Program Analysis of Sequential and Parallel Programs

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(Automatic) Program Analysis

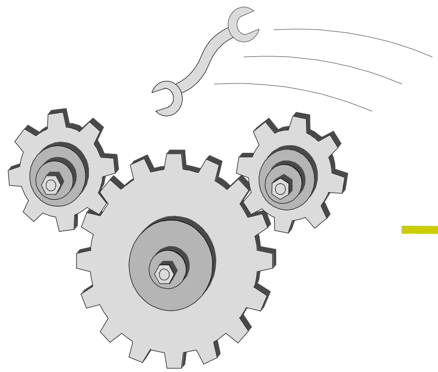
- What?
  - Compute (or prove) properties of programs automatically without executing the program: „static analysis“
- Why?
  - Validation/verification/debugging:
    - find errors in programs
    - guarantee correctness
    - check programs from untrusted source (e.g. Internet)
    - ...
  - Programming systems:
    - optimizing compilers
    - IDEs (integrated development environments)
    - CASE tools (CASE = Computer Aided Software Engineering)
    - ...

Dream of Automatic Analysis

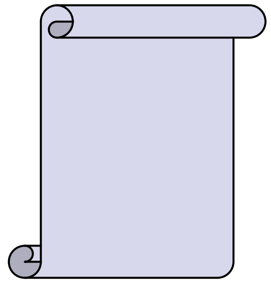
program

```
main()
{ x=17;
  if (x>63)
  { y=17;x=10;x=x+1;}
  else
  { x=42;
    while (y<99)
    { y=x+y;x=y+1;}
    y=11;}
  x=y+1;
  printf(x);
}
```

analyzer



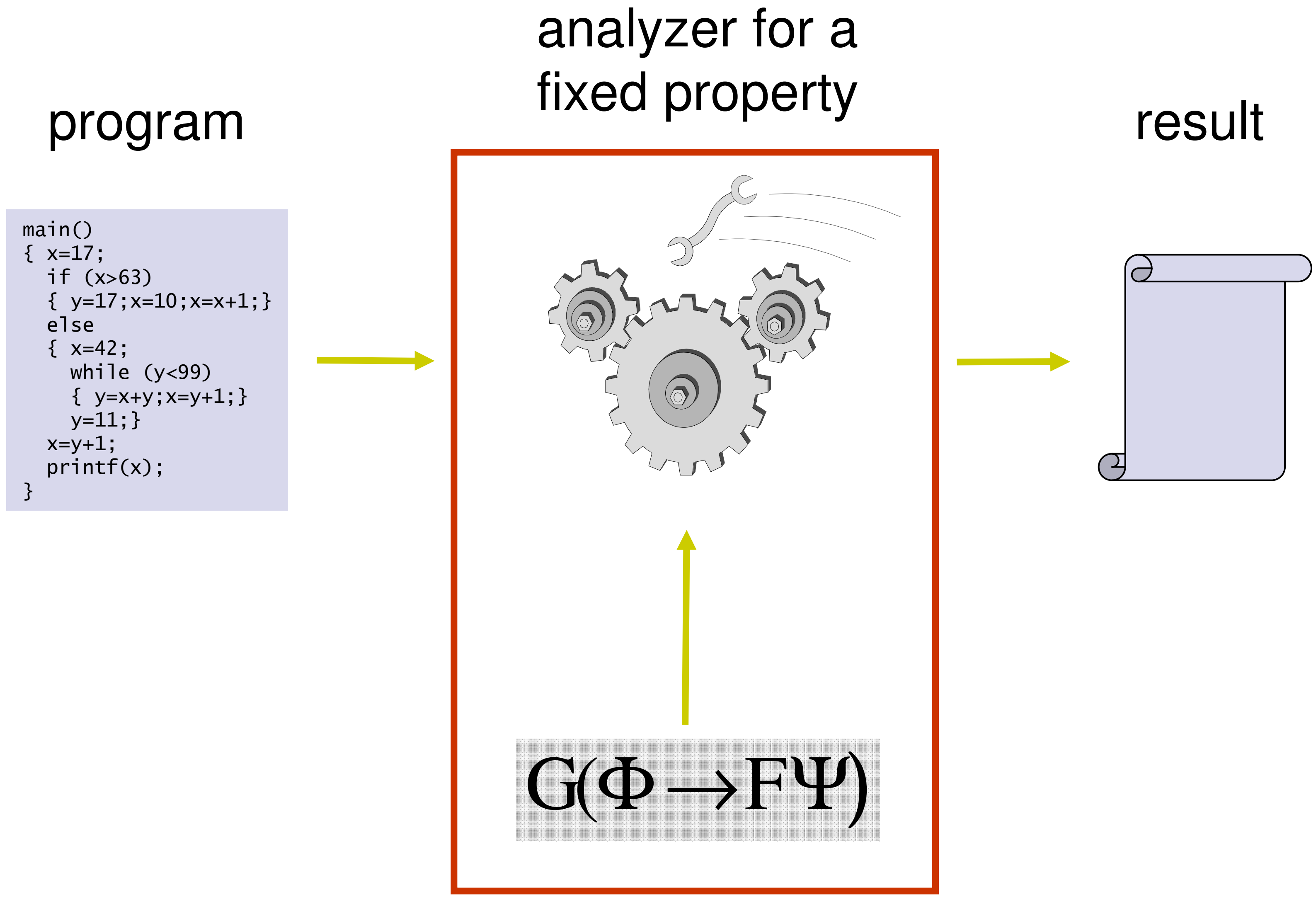
result



$$G(\Phi \rightarrow F\Psi)$$

specification of property

Dream of Automatic Analysis



Fundamental Problem

Rice's Theorem (informal version):

All non-trivial semantic properties of programs from a Turing-complete programming language are undecidable.

Consequence:

For Turing-complete programming languages:

Automatic analyzers of semantic properties, which are both correct and complete are impossible.

☹

What can we do about it?

- Give up „automatic“: interactive approaches:
    - proof calculi, theorem provers, …

- Give up „sound“: ???

- Give up „complete“: approximative approaches:

    - Approximate analyses:
        - type checking, flow analysis, abstract interpretation, …

    - Analyse weaker formalism:
        - model checking, reachability analysis, equivalence- or preorder-checking, …

What can we do about it?

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  - Approximate analyses:
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  - Analyse weaker formalism:
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Overview

- Introduction
- Fundamentals of Program Analysis
- Interprocedural Analysis
- Analysis of Parallel Programs
- Conclusion

Apology for not giving proper credit in these lectures !

Overview

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- Fundamentals of Program Analysis
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From Programs to Flow Graphs

```
main()
{ x=17;
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  { y=17;x=10;x=x+1;}
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```
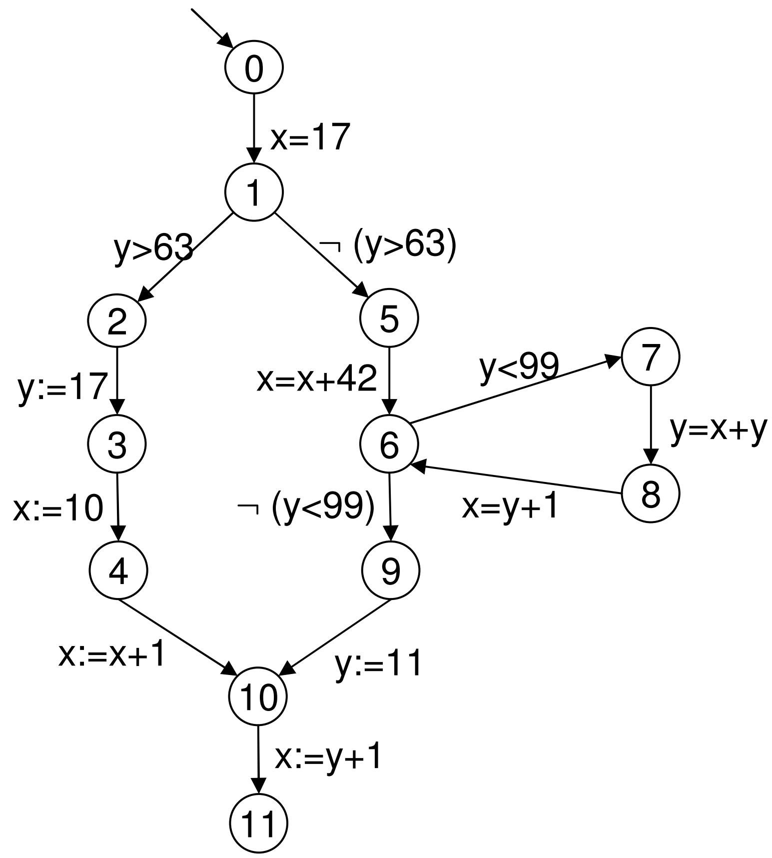



Dead Code Elimination

Goal:

find and eliminate assignments that compute values which are never used

Fundamental problem:

undecidability

$\rightarrow$ use approximate algorithm:

e.g.: ignore that guards prohibit certain execution paths

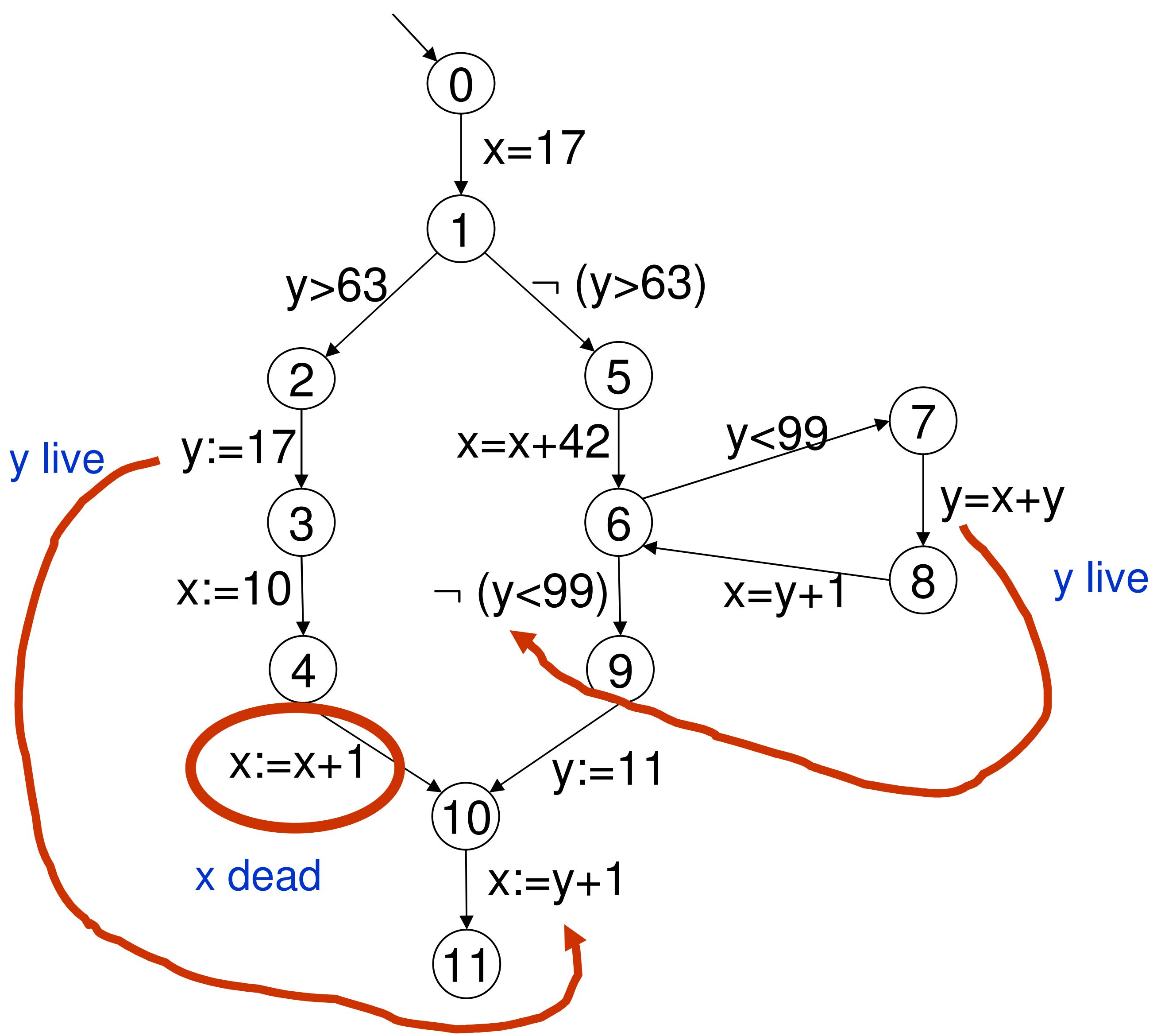
Technique:

1) perform *live variables* analyses:

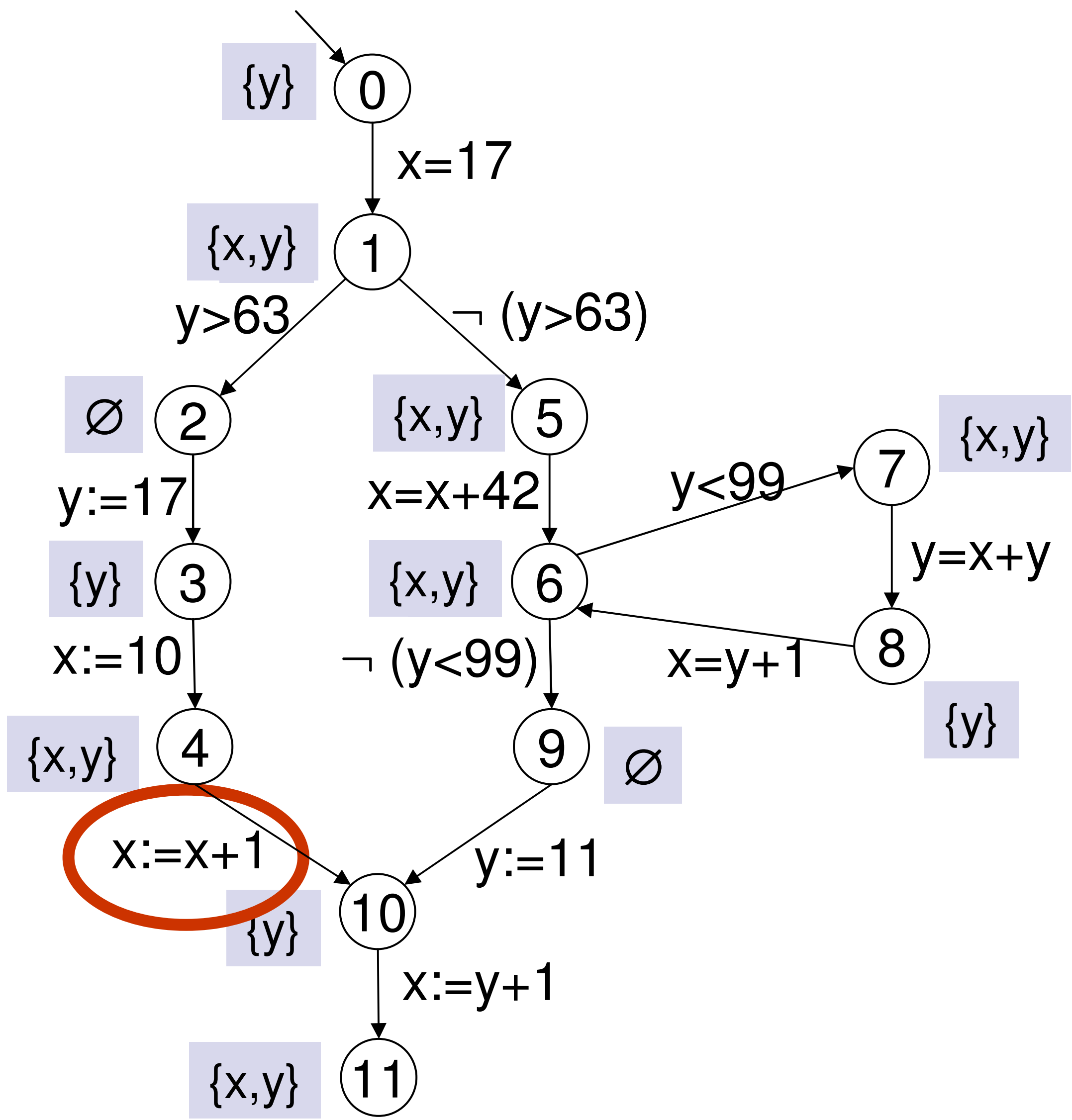
variable x is live at program point u iff
there is a path from u on which x is used before it is modified

2) eliminate assignments to variables that are not live at the target point

Live Variables



Live Variables Analysis



Remarks on Data-Flow Analysis

- Forward vs. backward analyses
- (Separable) bitvector analyses
  - forward: reaching definitions, available expressions, ...
  - backward: live/dead variables, very busy expressions, ...

Complete Lattice

Complete lattice $(L,\sqsubseteq)$:

- a partial order $(L,\sqsubseteq)$ for which the least upper bound, $\sqcup X$, exists for all $X \subseteq L$.

In a complete lattice $(L,\sqsubseteq)$:

- $\sqcap X$ exists for all $X \subseteq L$: $\quad \sqcap X = \sqcup \{ x \in L \mid x \sqsubseteq X \}$
- least element $\perp$ exists: $\quad \perp = \sqcup L = \sqcap \emptyset$
- greatest element $\top$ exists: $\quad \top = \sqcup \emptyset = \sqcap L$

Example:

- for any set A let $P(A) = \{X \mid X \subseteq A\}$ (power set of A).
- $(P(A),\subseteq)$ is a complete lattice.
- $(P(A),\supseteq)$ is a complete lattice.

Interpretation in Approximate Program Analysis

$x \sqsubseteq y$:

- x is more precise information than y.
- y is a correct approximation of x.

$\sqcup X$ for $X \subseteq L$:

the most precise information consistent with all informations $x \in X$.

Remark:

often dual interpretation in the literature !

Example:

lattice for live variables analysis:

- $(P(\mathrm{Var}), \subseteq)$ with Var = set of variables in the program

Specifying Live Variables Analysis by a Constraint System

Compute (smallest) solution over $(L,\sqsubseteq) = (\mathrm{P}(\mathrm{Var}),\subseteq)$ of:

$$V^\#[fin] \sqsupseteq init, \qquad \text{for } fin\text{, the termination node}$$

$$V^\#[u] \sqsupseteq f_e(V^\#[v]), \qquad \text{for each edge } e = (u,s,v)$$

where $init = \mathrm{Var}$,

$f_e:\mathrm{P}(\mathrm{Var}) \rightarrow \mathrm{P}(\mathrm{Var})$, $f_e(x) = x \backslash \mathrm{kill}_e \cup \mathrm{gen}_e$, with

