedge \beta \vdash \gamma$$

    and

$$\alpha \vdash \beta \rightarrow \gamma$$

- linear logic:
  - symmetric monoidal closed structure for $\otimes$ and $\multimap$
  - $B \multimap -$ defined as right-adjoint of $- \otimes B$
  - corresponds to equivalence of

$$\alpha \otimes \beta \vdash \gamma$$

    and

$$\alpha \vdash \beta \multimap \gamma$$

Structure for !

- symmetric lax monoidal functor structure:

$$!\alpha \otimes !\beta \vdash !(\alpha \otimes \beta)$$
$$1 \vdash !1$$

- comonad structure:

$$!\alpha \vdash \alpha$$
$$!\alpha \vdash !!\alpha$$

- commutative comonoid structure:

$$!\alpha \vdash !\alpha \otimes !\alpha$$
$$!\alpha \vdash 1$$

- some additional coherence conditions

Linear and non-linear models

- for now:
  - non-linear logic with only $\wedge$, $\top$, and $\rightarrow$
  - linear logic with only $\otimes$, 1, and $\multimap$
- categorical models:

  non-linear logic cartesian closed category:

  $$(\mathcal{C}, \times, 1, \rightarrow)$$

  linear logic symmetric monoidal closed category:

  $$(\mathcal{L}, \otimes, I, \multimap)$$

- beware:

  $$\text{proposition } \top \triangleq \text{object } 1$$

  $$\text{proposition } 1 \triangleq \text{object } I$$

Interaction

- symmetric lax monoidal adjunction $(F, \varphi, \psi) \dashv (G, \upsilon, \nu)$ between $(\mathcal{L}, \otimes, I)$ and $(\mathcal{C}, \times, 1)$:
  - adjunction $F \dashv G$ between $\mathcal{L}$ and $\mathcal{C}$:

$$F : \mathcal{C} \to \mathcal{L}$$
$$G : \mathcal{L} \to \mathcal{C}$$

  - (F, φ, ψ) and (G, υ, ν) are symmetric lax monoidal functors between $(\mathcal{L}, \otimes, I)$ and $(\mathcal{C}, \times, 1)$:

$$\varphi_{X,Y} : FX \otimes FY \to F(X \times Y) \qquad \psi : I \to F1$$
$$\upsilon_{A,B} : GA \times GB \to G(A \otimes B) \qquad \nu : 1 \to GI$$

  - unit and counit of $F \dashv G$ are monoidal transformations

Isomorphisms

Theorem

If $(F, \varphi, \psi) \dashv (G, \upsilon, \nu)$ is a lax monoidal adjunction, then φ and ψ are isomorphisms.

- inverses:

$$\varphi^{-1}_{X,Y} : F(X \times Y) \to FX \otimes FY$$
$$\varphi^{-1}_{X,Y} = \Phi^{-1}(\upsilon_{FX,FY} \circ (\eta_X \times \eta_Y))$$

$$\psi^{-1} : F1 \to I$$
$$\psi^{-1} = \Phi^{-1}(\nu)$$

- closer relationship between $\times$ and $\otimes$ as well as 1 and I:

$$FX \otimes FY \cong F(X \times Y)$$
$$I \cong F1$$

Derived structure for !

- adjunction $F \dashv G$ gives rise to a comonad $(!, \varepsilon, \delta)$:

$$! : \mathcal{L} \to \mathcal{L} \qquad \delta : FG \to FGFG$$
$$! = FG \qquad \delta = F\eta G$$

- symmetric monoidal functor structures for F and G give rise to a symmetric monoidal functor structure for !
- commutative comonoid structure can be derived:

$$\xi_A : FGA \to FGA \otimes FGA$$
$$\xi_A = \varphi^{-1}_{GA,GA} \circ F\Delta_{GA}$$

$$\chi_A : FGA \to I$$
$$\chi_A = \psi^{-1} \circ F!_{GA}$$

- further coherence conditions follow

More structure

- more structure can be required:
  - finite products in $\mathcal{L}$ for & and $\top$:

$$(\mathcal{L}, \&, \top)$$

  - finite coproducts in $\mathcal{C}$ for $\vee$ and $\bot$:

$$(\mathcal{C}, +, 0)$$

  - finite coproducts in $\mathcal{L}$ for $\oplus$ and 0:

$$(\mathcal{L}, \oplus, 0)$$

- no additional coherence conditions
- interesting properties can still be derived

More isomorphisms

- right-adjoints preserve limits:

$$GA \times GB \cong G(A \,\&\, B)$$
$$1 \cong G\top$$

- consequence:

$$!A \otimes !B \cong !(A \,\&\, B)$$
$$I \cong !\top$$

- left-adjoints preserve colimits:

$$FX \oplus FY \cong F(X + Y)$$
$$0 \cong F0$$

References

P. N. Benton.
A mixed linear and non-linear logic: Proofs, terms and models.
Technical Report UCAM-CL-TR-352, University of Cambridge, Oct. 1994.