- kill_e = variables assigned at e
- gen_e = variables used in an expression evaluated at e

Specifying Live Variables Analysis by a Constraint System

Remarks:

1. Every solution is „correct“.
2. The smallest solution is called MFP-solution;
   it comprises a value MFP[u] $\in$ L for each program point u.
3. (MFP abbreviates „maximal fixpoint“ for traditional reasons.)
4. The MFP-solution is the most precise one.

Data-Flow Frameworks

Correctness

- generic properties of frameworks can be studied and proved

Implementation

- efficient, generic implementations can be constructed

Questions

- Do (smallest) solutions always exist ?
- How to compute the (smallest) solution ?
- How to justify that a solution is what we want ?

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Knaster-Tarski Fixpoint Theorem

Definitions:

Let $(L,\sqsubseteq)$ be a partial order.

- $f : L \rightarrow L$ is *monotonic* iff $\forall\, x,y \in L : \quad x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.
- $x \in L$ is a *fixpoint* of f iff $f(x)=x$.

Fixpoint Theorem of Knaster-Tarski:

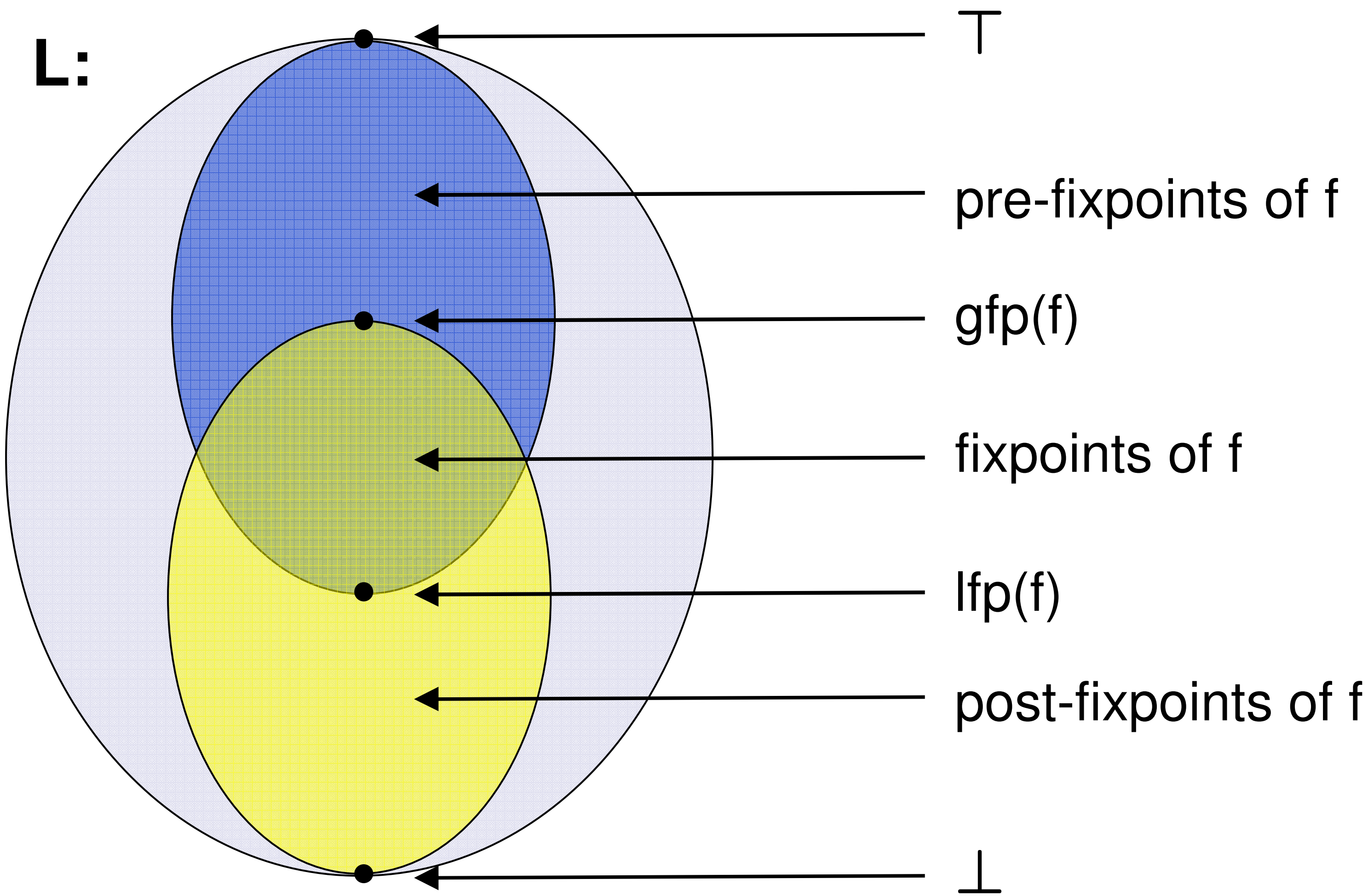
Every monotonic function f on a complete lattice L has a least fixpoint lfp(f) and a greatest fixpoint gfp(f).

More precisely,

$$\mathrm{lfp}(f) = \sqcap \{ x \in L \mid f(x) \sqsubseteq x \} \qquad \textit{least pre-fixpoint}$$

$$\mathrm{gfp}(f) = \sqcup \{ x \in L \mid x \sqsubseteq f(x) \} \qquad \textit{greatest post-fixpoint}$$

Knaster-Tarski Fixpoint Theorem



Picture from: Nielson/Nielson/Hankin, *Principles of Program Analysis*

Smallest Solutions Exist Always

- Define functional $F : L^n \rightarrow L^n$ from right hand sides of constraints such that:
  - σ solution of constraint system iff σ pre-fixpoint of F

- Functional F is monotonic.

- By Knaster-Tarski Fixpoint Theorem:
  - F has a least fixpoint which equals its least pre-fixpoint.

☺

Questions

- Do (smallest) solutions always exist ?
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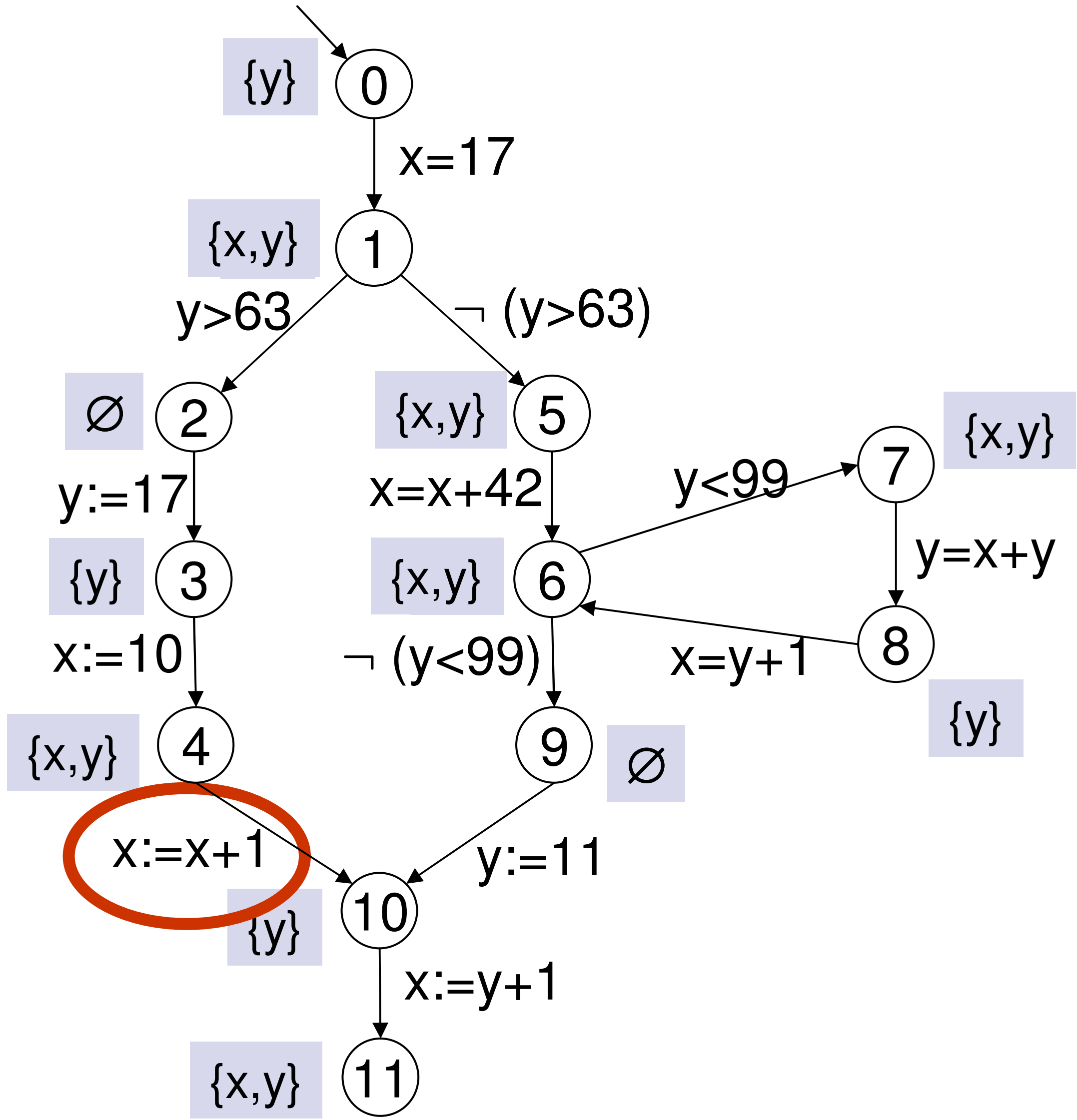
Workset-Algorithm

$$
\begin{array}{l}
W = \emptyset; \\
\textbf{forall}\ (\text{program points } v)\ \ \{\ A[v] = \bot;\ W = W \cup \{v\};\ \} \\
A[\mathit{fin}] = \mathit{init}; \\
\textbf{while}\ W \neq \emptyset\ \ \{ \\
\quad v = \mathit{Extract}(W); \\
\quad \textbf{forall}\ (u,s\ \textbf{with}\ e = (u,s,v)\ \text{edge})\ \ \{ \\
\quad\quad t = f_e(A[v]); \\
\quad\quad \textbf{if}\ \neg(t \sqsubseteq A[u])\ \ \{ \\
\quad\quad\quad A[u] = A[u] \sqcup t; \\
\quad\quad\quad W = W \cup \{u\}; \\
\quad\quad \} \\
\quad \} \\
\}
\end{array}
$$

Live Variables Analysis



Invariants of the Main Loop

a) $A[u] \sqsubseteq \text{MFP}[u]$ f.a. prg. points u

b1) $A[fin] \sqsupseteq init$

b2) $v \notin W \;\Rightarrow\; A[u] \sqsupseteq f_e(A[v])$ f.a. edges $e = (u,s,v)$

If and when workset algorithm terminates:

A is a solution of the constraint system by b1)&b2)

$\Rightarrow \quad A[u] \sqsupseteq MFP[u]$ f.a. u

Hence, with a): $A[u] = MFP[u]$ f.a. u ☺

How to Guarantee Termination

- Lattice $(L,\sqsubseteq)$ has finite heights
  $\Rightarrow$ algorithm terminates after at most
  #prg points · (heights(L)+1)
  iterations of main loop

- Lattice $(L,\sqsubseteq)$ has no infinite ascending chains
  $\Rightarrow$ algorithm terminates

- Lattice $(L,\sqsubseteq)$ has infinite ascending chains:
  $\Rightarrow$ algorithm may not terminate;
  use *widening operators* in order to enforce termination

Widening Operator

[Cousot]

$\nabla: L \times L \to L$ is called a *widening operator* iff

1) $\forall\, x,y \in L: x \sqcup y \sqsubseteq x \nabla y$

2) for all sequences $(l_n)_n$, the (ascending) chain $(w_n)_n$

$$w_0 = l_0, \quad w_{i+1} = w_i \nabla l_{i+1} \text{ for } i > 0$$

stabilizes eventually.

Workset-Algorithm with Widening

$W = \emptyset;$

forall $(\text{program points } v)$ $\{ A[v] = \bot;\ W = W \cup \{v\}; \}$

$A[fin] = init;$

while $W \neq \emptyset$ $\{$

$\quad v = Extract(W);$

$\quad$ **forall** $(u, s$ **with** $e = (u, s, v) \text{ edge})$ $\{$

$\quad\quad t = f_e(A[v]);$

$\quad\quad$ **if** $\neg(t \sqsubseteq A[u])$ $\{$

$\quad\quad\quad A[u] = A[u] \triangledown t;$

$\quad\quad\quad W = W \cup \{u\};$

$\quad\quad \}$

$\quad \}$

$\}$

Invariants of the Main Loop

a) ~~$A[u] \sqsubseteq MFP[u]$~~ f.a. prg. points u

b1) $A[fin] \sqsupseteq init$

b2) $v \notin W \;\Rightarrow\; A[u] \sqsupseteq f_e(A[v])$ f.a. edges $e = (u,s,v)$

With a widening operator we enforce termination but we loose invariant a).

Upon termination, we have:

A is a solution of the constraint system by b1)&b2)

$\Rightarrow \quad A[u] \sqsupseteq MFP[u]$ f.a. u

Compute a sound upper approximation (only) ! 😐

Example of a Widening Operator: Interval Analysis

The goal

Find save interval for the values of program variables, e.g. of *i* in:

```
for (i=0; i<42; i++)
   if (0<=i and i<42)
  {
      A1 = A+i;
      M[A1] = i;
   }
```

..., e.g., in order to remove the redundant array range check. ☺

Example of a Widening Operator: Interval Analysis

The lattice...

$$(L, \sqsubseteq) \;=\; \left(\{\, [l,u] \mid l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{+\infty\}, l \le u \} \cup \{\varnothing\},\ \subseteq \right)$$

... has infinite ascending chains, e.g.:

$$[0,0] \subset [0,1] \subset [0,2] \subset \ldots$$

A widening operator:

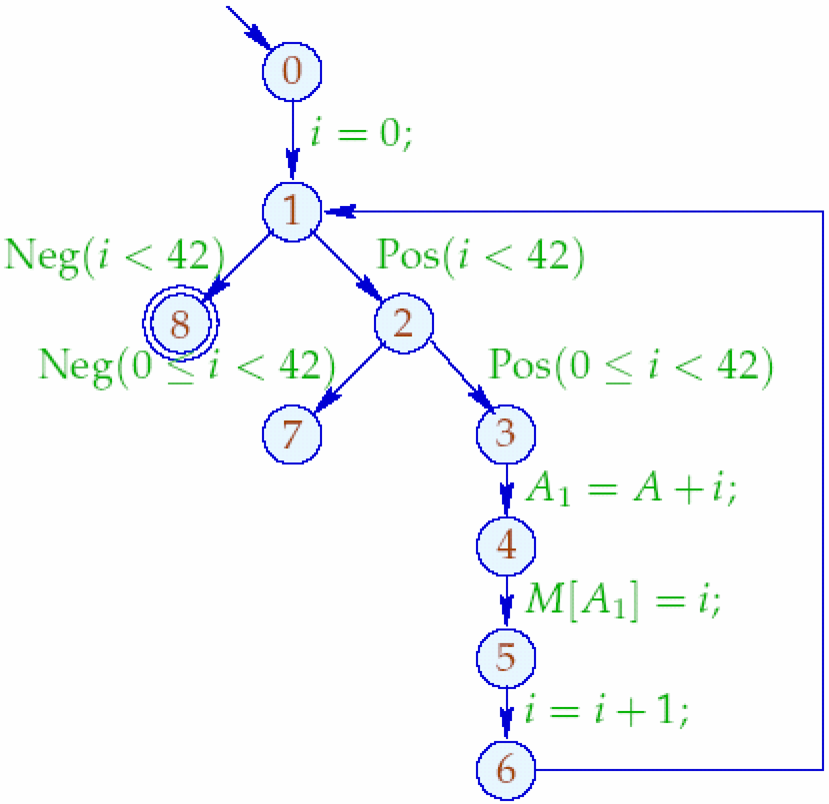
$[l_0,u_0] \mathbin{\triangledown} [l_1,u_1] = [l_2,u_2]$, where

$$l_2 = \begin{cases} l_0 & \text{if } l_0 \le l_1 \\ -\infty & \text{otherwise} \end{cases} \quad \text{and} \quad u_2 = \begin{cases} u_0 & \text{if } u_0 \ge u_1 \\ +\infty & \text{otherwise} \end{cases}$$

A chain of maximal length arising with this widening operator:

$$\varnothing \subset [3,7] \subset [3,+\infty] \subset [-\infty,+\infty]$$

Analyzing the Program with the Widening Operator



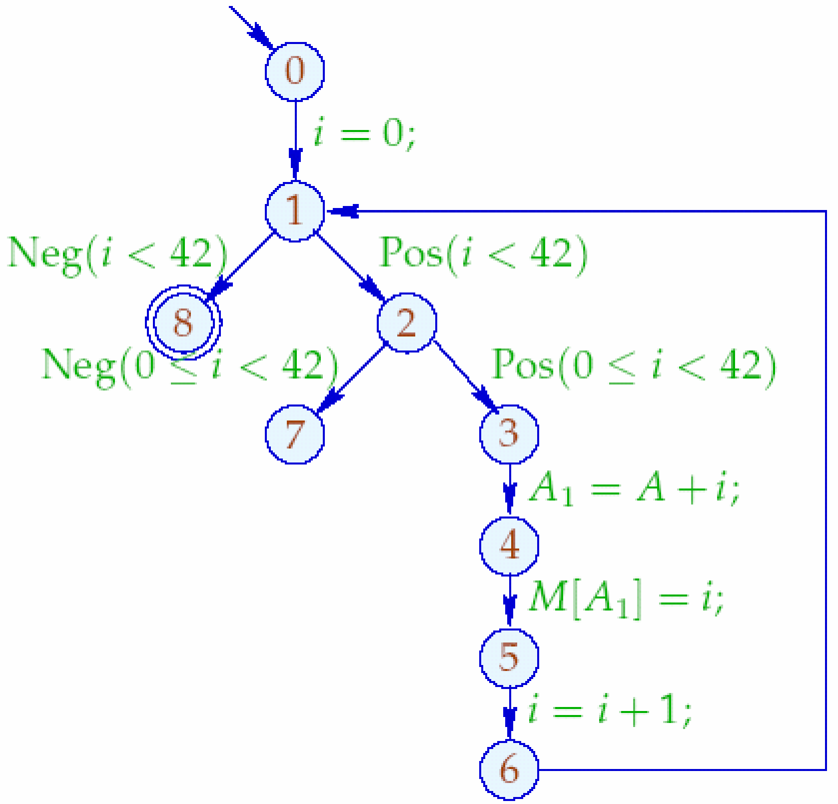
	1		2		3	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$		
1	0	0	0	$+\infty$		
2	0	0	0	$+\infty$		
3	0	0	0	$+\infty$		
4	0	0	0	$+\infty$	dito	
5	0	0	0	$+\infty$		
6	1	1	1	$+\infty$		
7	$\perp$		42	$+\infty$		
8	$\perp$		42	$+\infty$		

$\Rightarrow$ Result is far too imprecise ! ☹

Example taken from: H. Seidl, Vorlesung „Programmoptimierung“

Remedy 1: Loop Separators

- Apply the widening operator only at a „*loop separator*“ (a set of program points that cuts each loop).
- We use the loop separator {1} here.

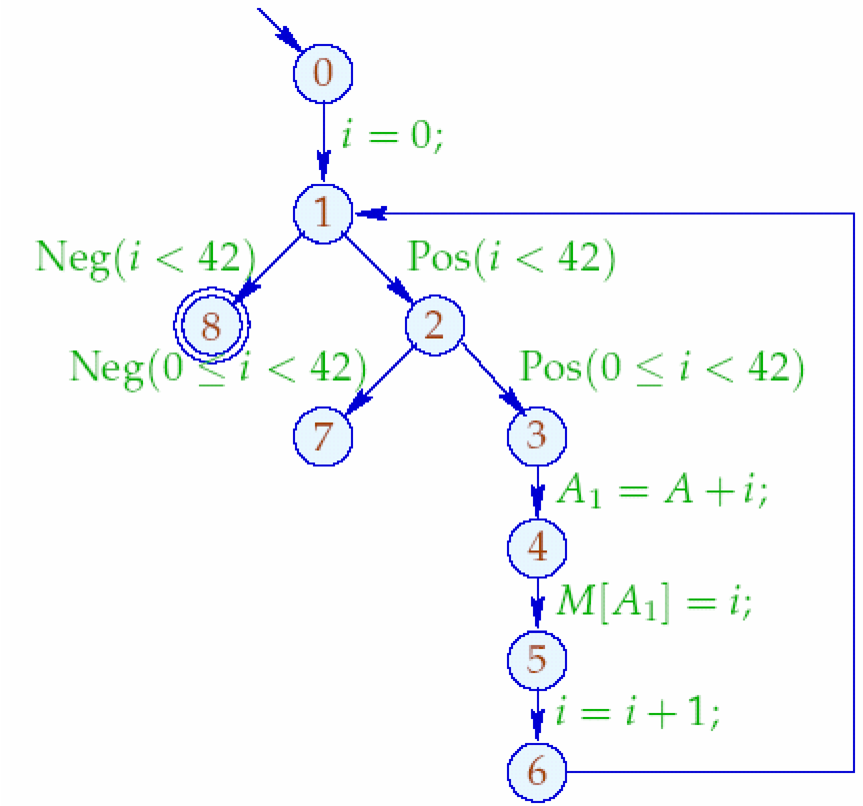


	1		2		3	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$		
1	0	0	0	$+\infty$		
2	0	0	0	41		
3	0	0	0	41		
4	0	0	0	41	dito	
5	0	0	0	41		
6	1	1	1	42		
7	$\perp$		$\perp$			
8	$\perp$		42	$+\infty$		

$\Rightarrow$ Identify condition at edge from 2 to 3 as redundant ! ☺

Remedy 2: Narrowing

- Iterate again from the result obtained by widening
  --- Iteration from a prefix-point stays above the least fixpoint ! ---



	0		1		2	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$
1	0	$+\infty$	0	$+\infty$	0	42
2	0	$+\infty$	0	41	0	41
3	0	$+\infty$	0	41	0	41
4	0	$+\infty$	0	41	0	41
5	0	$+\infty$	0	41	0	41
6	1	$+\infty$	1	42	1	42
7	42	$+\infty$	$\perp$		$\perp$	
8	42	$+\infty$	42	$+\infty$	42	42

$\Rightarrow$ We get the exact result in this example (but not guaranteed) ! ☺

Remarks

- Can use work-list instead of work-set
- Special iteration strategies in special situations
- Semi-naive iteration

Program Analysis of Sequential and Parallel Programs II

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Recall: Specifying Live Variables Analysis by a Constraint System

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where $\mathit{init} = \mathrm{Var}$,

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Recall: Questions

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  - MOP vs MFP-solution
  - Abstract interpretation

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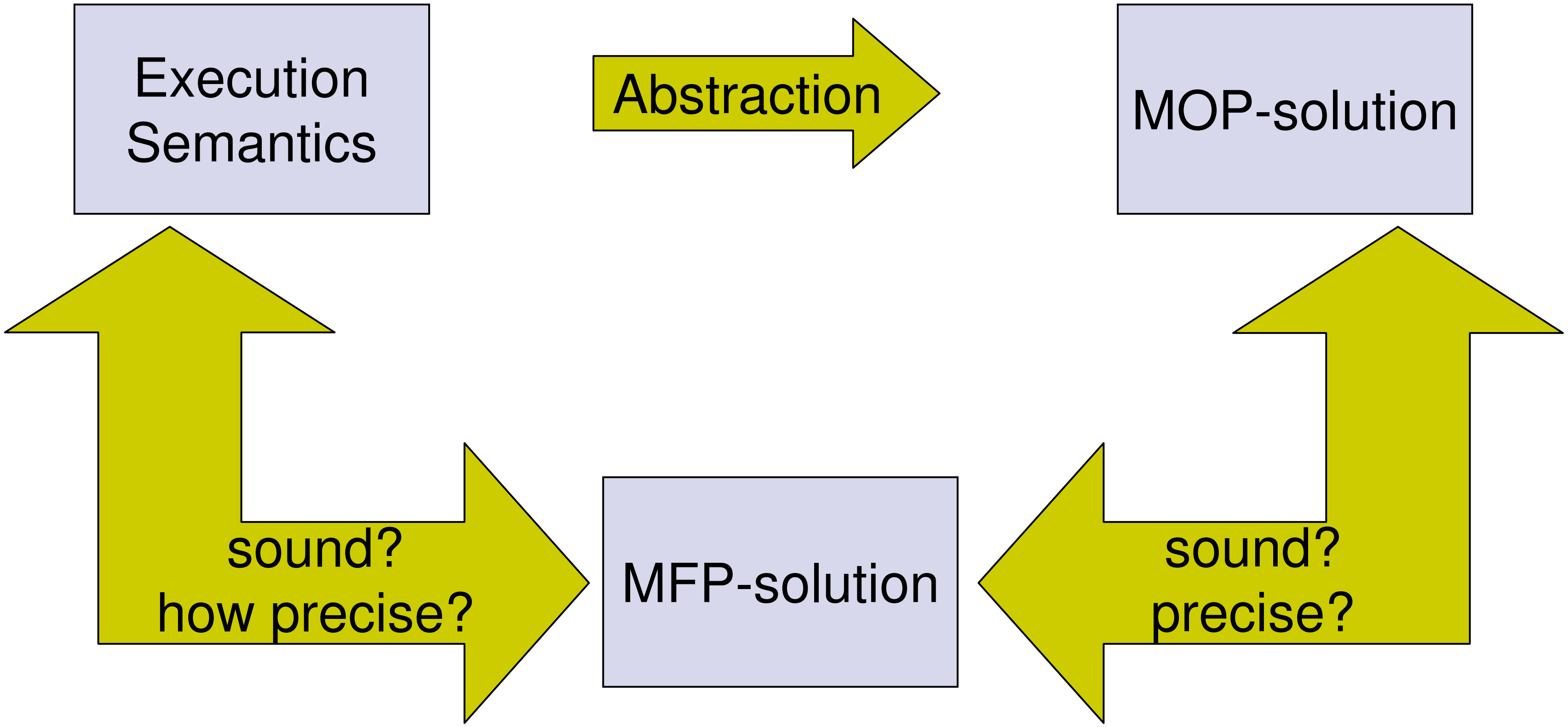
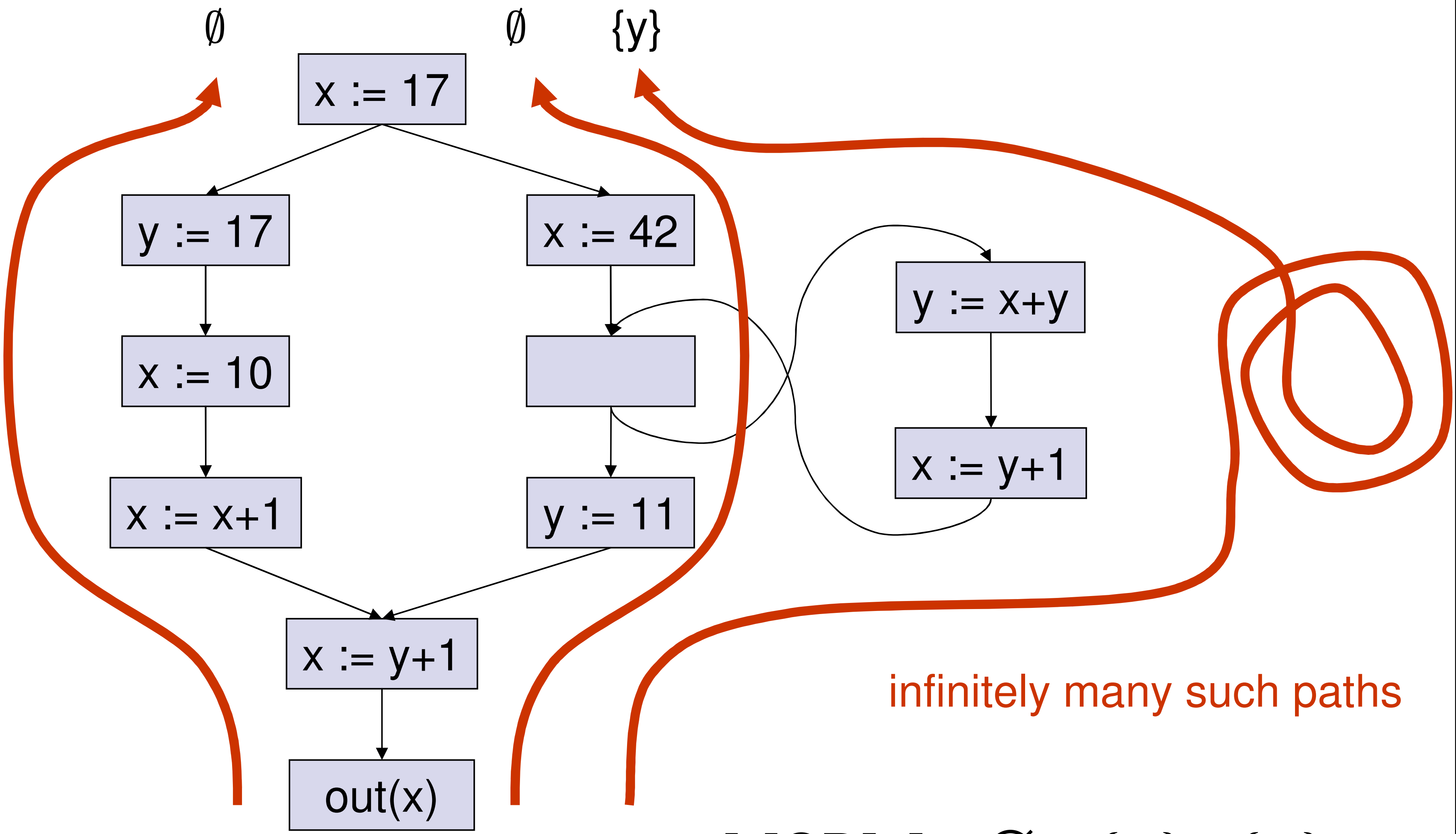
Assessing Data Flow Frameworks



Live Variables



$$\mathrm{MOP}[v] = \varnothing \cup \{y\} = \{y\}$$

Meet-Over-All-Paths Solution (MOP)

- Forward Analysis

$$\mathrm{MOP}[u] := \bigsqcup_{p \in \mathrm{Paths}[entry,u]} \mathrm{F}_p(init)$$

- Backward Analysis

$$\mathrm{MOP}[u] := \bigsqcup_{p \in \mathrm{Paths}[u,exit]} \mathrm{F}_p(init)$$

- Here: „Join-over-all-paths"; MOP traditional name

Coincidence Theorem

Definition:

A framework is positively-distributive if

$$f(\sqcup X) = \sqcup\{ f(x) \mid x \in X\} \text{ for all } \emptyset \neq X \subseteq L,\ f \in F.$$

Theorem:

For any instance of a positively-distributive framework:

$$MOP[u] = MFP[u] \qquad \text{for all program points } u.$$

Remark:

A framework is positively-distributive if a) and b) hold:

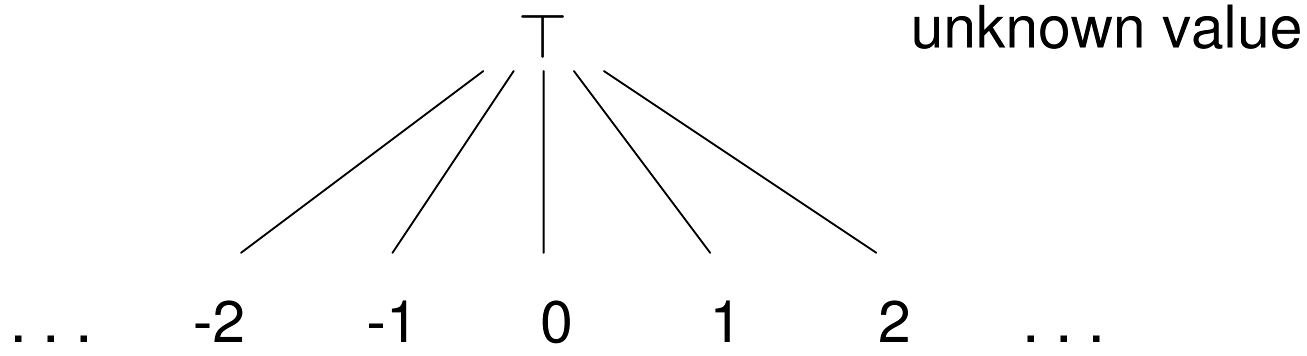
(a) it is distributive: $f(x \sqcup y) = f(x) \sqcup f(y)$ f.a. $f \in F$, $x,y \in L$

(b) it is effective: L does not have infinite ascending chains.

Remark:

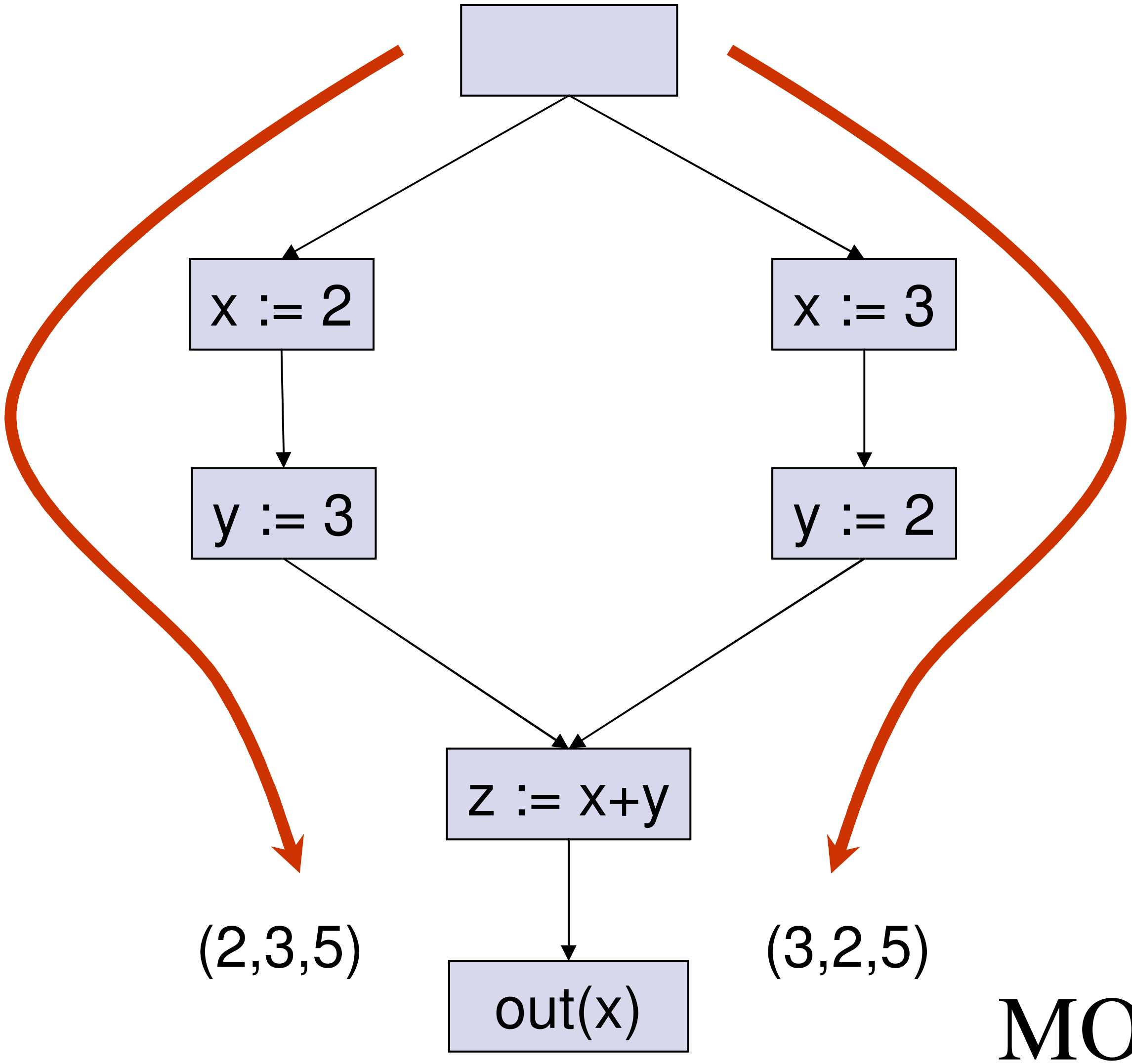
All bitvector frameworks are distributive and effective.

Lattice for Constant Propagation



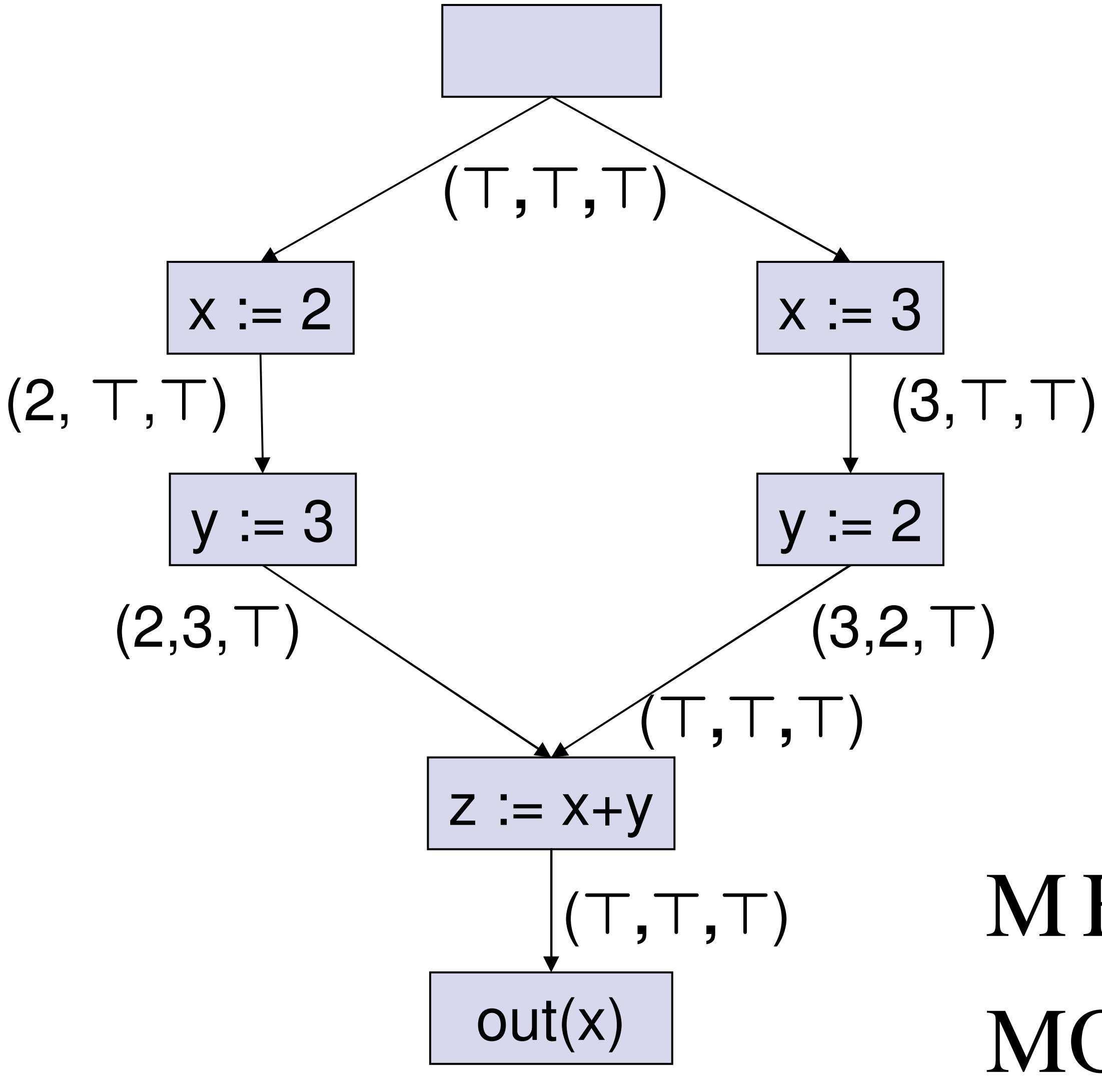
lattice L	$\{\rho \mid \rho : \mathrm{Var} \to (\mathbb{Z} \cup \{\top\})\} \cup \{\bot\}$
$\sqsubseteq$	$\rho \sqsubseteq \rho' \;:\Leftrightarrow\; \rho = \bot \;\vee$
	$(\rho, \rho' \neq \bot \wedge \forall x : \rho(x) \sqsubseteq \rho'(x))$
$\sqcup$	pointwise join
$\top$	$\top(x) = \top$ f.a. $\mathrm{x} \in \mathrm{Var}$

$(\rho(x), \rho(y), \rho(z))$



$$\mathrm{MOP}[v] = (\top, \top, 5)$$

$$(\rho(x), \rho(y), \rho(z))$$



$$MFP[v] = (\top, \top, \top)$$

$$MOP[v] = (\top, \top, 5)$$

Correctness Theorem

Definition:

A framework is monotone if for all $f \in F$, $x,y \in L$:

$$x \sqsubseteq y \quad \Rightarrow \quad f(x) \sqsubseteq f(y) \,.$$

Theorem:

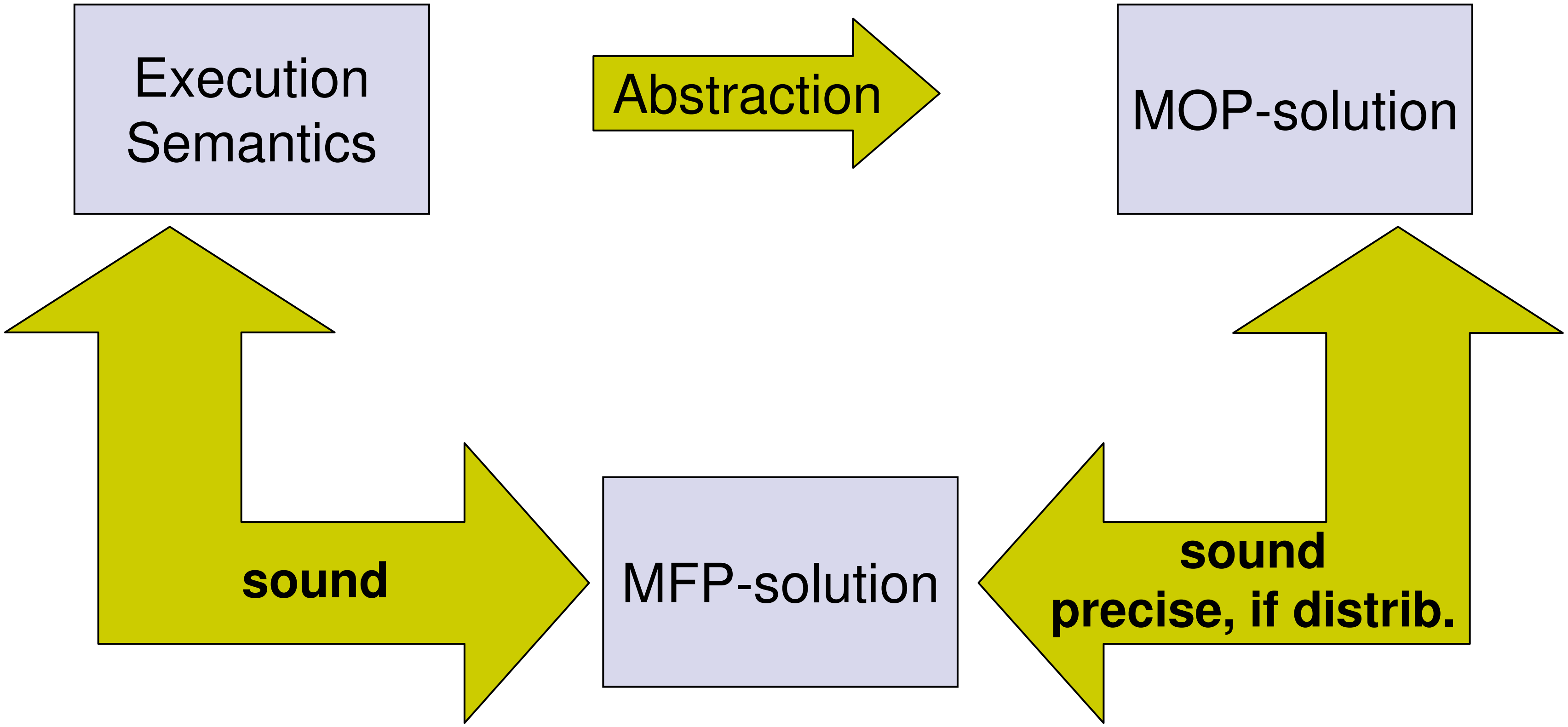
In any monotone framework:

$MOP[i] \sqsubseteq MFP[i]$ for all program points i.

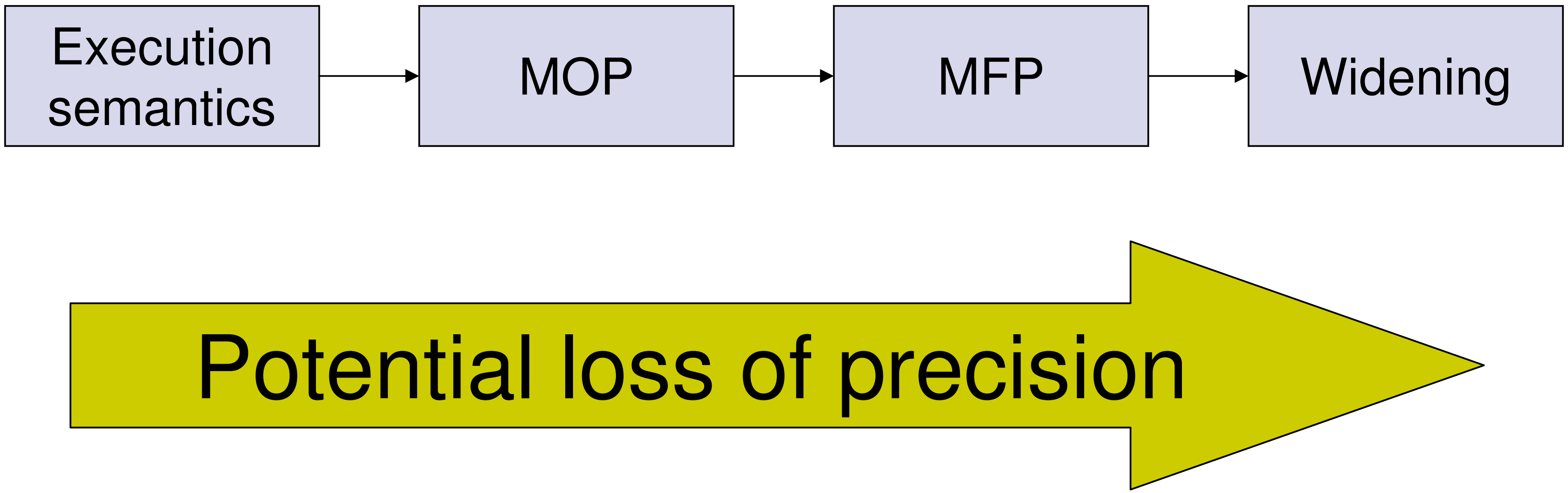
Remark:

Any "reasonable" framework is monotone. ☺

Assessing Data Flow Frameworks



Where Flow Analysis Looses Precision



Questions

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- How to justify that a solution is what we want ?
  - MOP vs MFP-solution
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Abstract Interpretation

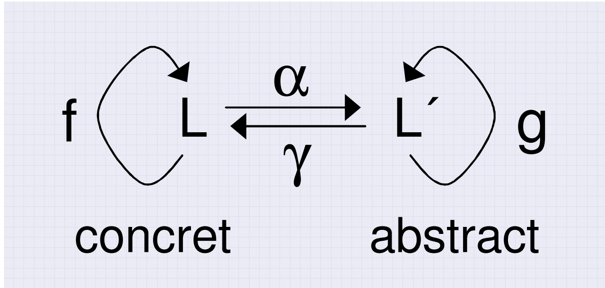
[Cousot]

constraint system for **Reference Semantics** on concrete lattice $(D,\sqsubseteq)$	**Replace** concrete operators o by abstract operators $o^\#$ $\Rightarrow$	constraint system for **Analysis** on abstract lattice $(D^\#,\sqsubseteq^\#)$
MFP		$MFP^\#$

Often used as reference semantics:

- sets of reaching runs:
  $(D,\sqsubseteq) = (P(Edges^*),\subseteq)$ or $(D,\sqsubseteq) = (P(Stmt^*),\subseteq)$
- sets of reaching states (*collecting semantics*):
  $(D,\sqsubseteq) = (P(\Sigma^*),\subseteq)$ with $\Sigma = Var \rightarrow Val$

Transfer Lemma



Situation:

complete lattices $(L,\sqsubseteq)$, $(L',\sqsubseteq')$
montonic functions $f{:}L\to L$, $g{:}\ L'\to L'$, $\alpha{:}L\to L'$

Definition:

Let $(L,\sqsubseteq)$ be a complete lattice.
$\alpha : L\to L$ is called *universally-disjunctive* iff $\forall X\subseteq L{:}\ \alpha(\sqcup X) = \sqcup \{ \alpha(x) \mid x\in X\}$.

Remark:

- (α,γ) is called *Galois connection* iff $\forall x\in L, x'\in L'{:}\ \ \alpha(x) \sqsubseteq' y \Leftrightarrow x \sqsubseteq \gamma(y)$.
- α is universally-disjunctive iff $\exists \gamma{:}L'\to L : (\alpha,\gamma)$ is Galois connection.

Transfer Lemma:

Suppose α is universally-disjunctive. Then:
(a) $\alpha \circ f \sqsubseteq' g \circ \alpha \Rightarrow \alpha(\mathrm{lfp}(f)) \sqsubseteq' \mathrm{lfp}(g)$.
(b) $\alpha \circ f = g \circ \alpha \Rightarrow \alpha(\mathrm{lfp}(f)) = \mathrm{lfp}(g)$.

Abstract Interpretation

Assume a universally-disjunctive abstraction function $\alpha : D \to D^{\#}$.

Correct abstract interpretation:

Show $\alpha(o(x_1,\ldots,x_k)) \sqsubseteq^{\#} o^{\#}(\alpha(x_1),\ldots,\alpha(x_k))$ f.a. $x_1,\ldots,x_k \in L$, operators o

Then $\alpha(MFP[u]) \sqsubseteq^{\#} MFP^{\#}[u]$ f.a. u

Correct and precise abstract interpretation:

Show $\alpha(o(x_1,\ldots,x_k)) = o^{\#}(\alpha(x_1),\ldots,\alpha(x_k))$ f.a. $x_1,\ldots,x_k \in L$, operators o

Then $\alpha(MFP[u]) = MFP^{\#}[u]$ f.a. u

Use this as guideline for designing correct (and precise) analyses !

Abstract Interpretation

Constraint system for reaching runs:

$$R[st] \supseteq \{\varepsilon\}, \qquad \text{for } st\text{, the start node}$$

$$R[v] \supseteq R[u] \cdot \{\langle e \rangle\}, \qquad \text{for each edge } e = (u,s,v)$$

Operational justification:

Let $\underline{R}[u]$ be components of smallest solution over P(Edges*). Then

$$\underline{R}[u] = R^{op}[u] =_{def} \{r \in Edges^* \mid st \xrightarrow{r} u\} \qquad \text{for all } u$$

Prove:

a) $R^{op}[u]$ satisfies all constraints (direct)

$\Rightarrow$ $\underline{R}[u] \subseteq R^{op}[u]$ f.a. u

b) $w \in R^{op}[u] \Rightarrow w \in \underline{R}[u]$ (by induction on |w|)

$\Rightarrow$ $R^{op}[u] \subseteq \underline{R}[u]$ f.a. u

Abstract Interpretation

Constraint system for reaching runs:

$$R[st] \supseteq \{\varepsilon\}, \qquad \text{for } st\text{, the start node}$$

$$R[v] \supseteq R[u] \cdot \{\langle e \rangle\}, \qquad \text{for each edge } e = (u, s, v)$$

Derive the analysis:

Replace

$\{\varepsilon\}$ by init

$(\bullet) \cdot \{\langle e \rangle\}$ by f_e

Obtain abstracted constraint system:

$$R^\#[st] \sqsupseteq init, \qquad \text{for } st\text{, the start node}$$

$$R^\#[v] \sqsupseteq f_e(R^\#[u]), \qquad \text{for each edge } e = (u, s, v)$$

Abstract Interpretation

MOP-Abstraction:

Define α_{MOP} : P(Edges$^{*)}$ $\to$ L by

$$\alpha_{\text{MOP}}(R) = \sqcup \{ f_r(init) \mid r \in R \} \quad \text{where } f_\varepsilon = Id,\ f_{s \cdot \langle e \rangle} = f_e \circ f_s$$

Remark:

For all monotone frameworks the abstraction is correct:

$\alpha_{\text{MOP}}(\underline{R}[u]) \sqsubseteq \underline{R}^{\#}[u]$ f.a. prg. points u

For all universally-distributive frameworks the abstraction is correct and precise:

$\alpha_{\text{MOP}}(\underline{R}[u]) = \underline{R}^{\#}[u]$ f.a. prg. points u

Justifies MOP vs. MFP theorems (*cum grano salis*). ☺

Overview

- Introduction
- Fundamentals of Program Analysis

- Interprocedural Analysis
- Analysis of Parallel Programs
- Conclusion

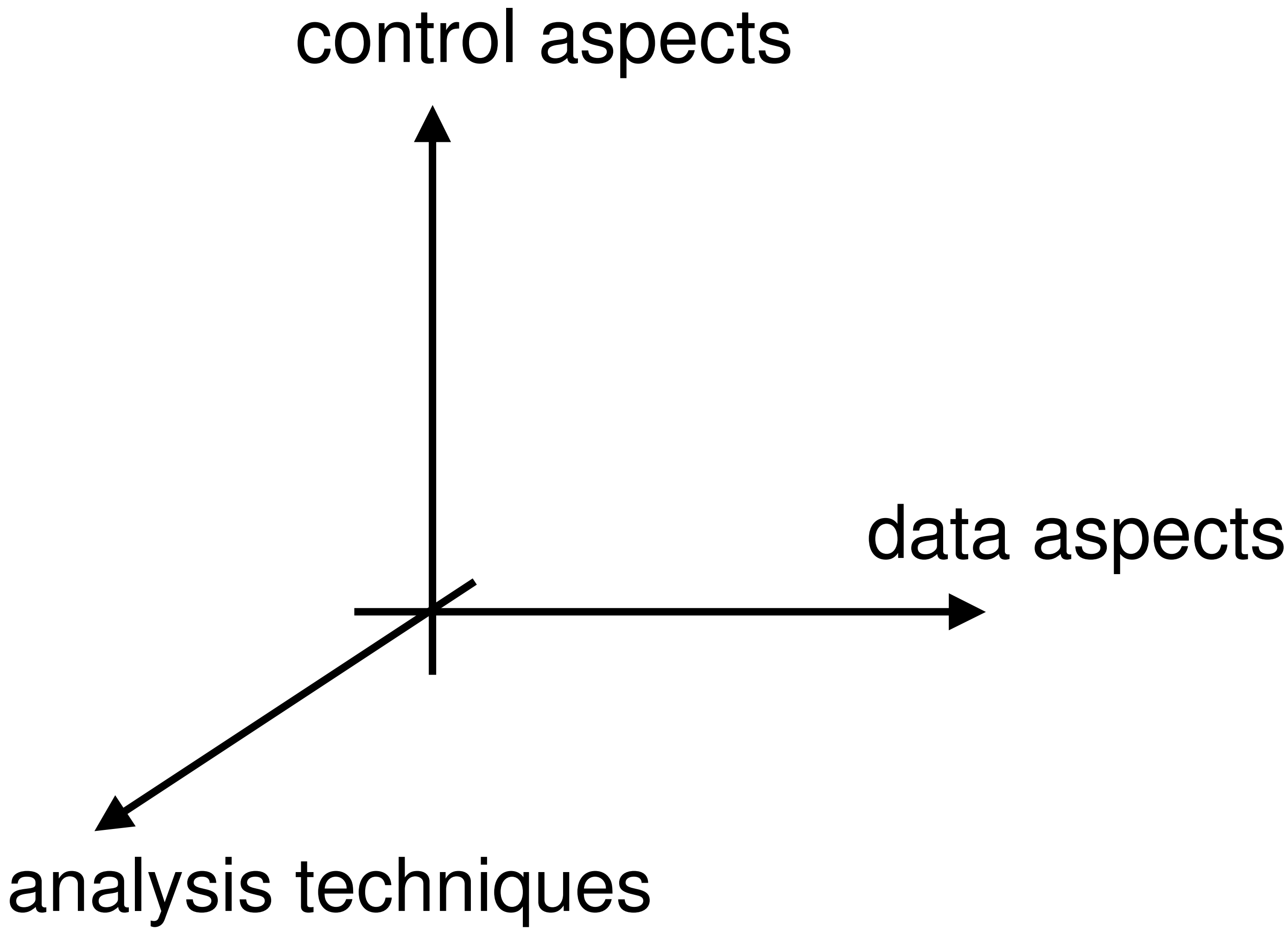
Challenges for Automatic Analysis

- Data aspects:
  - infinite number domains
  - dynamic data structures (e.g. lists of unbounded length)
  - pointers
  - ...

- Control aspects:
  - recursion
  - concurrency
  - creation of processes / threads
  - synchronization primitives (locks, monitors, communication stmts ...)
  - ...

$\Rightarrow$ infinite/unbounded state spaces

Classifying Analysis Approaches



(My) Main Interests of Recent Years

Data aspects:

- algebraic invariants over $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_m$ ($m = 2^n$) in sequential programs, partly with recursive procedures
- invariant generation relative to Herbrand interpretation

Control aspects:

- recursion
- concurrency with process creation / threads
- synchronization primitives

Technics:

- fixpoint-based
- automata-based
- (linear) algebra
- syntactic substitution-based techniques
- ...

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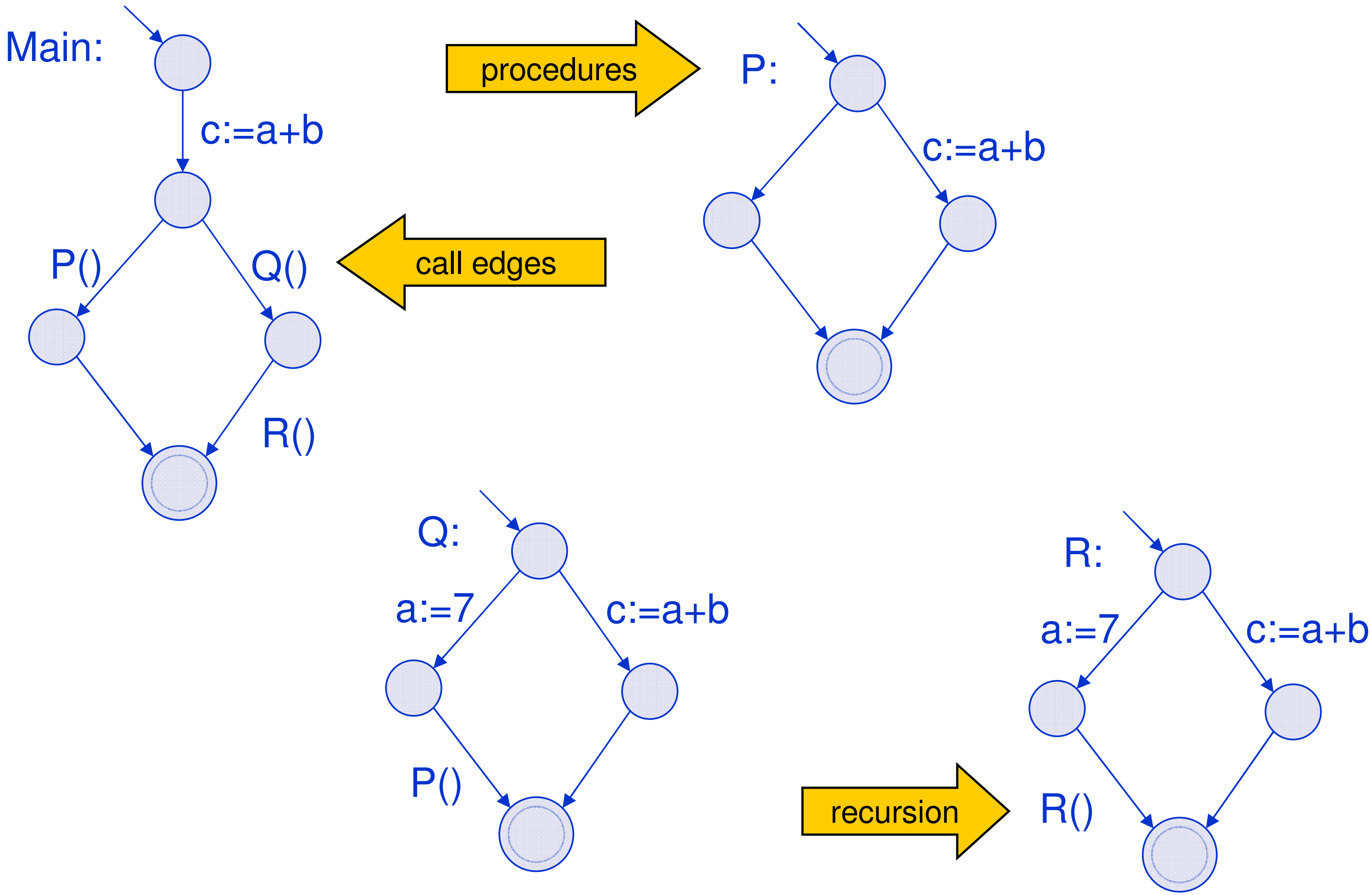
Interprocedural Analysis



Running Example:
(Definite) Availability of the single expression a+b

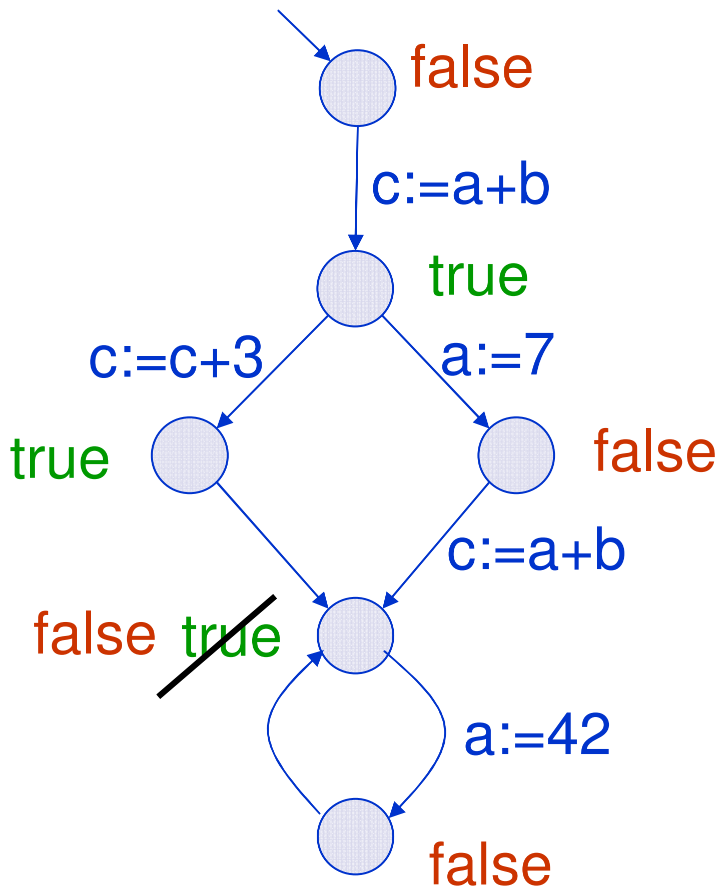
The lattice:

false — a+b not available

|

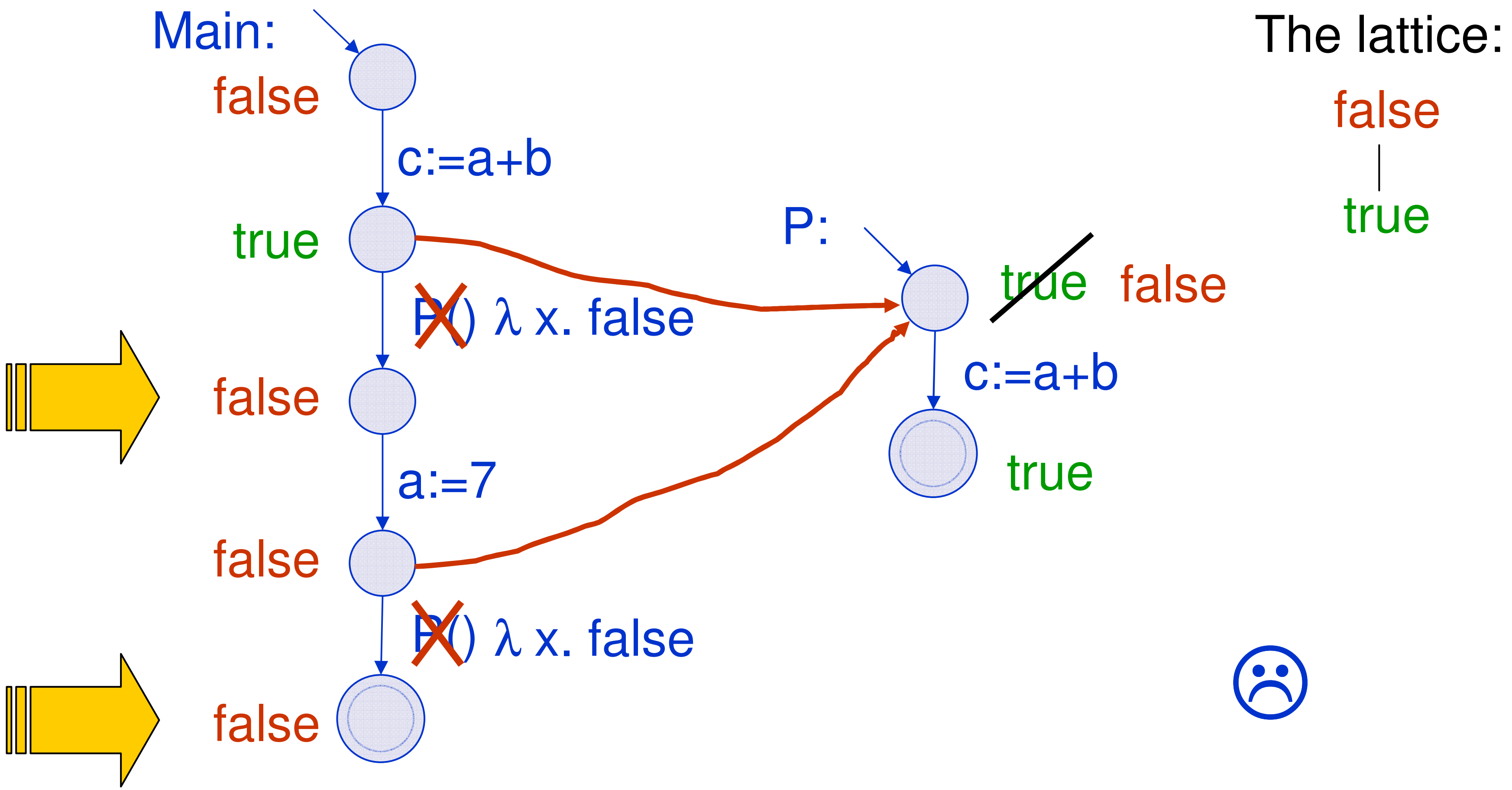
true — a+b available

Initial value: false



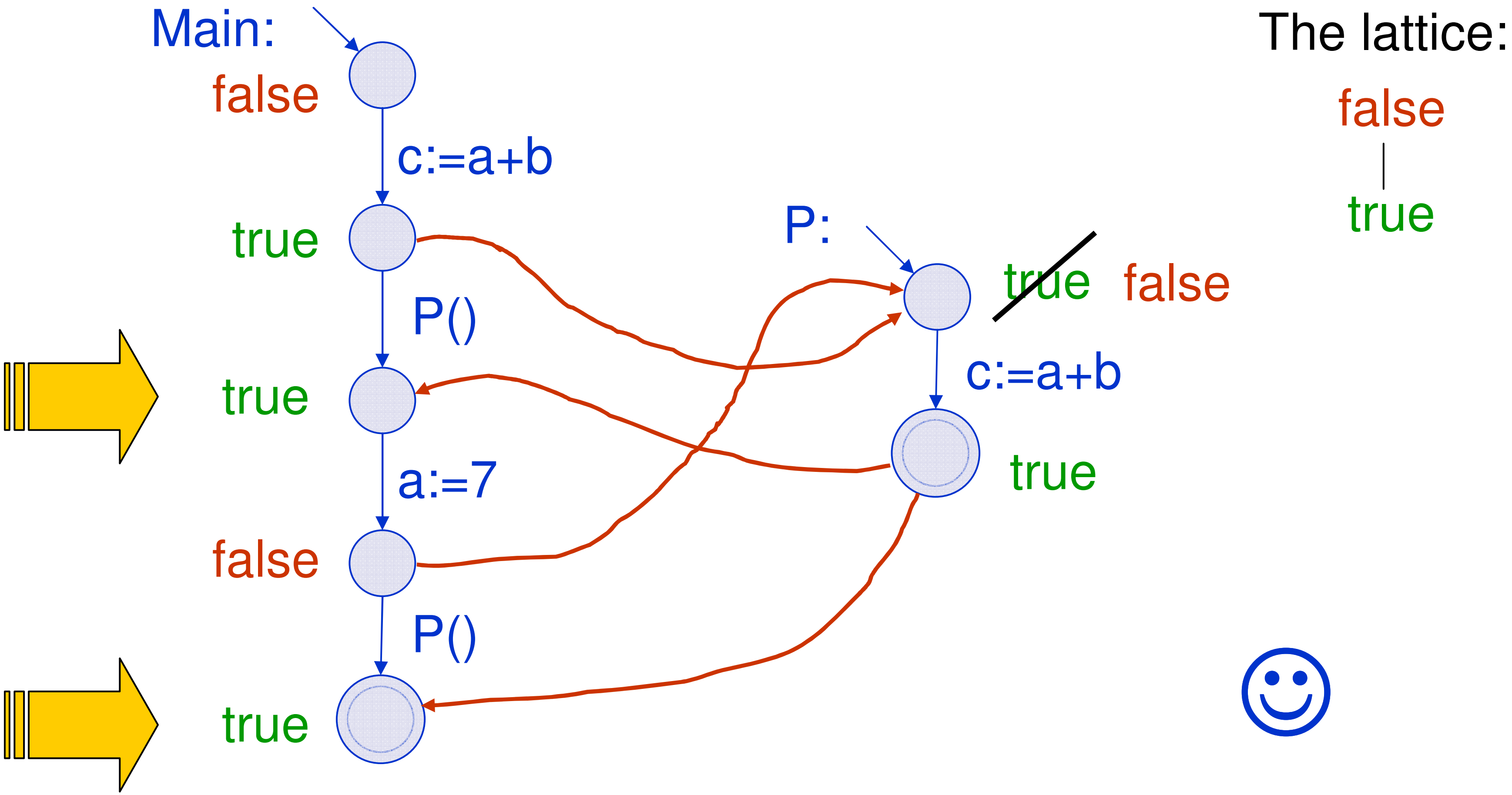
Intra-Procedural-Like Analysis

Conservative assumption: procedure destroys all information; information flows from call node to entry point of procedure



Context-Insensitive Analysis

Conservative assumption: Information flows from each call node to entry of procedure and from exit of procedure back to return point



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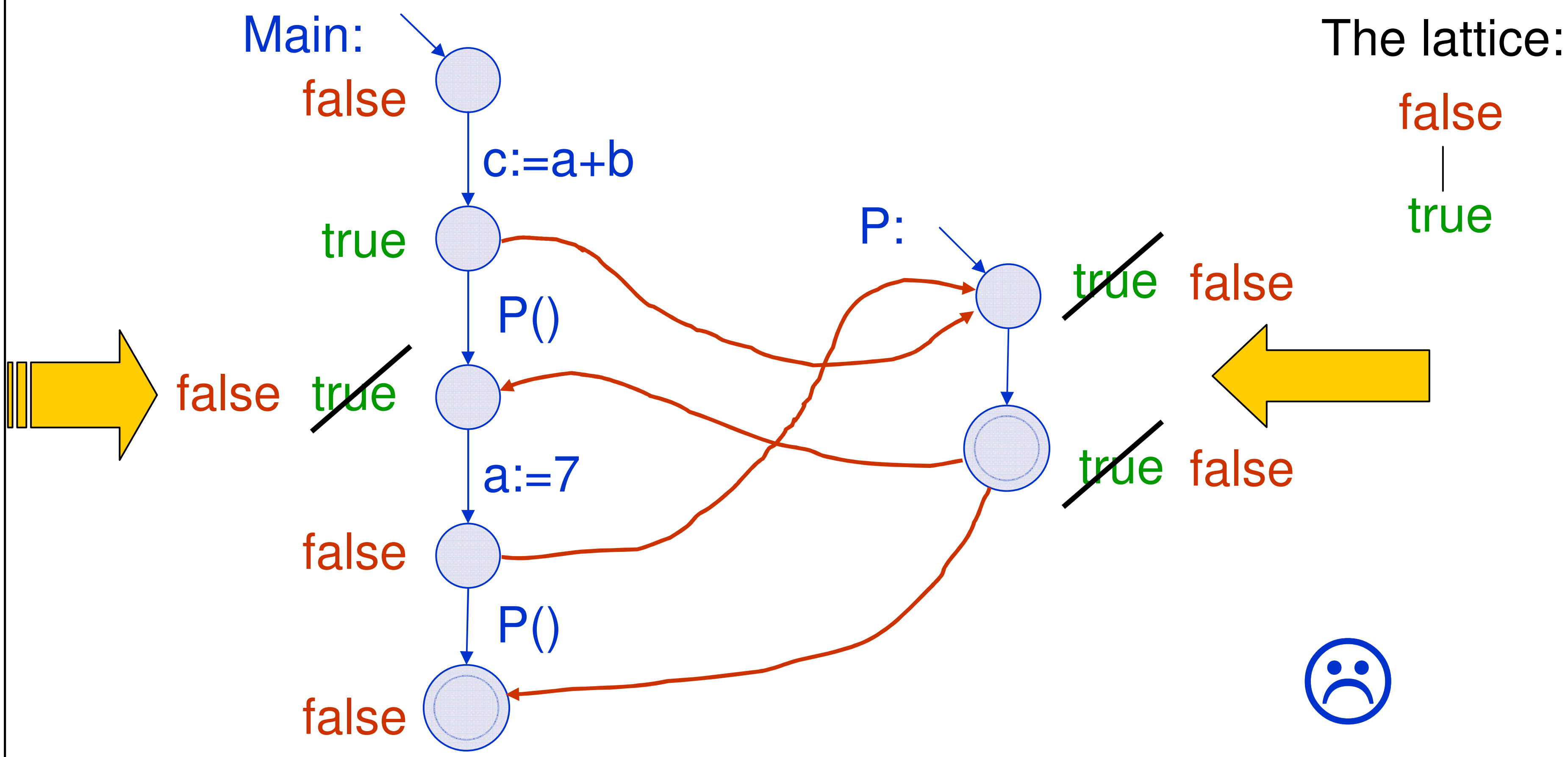


Let's Apply Our Abstract Interpretation Recipe: Constraint System for Feasible Paths

Operational justification:

$$\underline{S}(u) = \{ r \in \text{Edges}^* \mid st_p \xrightarrow{r} u \} \quad \text{for all } u \text{ in procedure } p$$
$$\underline{S}(p) = \{ r \in \text{Edges}^* \mid st_p \xrightarrow{r} \varepsilon \} \quad \text{for all procedures } p$$

$$\underline{R}(u) = \{ r \in \text{Edges}^* \mid \exists \omega \in \text{Nodes}^* : st_{Main} \xrightarrow{r} u\omega \} \quad \text{for all } u$$

Same-level runs:

$S(p) \supseteq S(r_p)$		r_p return point of p
$S(st_p) \supseteq \{\varepsilon\}$		st_p entry point of p
$S(v) \supseteq S(u) \cdot \{\langle e \rangle\}$		$e = (u,s,v)$ base edge
$\mathrm{S(v)} \supseteq S(u) \cdot S(p)$		$e = (u,p,v)$ call edge

Reaching runs:

$R(st_{Main}) \supseteq \{\varepsilon\}$	st_{Main} entry point of $Main$
$R(v) \supseteq R(u) \cdot \{\langle e \rangle\}$	$e = (u,s,v)$ basic edge
$R(v) \supseteq R(u) \cdot S(p)$	$e = (u,p,v)$ call edge
$R(st_p) \supseteq R(u)$	$e = (u,p,v)$ call edge, st_p entry point of p

Context-Sensitive Analysis

Idea:

Phase 1: Compute summary information for each procedure...
... as an abstraction of same-level runs

Phase 2: Use summary information as transfer functions for procedure calls...
... in an abstraction of reaching runs

Classic approaches for summary informations:

1) Functional approach: [Sharir/Pnueli 81, Knoop/Steffen: CC´92]
Use (monotonic) functions on data flow informations !

2) Relational approach: [Cousot/Cousot: POPL´77]
Use relations (of a representable class) on data flow informations !

3) Call string approach: [Sharir/Pnueli 81], [Khedker/Karkare: CC´08]
Analyse relative to finite portion of call stack !

Formalization of Functional Approach

Abstractions:

Abstract same-level runs with $\alpha_{Funct} : \text{Edges}^* \to (L \to L)$:

$$\alpha_{Funct}(R) \;=\; \bigsqcup \{\, f_r \mid r \in R \,\} \qquad \text{for } R \subseteq \text{Edges}^*$$

Abstract reaching runs with $\alpha_{MOP} : \text{Edges}^* \to L$:

$$\alpha_{MOP}(R) \;=\; \bigsqcup \{\, f_r(init) \mid r \in R \,\} \qquad \text{for } R \subseteq \text{Edges}^*$$

1. Phase: Compute summary informations, i.e., functions:

$S^\#(p)$	$\sqsupseteq$	$S^\#(r_p)$	r_p return point of p
$S^\#(st_p)$	$\sqsupseteq$	id	st_p entry point of p
$S^\#(v)$	$\sqsupseteq$	$f_e^\# \circ S^\#(u)$	$e = (u,s,v)$ base edge
$S^\#(v)$	$\sqsupseteq$	$S^\#(p) \circ S^\#(u)$	$e = (u,p,v)$ call edge

2. Phase: Use summary informations; compute on data flow informations:

$R^\#(st_{Main})$	$\sqsupseteq$	$init$	st_{Main} entry point of $Main$
$R^\#(v)$	$\sqsupseteq$	$f_e^\#(R^\#(u))$	$e = (u,s,v)$ basic edge
$R^\#(v)$	$\sqsupseteq$	$S^\#(p)(R^\#(u))$	$e = (u,p,v)$ call edge
$R^\#(st_p)$	$\sqsupseteq$	$R^\#(u)$	$e = (u,p,v)$ call edge, st_p entry point of p

Functional Approach

Theorem:

Correctness: For any monotone framework:

$$\alpha_{MOP}(\underline{R}[u]) \sqsubseteq \underline{R}^{\#}[u] \quad \text{f.a. } u$$

Completeness: For any universally-distributive framework:

$$\alpha_{MOP}(\underline{R}[u]) = \underline{R}^{\#}[u] \quad \text{f.a. } u$$

Alternative condition:
framework positively-distributive & all prog. point dyn. reachable

Remark:

a) Functional approach is effective, if L is finite...

b) ... but may lead to chains of length up to $|L| \cdot \text{height}(L)$ at each program point (in general).

Functional Approach for Availability of Single Expression Problem

Observations:

Just three montone functions on lattice L: $\begin{array}{c}\textbf{false}\\ | \\ \textbf{true}\end{array}$

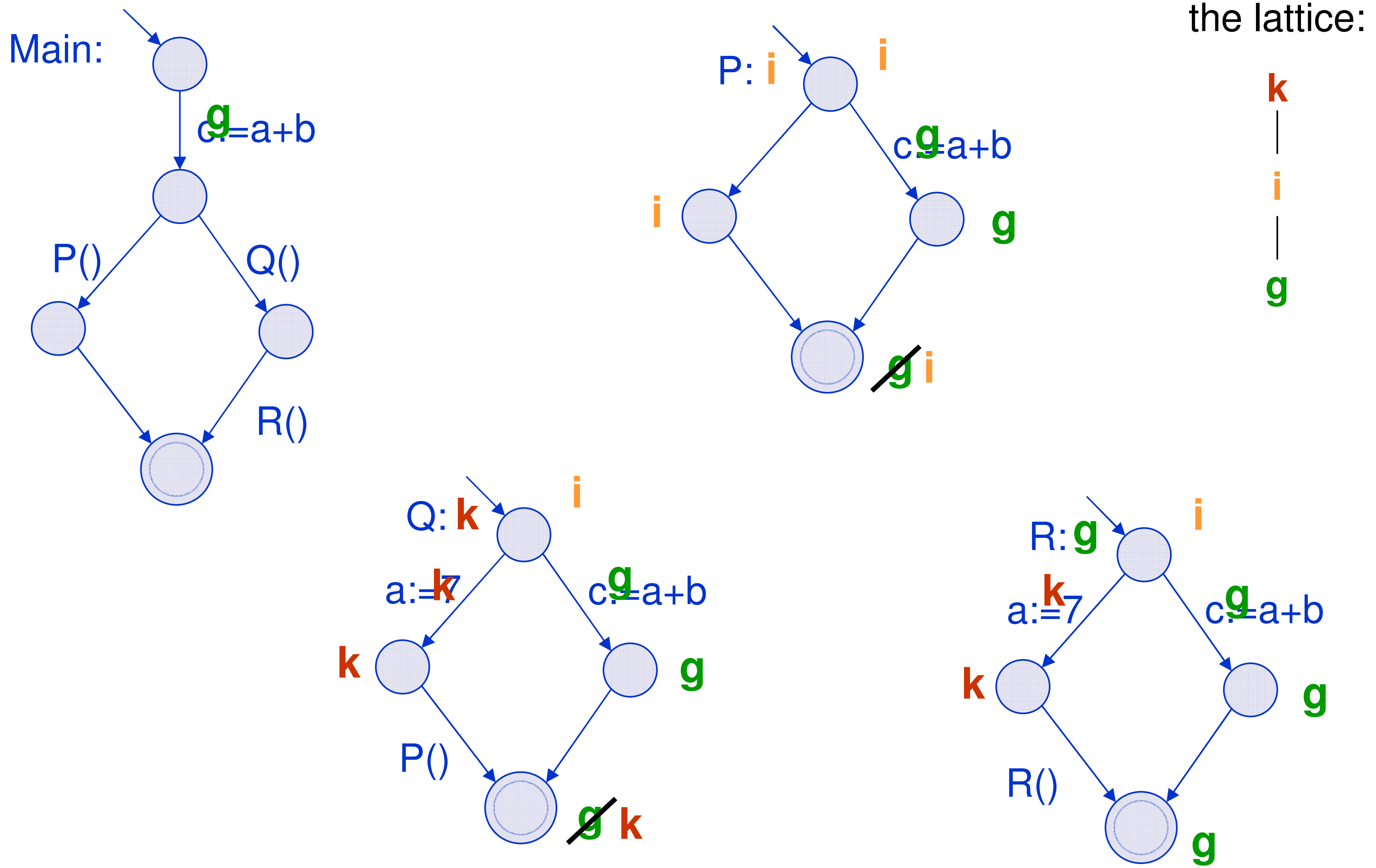
$$\begin{array}{ccc}\lambda\, \mathbf{x}\,.\, \mathbf{false} & \qquad & \mathbf{k}\ \mathbf{(ill)} \\ | & & | \\ \lambda\, \mathbf{x}\,.\, \mathbf{x} & & \mathbf{i}\ \mathbf{(gnore)} \\ | & & | \\ \lambda\, \mathbf{x}\,.\, \mathbf{true} & & \mathbf{g}\ \mathbf{(enerate)}\end{array}$$

Functional composition of two such functions $f,g : L \rightarrow L$:

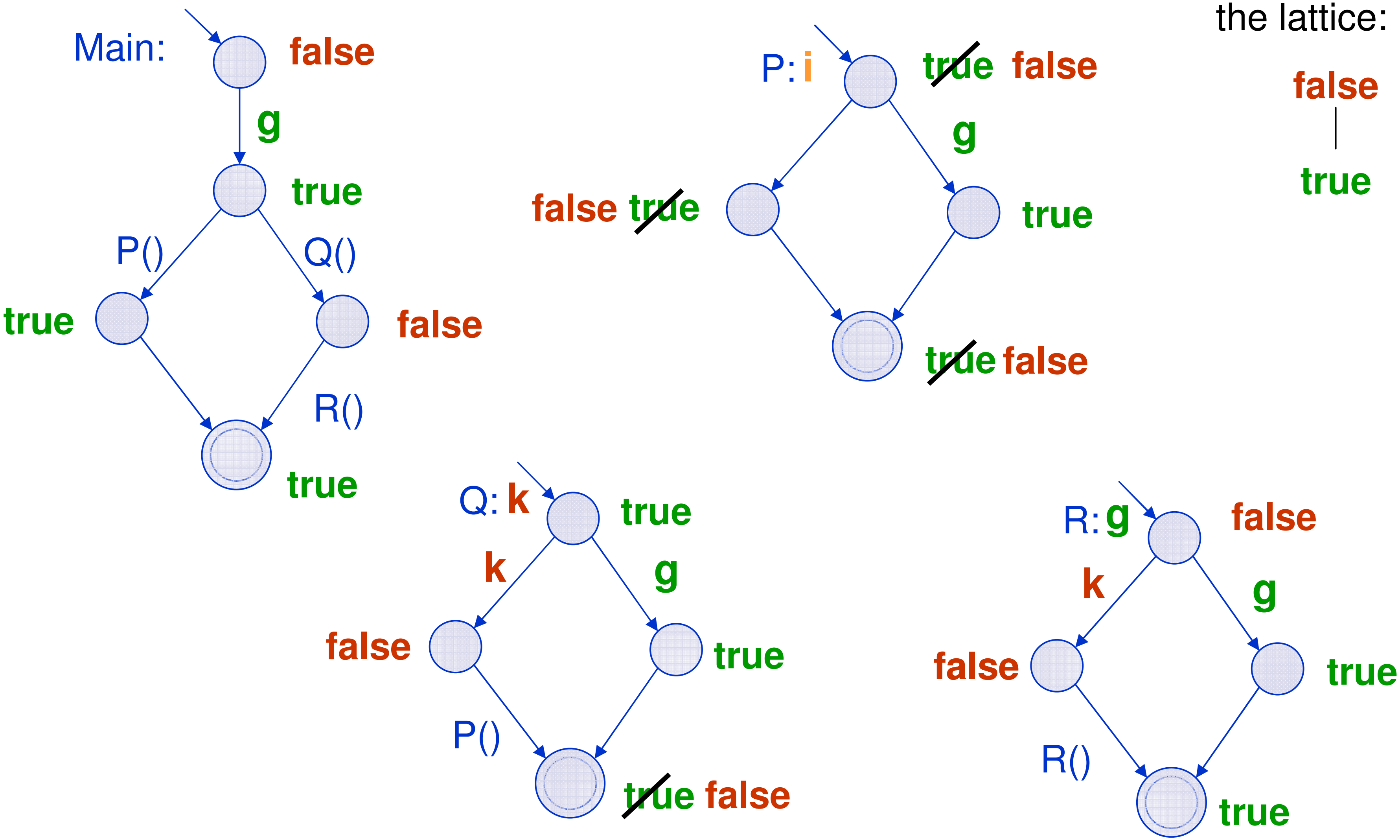
$$h \circ f = \begin{cases} f & \text{if } h = \mathrm{i} \\ h & \text{if } h \in \{\mathrm{g}, \mathrm{k}\} \end{cases}$$

Analogous: precise interprocedural analysis for
all (separable) bitvector problems
in time linear in program size. ☺

Context-Sensitive Analysis, 1. Phase



Context-Sensitive Analysis, 2. Phase



Functional Approach

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a) Functional approach is effective, **if *L* is finite ...** 😐

b) ... but may lead to chains of length up to $|L| \cdot$ height(L) at each program point.

Program Analysis of Sequential and Parallel Programs III

Markus Müller-Olm
Westfälische Wilhelms-Universität Münster, Germany

14th Estonian Winter School in Computer Science
Palmse, Estonia, March, 1-6, 2009

Outline of this Lecture

- Precise interprocedural analysis through linear algebra
  - algebraic invariants, fixpoints, recursion, linear algebra
- Regular symbolic analysis of dynamic networks of pushdown systems
  - automata-based, recursion, parallelism (thread creation)

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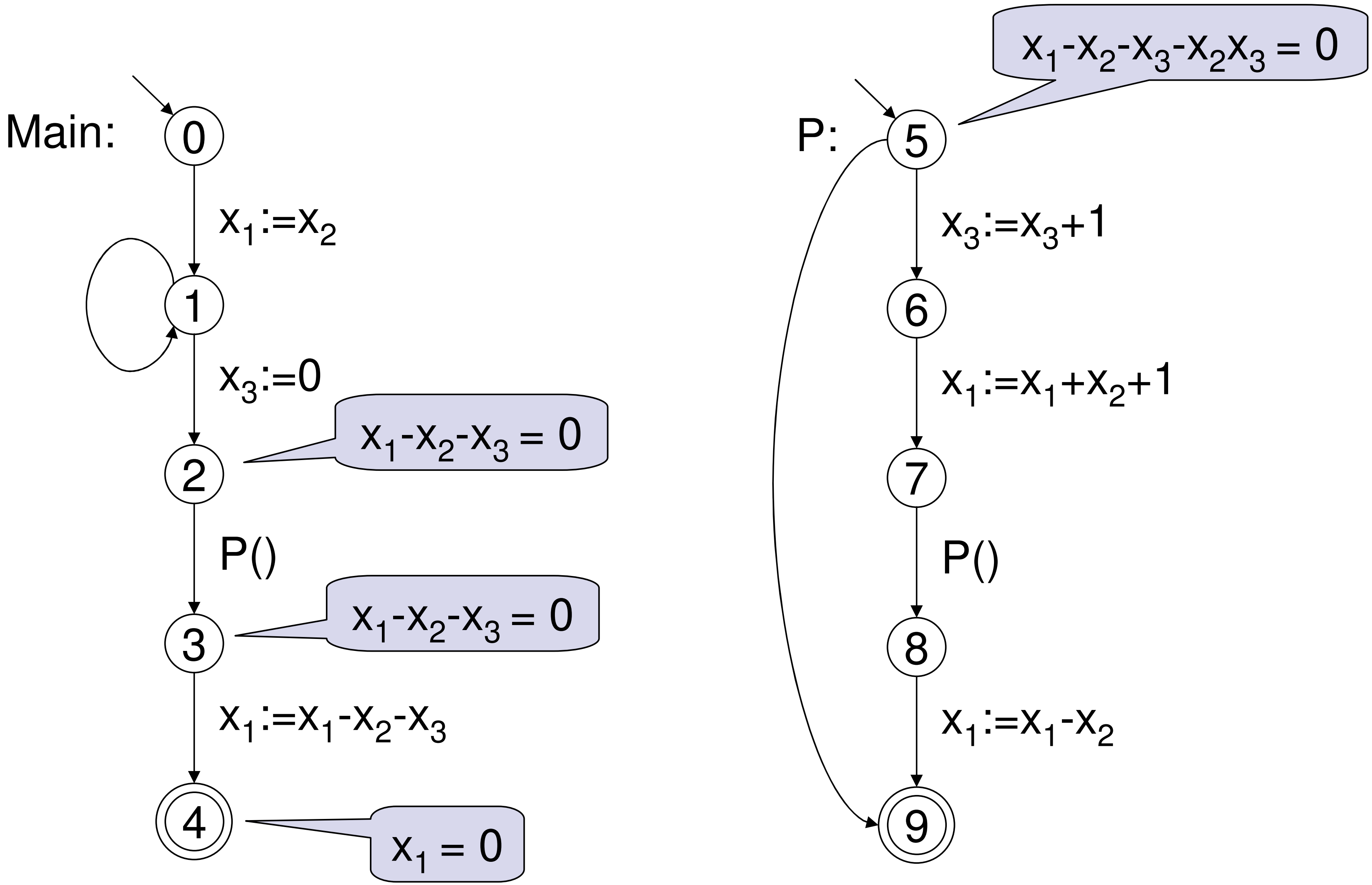
Finding Invariants...



… through Linear Algebra

- Linear Algebra
  - vectors
  - vector spaces, sub-spaces, bases
  - linear maps, matrices
  - vector spaces of matrices
  - Gaussian elimination
  - ...

Applications

- definite equalities: $x = y$
- constant propagation: $x = 42$
- discovery of symbolic constants: $x = 5yz+17$
- complex common subexpressions: $xy+42 = y^2+5$
- loop induction variables
- program verification
- ...

A Program Abstraction

Affine programs:

- affine assignments: $x_1 := x_1 - 2x_3 + 7$
- unknown assignments: $x_i := ?$

  $\to$ abstract too complex statements!
- non-deterministic instead of guarded branching

The Challenge

Given an affine program

(with procedures, parameters, local and global variables, ...)

over R :

(R the field $\mathbb{Q}$ or $\mathbb{Z}_p$, a modular ring $\mathbb{Z}_m$, the ring of integers $\mathbb{Z}$, an effective PIR,...)

- determine all valid affine relations:

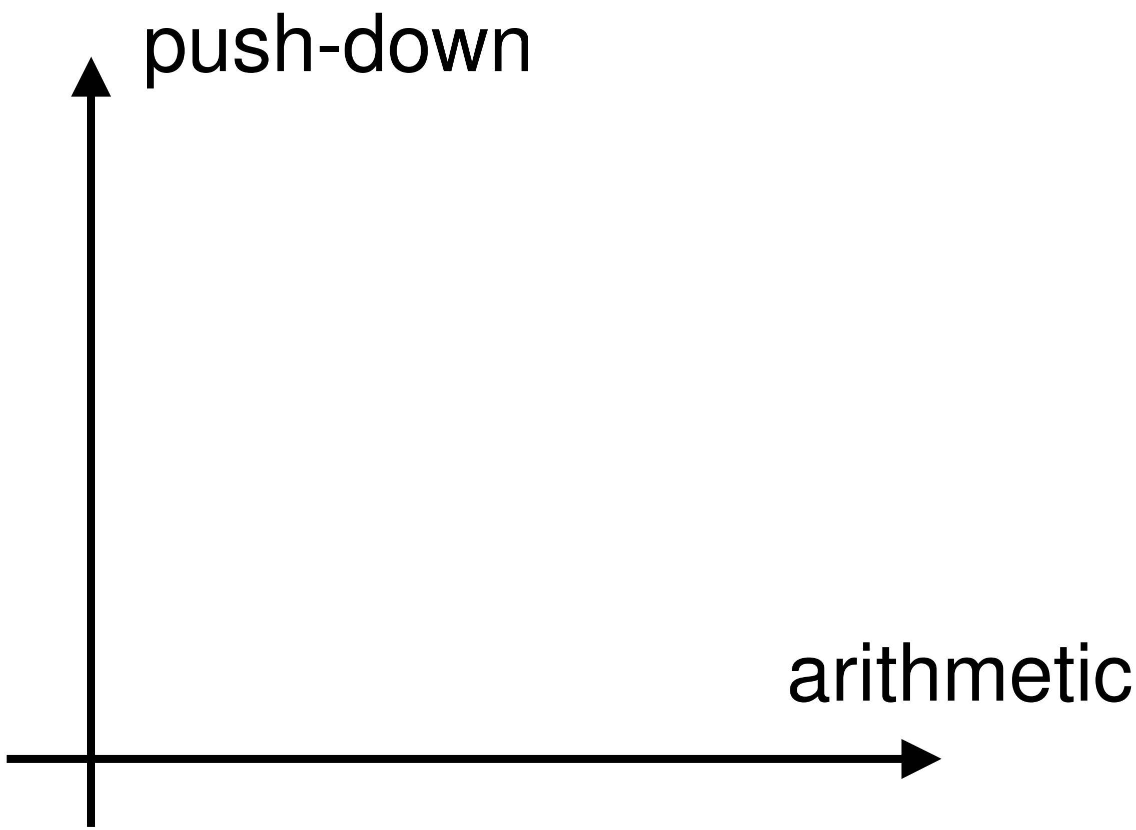
  $a_0 + \sum a_i x_i = 0$ $\quad a_i \in R$ $\quad 5x+7y-42=0$

- determine all valid polynomial relations (of degree $\leq$ d):

  $p(x_1,\ldots,x_k) = 0$ $\quad p \in R[x_1,\ldots,x_n]$ $\quad 5xy^2+7z^3-42=0$

… and all this in polynomial time (unit cost measure) !!!

Infinity Dimensions



Use a Standard Approach for Interprocedural Generalization of Karr ?

Functional approach [Sharir/Pnueli, 1981], [Knoop/Steffen, 1992]

- Idea: summarize each procedure by function on data flow facts
- Problem: not applicable

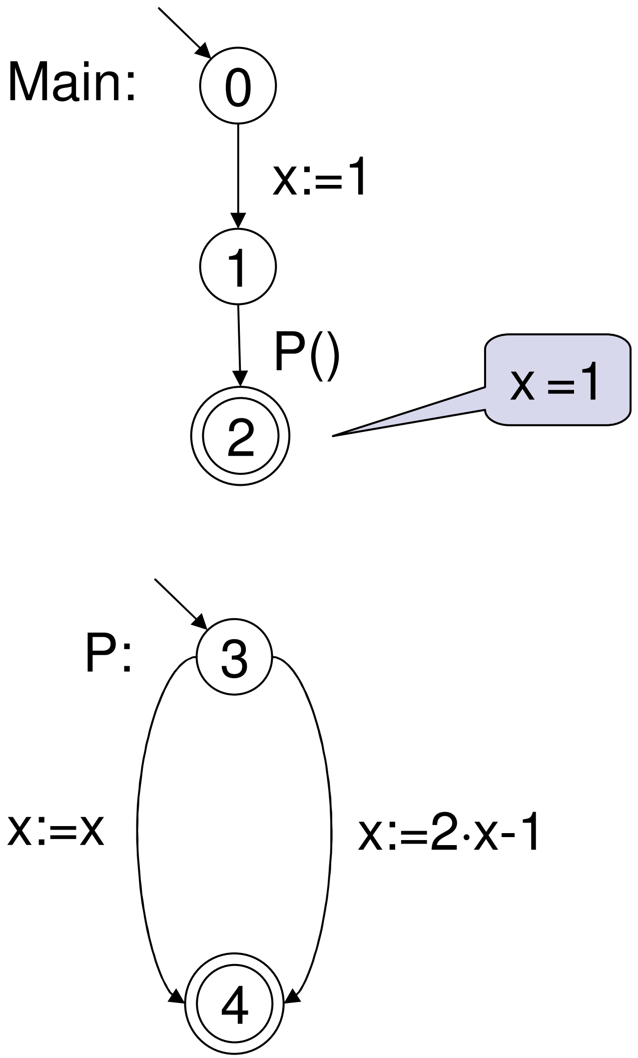
Call-string approach [Sharir/Pnueli, 1981] , [Khedker/Karkare: CC´08]

- Idea: take just a finite piece of run-time stack into account
- Problem: not exact

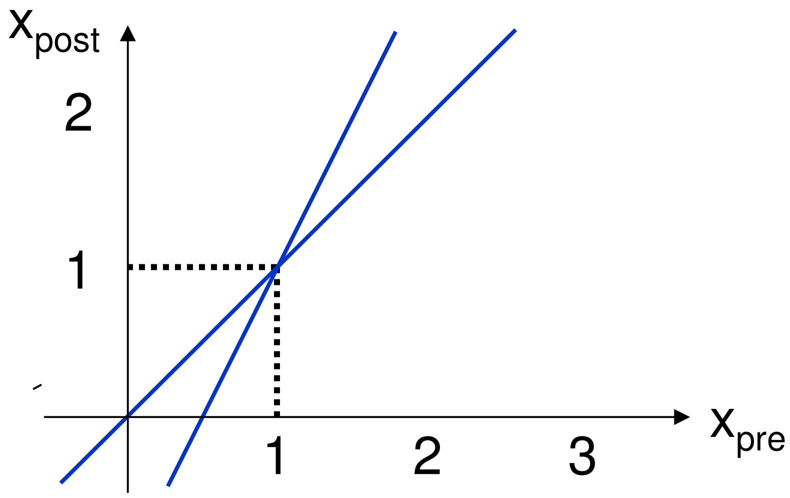
Relational approach [Cousot/Cousot, 1977]

- Idea: summarize each procedure by approximation of I/O relation
- Problem: not exact (next slide)

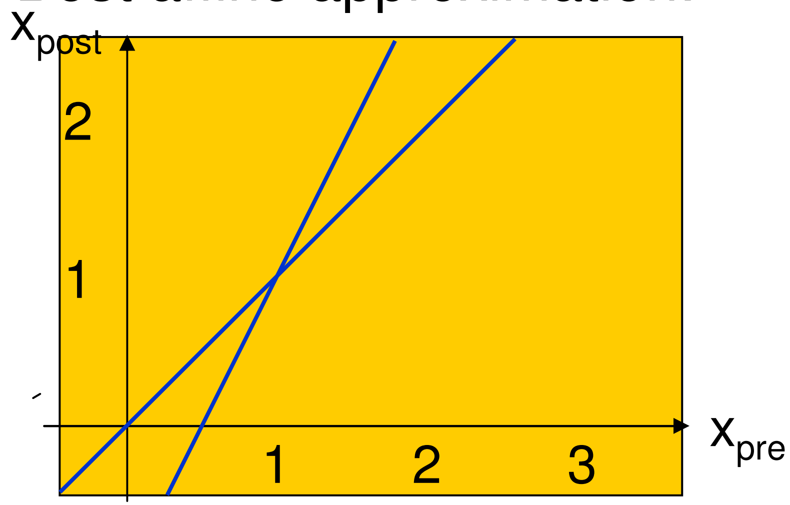
Relational Analysis is Not Strong Enough



True relational semantics of P:



Best affine approximation:



Towards the Algorithm ...

Concrete Semantics of an Execution Path

- Every execution path π induces an **affine transformation** of the program state:

$$
\begin{aligned}
& [\![x_1 := x_1 + x_2 + 1;\ x_3 := x_3 + 1]\!](v) \\
= \ & [\![x_3 := x_3 + 1]\!]\left([\![x_1 := x_1 + x_2 + 1]\!](v) \right) \\
= \ & [\![x_3 := x_3 + 1]\!]\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \\
= \ & \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)
\end{aligned}
$$

Affine Relations

- An affine relation can be viewed as a vector:

$$x_1 - 3x_2 + 5 = 0 \quad \text{corresponds to} \quad a = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

Affine Assignments induce linear wp- Transformations on Affine Relations

$$\{x_2 + x_3 + 5 = 0\} \quad x_1 := 4x_2 + x_3 + 3 \quad \{x_1 - 3x_2 + 2 = 0\}$$

weakest precondition!

A linear transformation:

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

WP of Affine Relations

- Every execution path π induces a linear transformation of affine post-conditions into their weakest pre-conditions:

$$[\![x_1 := x_1 + x_2 + 1;\ x_3 := x_3 + 1]\!]^{\mathsf{T}}(a)$$

$$= [\![x_1 := x_1 + x_2 + 1]\!]^{\mathsf{T}}\left([\![x_3 := x_3 + 1]\!]^{\mathsf{T}}(a)\right)$$

$$= [\![x_1 := x_1 + x_2 + 1]\!]^{\mathsf{T}}\left(\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}\right)$$

$$= \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}$$

Observations

- Only the zero relation is valid at program start:

$$\mathbf{0} : \; 0+0x_1+\ldots+0x_k = 0$$

- Thus, relation $a_0+a_1x_1+\ldots+a_kx_k=0$ is valid at program point v

iff

$$M\,a = \mathbf{0} \quad \text{for all } M \in \{[\![\pi]\!]^T \mid \pi \text{ reaches } v\}$$

iff

$$M\,a = \mathbf{0} \quad \text{for all } M \in \mathit{Span}\,\{[\![\pi]\!]^T \mid \pi \text{ reaches } v\}$$

iff

$$M\,a = \mathbf{0} \quad \text{for all } M \text{ in a basis of } \mathit{Span}\,\{[\![\pi]\!]^T \mid \pi \text{ reaches } v\}$$

- Matrices M form a vector space of dimension (k+1) x (k+1)

- Sub-spaces form a complete lattice of height $O(k^2)$.

Let's Apply Our Abstract Interpretation Recipe: Constraint System for Feasible Paths

Operational justification:

$$\underline{S}(u) \;=\; \{\, r \in \text{Edges}^* \mid st_p \xrightarrow{r} u \,\} \quad \text{for all } u \text{ in procedure } p$$

$$\underline{S}(p) \;=\; \{\, r \in \text{Edges}^* \mid st_p \xrightarrow{r} \varepsilon \,\} \quad \text{for all procedures } p$$

$$\underline{R}(u) \;=\; \{\, r \in \text{Edges}^* \mid \exists \omega \in \text{Nodes}^* : st_{Main} \xrightarrow{r} u\omega \,\} \quad \text{for all } u$$

Same-level runs:

$S(p)$	$\supseteq$	$S(r_p)$	r_p return point of p
$S(st_p)$	$\supseteq$	$\{\varepsilon\}$	st_p entry point of p
$S(v)$	$\supseteq$	$S(u) \cdot \{\langle e \rangle\}$	$e = (u, s, v)$ base edge
$S(v)$	$\supseteq$	$S(u) \cdot S(p)$	$e = (u, p, v)$ call edge

Reaching runs:

$R(st_{Main})$	$\supseteq$	$\{\varepsilon\}$	st_{Main} entry point of *Main*
$R(v)$	$\supseteq$	$R(u) \cdot \{\langle e \rangle\}$	$e = (u, s, v)$ basic edge
$R(v)$	$\supseteq$	$R(u) \cdot S(p)$	$e = (u, p, v)$ call edge
$R(st_p)$	$\supseteq$	$R(u)$	$e = (u, p, v)$ call edge, st_p entry point of p

Algorithm for Computing Affine Relations

1) Compute a basis B with:

   Span B = Span $\{[\![\pi]\!]^T \mid \pi$ reaches $v\}$

   for each program point by a precise abstract interpretation:

   Lattice: Subspaces of $\mathbb{F}^{(k+1) \times (k+1)}$

   Replace:

$\{\varepsilon\}$	*by*	$\{I\}$	(I identity matrix)
concatenation	*by*	matrix product	(lifted to subspaces)
$\{\langle e \rangle\}$	*by*	$\langle A_e \rangle$ for affine assignment edge $e = (u, s, v)$	

2) Solve the linear equation system:

   $M\, a = \mathbf{0}$ for all $M \in B$

Theorem

In an affine program:

- The following vector spaces of matrices can be computed precisely:

  $$\alpha(R(v)) = \text{Span}\ \{\ [\![\pi]\!]^T \mid \pi \in R(v)\ \}\ \text{ for each prg. point } v.$$

- The vector spaces

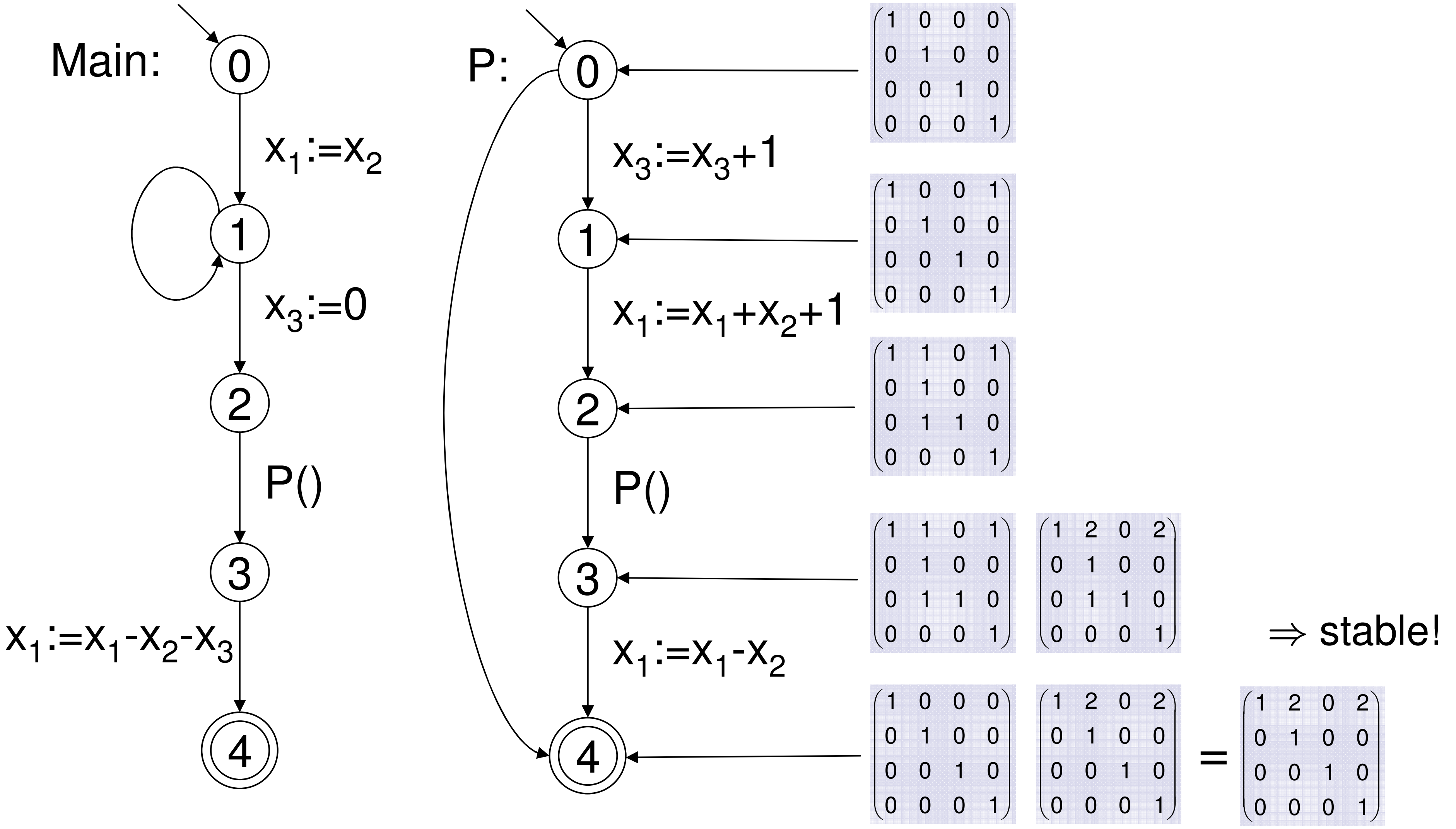
  $$\{\ a \in \mathbb{F}^{k+1} \mid \text{affine relation } a \text{ is valid at } v\ \}$$

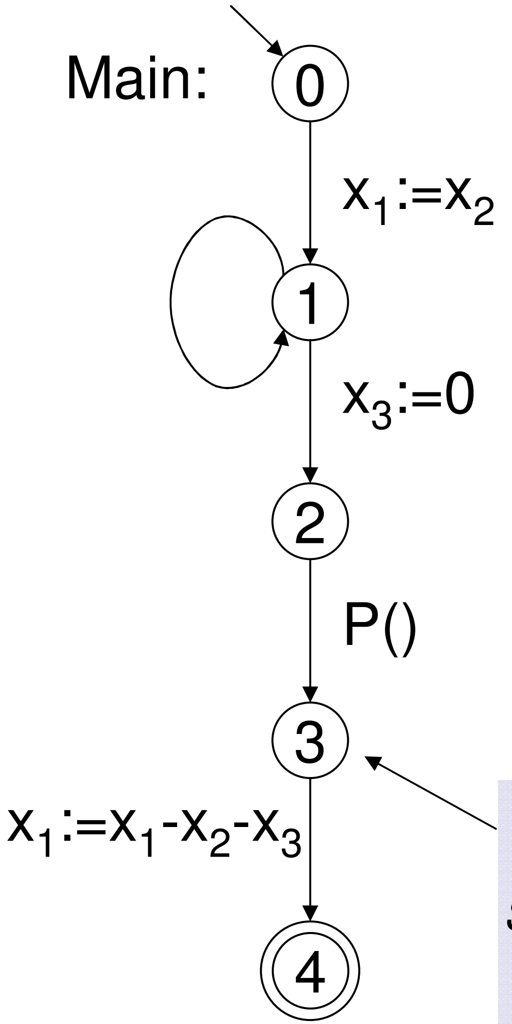
  can be computed precisely for all prg. points v.

- The time complexity is linear in the program size and polynomial in the number of variables: $\mathcal{O}(n \cdot k^8)$

  (n size of the program, k number of variables)

An Example


Main:
0
$x_1 := x_2$
1
$x_3 := 0$
2
P()
3
$x_1 := x_1 - x_2 - x_3$
4
P:
0
$x_3 := x_3 + 1$
1
$x_1 := x_1 + x_2 + 1$
2
P()
3
$x_1 := x_1 - x_2$
4
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\Rightarrow$ stable!
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



An Example

$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$ is valid at 3

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\Leftrightarrow a_0 = 0 \wedge a_2 = a_3 = -a_1$$

$\Rightarrow$ Just the affine relations of the form

$$a_1 x_1 - a_1 x_2 - a_1 x_3 = 0 \qquad (a_1 \in \mathbb{F})$$

are valid at 3 ☺

$$Span \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

Extensions

- In the paper (see handouts):
  - Local variables, value parameters, return values
  - Computing polynomial relations of degree $\leq$ d
  - Affine pre-conditions
  - Formalization as an abstract interpretation
- In follow-up papers (see webpage):
  - Computing over modular rings (e.g. modulo 2^w) or PIRs
  - Forward algorithm

Outline of this Lecture

- Precise interprocedural analysis through linear algebra
  - algebraic invariants, fixpoints, recursion, linear algebra
- Regular symbolic analysis of dynamic networks of pushdown systems
  - automata-based, recursion, parallelism (thread creation)

DPNs: Dynamic Pushdown-Networks

A *dynamic pushdown-network* (over a finite set of actions Act) consists of:

- P, a finite set of control symbols
- Γ, a finite set of stack symbols
- Δ, a finite set of rules of the following form

$$p\gamma \xrightarrow{a} p_1 w_1$$

$$p\gamma \xrightarrow{a} p_1 w_1 \rhd p_2 w_2$$

(with $p, p_1, p_2 \in$ P, $\gamma \in \Gamma$, $w_1, w_2 \in \Gamma^*$, a$\in$ Act).

DPNs: Dynamic Pushdown-Networks

A State of a DPN is a word in $(P\Gamma^*)^+$:

$$p_1 w_1 p_2 w_2 \cdots p_k w_k \qquad (\text{with } p_i \in P, w_i \in \Gamma^*, k > 0)$$

... an infinite state space

The transition relation of a DPN:

$$\left(p\gamma \xrightarrow{a} p_1 w_1\right) \in \Delta: \qquad u\,p\,\gamma v \xrightarrow{a} u\,p_1\,w_1\,v$$

$$\left(p\gamma \xrightarrow{a} p_1 w_1 \rhd p_2 w_2\right) \in \Delta: \qquad u\,p\,\gamma v \xrightarrow{a} u\,p_2\,w_2\,p_1\,w_1\,v$$

Example

Consider the following DPN with a single rule

$$p\gamma \xrightarrow{a} p\gamma\gamma \triangleright q\gamma$$

Transitions:

$$p\gamma$$
$$q\gamma p\gamma\gamma$$
$$q\gamma q\gamma p\gamma\gamma\gamma$$
$$q\gamma q\gamma q\gamma p\gamma\gamma\gamma\gamma$$
$$q\gamma q\gamma q\gamma q\gamma p\gamma\gamma\gamma\gamma\gamma$$
$$\vdots$$

Reachability Analysis

Given:

- Model of a system: M
- Set of system states: Bad

Reachability analysis:

- Can a state from Bad be reached from an initial states of the system?

$$\exists \sigma_0, \ldots, \sigma_k : \ \mathrm{Init} \ni \sigma_0 \to \cdots \to \sigma_k \in \mathrm{Bad} \quad ?$$

Applications:

- Check safety properties:

  Bad is a set of states to be avoided

- More applications by iterated computation of reachability sets for sub-models of the system model, e.g. data-flow analysis... ☺

Reachability Analysis

Given:

- Model of a system: M
- Set of system states: Bad

Reachability analysis:

- Can a state from Bad be reached from an initial state of the system?

$$\exists \sigma_0, \ldots, \sigma_k : \ \mathrm{Init} \ni \sigma_0 \rightarrow \cdots \rightarrow \sigma_k \in \mathrm{Bad} \quad ?$$

Def.: - pre*(X) $=_{df} \{ \sigma \mid \exists\, \sigma' \in X : \sigma \rightarrow^{*} \sigma' \}$

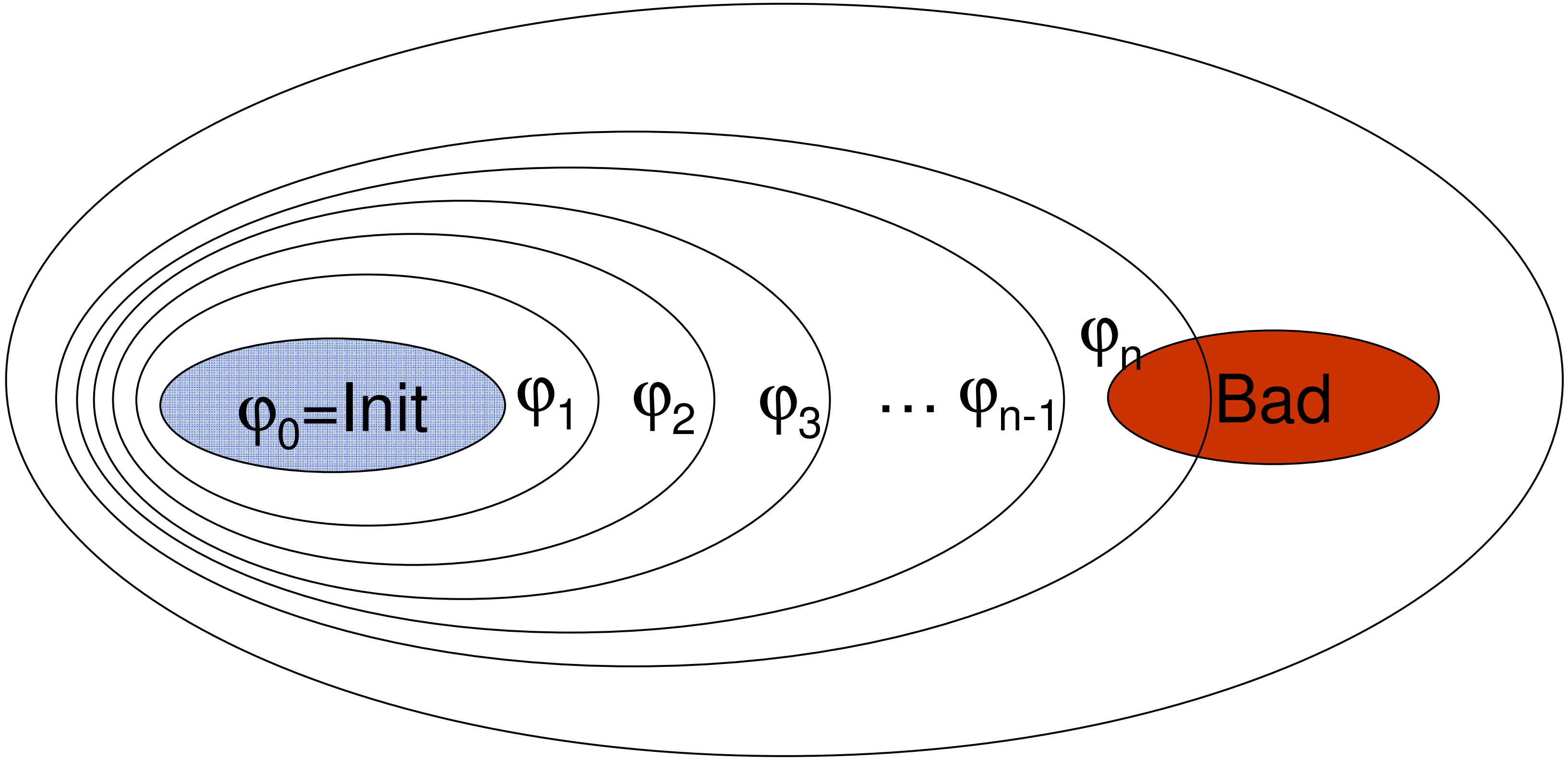
- post*(X) $=_{df} \{ \sigma \mid \exists\, \sigma' \in X : \sigma' \rightarrow^{*} \sigma \}$

Equivalent formulations of reachability analysis:

- pre*(Bad) $\cap$ Init $\neq \emptyset$
- post*(Init) $\cap$ Bad $\neq \emptyset$

$\Rightarrow$ Computation of pre* or post* is key to reachability analysis

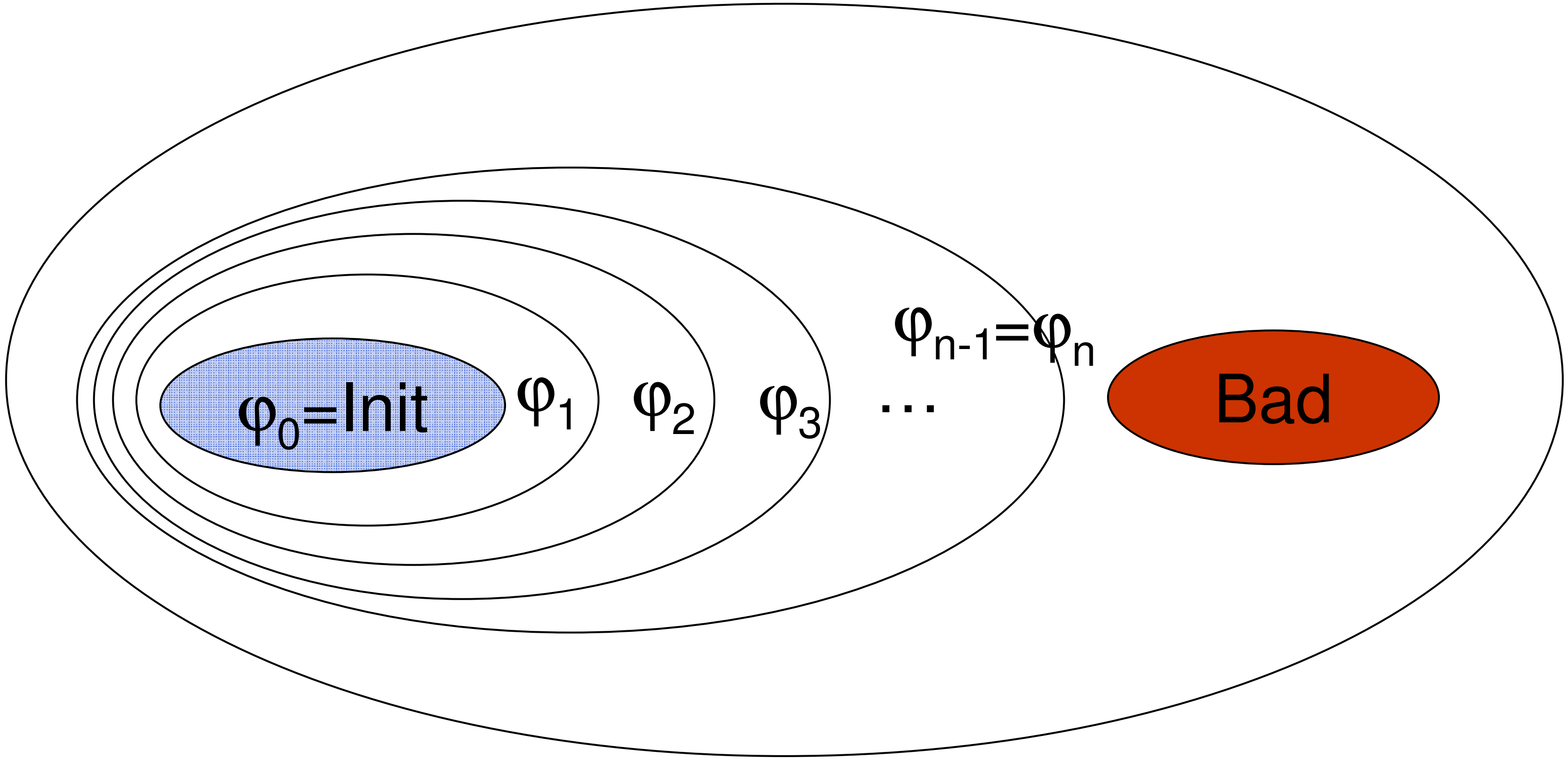
Reachability Analysis of Finite State Systems



$$\begin{aligned}
\varphi_0 &=_{df} \text{Init}\\
\varphi_{i+1} &=_{df} \varphi_i \cup \text{post}(\varphi_i)\\
\text{post}(X) &=_{df} \{\sigma \mid \exists \sigma' \in X : \sigma \to \sigma'\}
\end{aligned}$$

$\Rightarrow$ Bad reachable from initial state

Reachability Analysis of Finite State Systems



$$\begin{aligned}
\varphi_0 &=_{df} \text{Init} \\
\varphi_{i+1} &=_{df} \varphi_i \cup \text{post}(\varphi_i) \\
\text{post}(X) &=_{df} \{\sigma \mid \exists \sigma' \in X : \sigma \to \sigma'\}
\end{aligned}$$

$\Rightarrow$ Bad **not** reachable from initial state

Problems with Infinite-State Systems

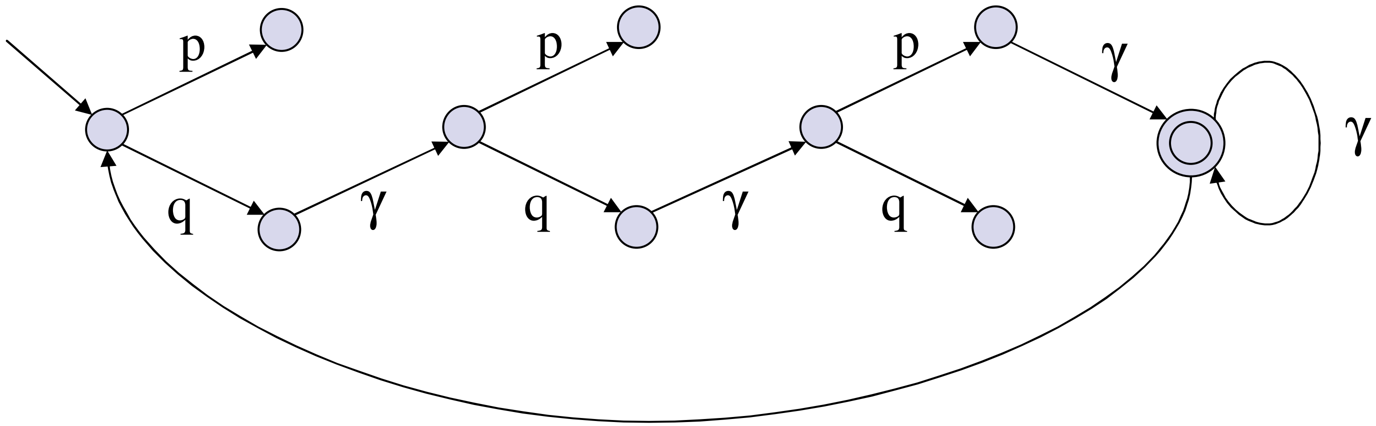
- State sets φ_i can be infinite

  $\Rightarrow$ symbolic representation of (certain) infinite state sets

  Here: by finite automata

Example: Representation of an Infinite State Set of a DPN by a Word Automaton

An automaton A:



The *regular* set of states represented by A:

$$L(A) = \left(q\gamma q\gamma p\gamma^*\right)^*$$

... an infinite set of states. ☺

Problems with Infinite-State Systems

- State sets φ_i can be infinite

  $\Rightarrow$ symbolic representation of (certain) infinite state sets

  Here: by finite (word) automata

- Iterated computation of reachability sets does not terminate in general

  $\Rightarrow$ Methods for *acceleration* of the computation

  Here: by computing with finite automata

Computing pre* for DPNs with Finite Automata

Theorem [Bouajjani, MO, Touili, 2005]

For every DPN and every regular state set R,
pre*(R) is regular and can be computed in polynomial time.

[Bouajjani/Esparza/Maler, 1997]

Proof:

Generalization of a known technique for single pushdown systems:
saturation of an automaton for R.

⇒ Reachability analysis is effective for regular sets Bad of states !

Example: Reachability Analysis for DPNs

Consider again DPN with the rule

$$p\gamma \xrightarrow{a} p\gamma\gamma \rhd q\gamma$$

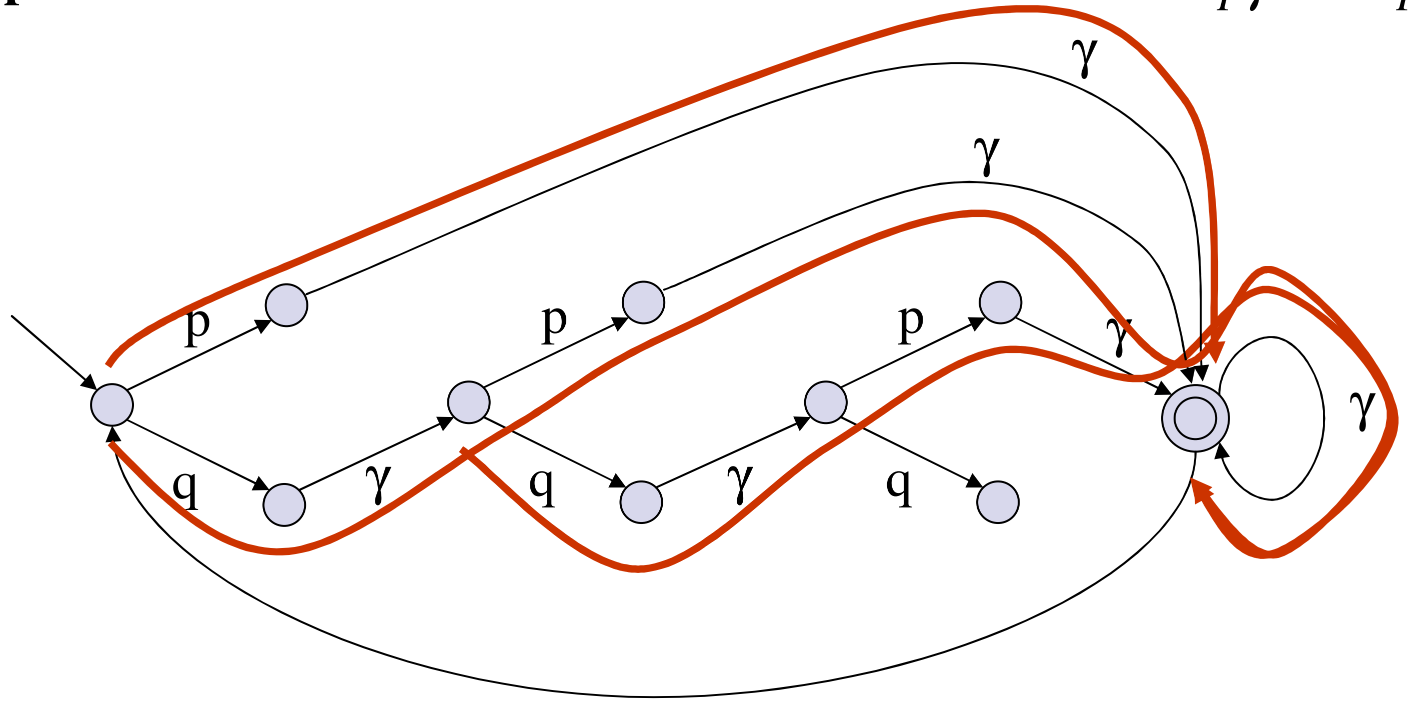
and the infinite set of states

$$\text{Bad} = \left(q\gamma q\gamma p\gamma^*\right)^* = L(A)$$

Analysis problem: can Bad be reached from $p\gamma$?

Example: Reachability Analysis for DPNs

1. Step: Saturate automaton for Bad with the DPN rule: $p\gamma \xrightarrow{a} p\gamma\gamma \rhd q\gamma$

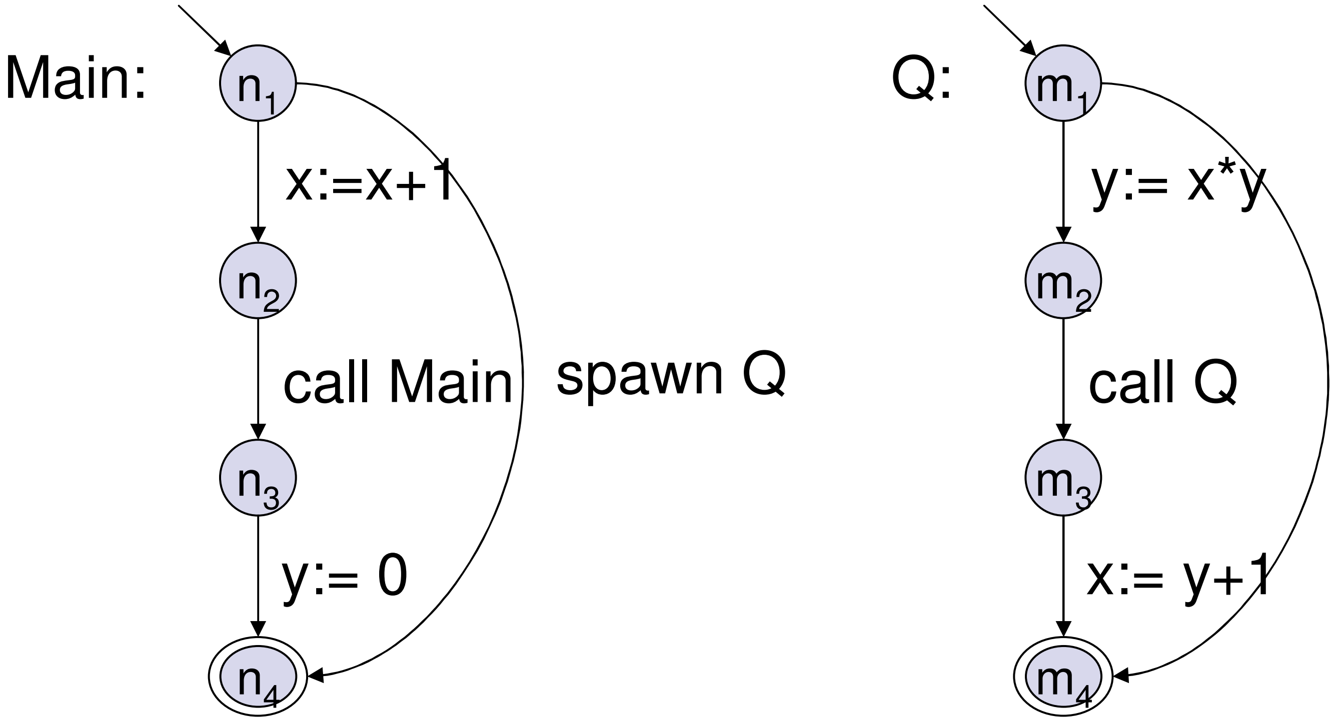


Resulting automaton A_{pre*} represents pre*(Bad) !

2. Step: Check, whether pγ is accepted by A_{pre*} or not

Result: Bad is reachable from pγ, as A_{pre*} accepts pγ.

Modelling Programs with Procedures and Threads by DPNs



$$\#N_1 \xrightarrow{x:=x+1} \#N_2 \qquad \#M_1 \xrightarrow{y:=x*y} \#M_2$$

$$\#N_2 \xrightarrow{call_P} \#N_1N_3 \qquad \#M_2 \xrightarrow{call_Q} \#M_1M_3$$

$$\#N_3 \xrightarrow{y:=0} \#N_4 \qquad \#M_3 \xrightarrow{x:=y+1} \#M_4$$

$$\#N_1 \xrightarrow{spawn_Q} \#N_4 \triangleright \#M_1 \qquad \#N_1 \xrightarrow{skip} \#M_4$$

Live Variables Analysis via Iterated pre[*]-computation

Esparza, Knoop

Steffen, Schmidt

Observation

Variable x is *live* at u

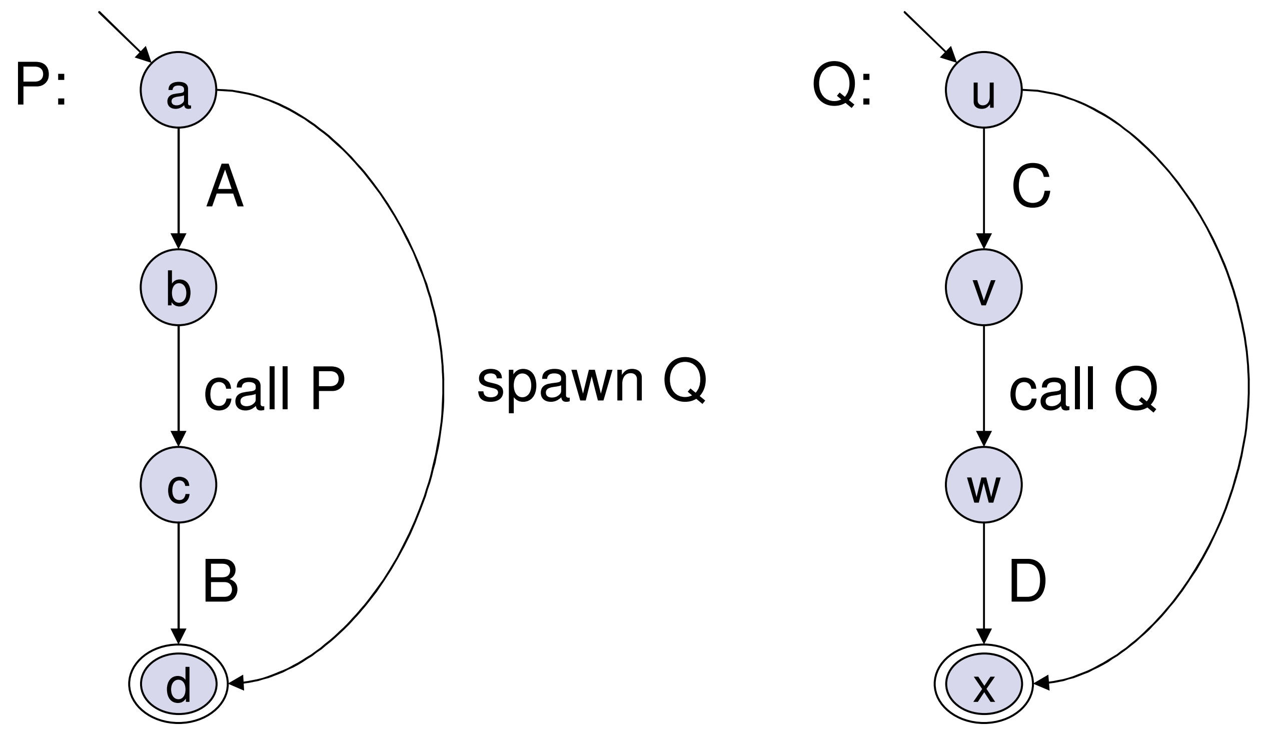
iff

$$e_{Main} \in pre^*(At_u \cap pre^*_{\Delta_{non-def}}(pre_{\Delta_{use}}(Conf)))$$

Remark

This condition can be checked by computing with automata

A Non-Representability Result



- P induces trace language: $L = \bigcup \{ A^n \cdot (B^m \otimes (C^i \cdot D^j)) \mid n \geq m \geq 0,\ i \geq j \geq 0 \}$
- L cannot be characterized by constraint system with operators „concatenation“ and „interleaving“

Forward Reachability Analysis of DPNs

Observation [Bouajjani, MO, Touili, 2005]

In general, post*(R) is not regular, not even if R is finite.

Example:

Consider DPN with the rule $p\gamma \xrightarrow{a} p\gamma\gamma \rhd q\gamma$

Recall:

$$p\gamma$$
$$q\gamma p\gamma\gamma$$
$$q\gamma q\gamma p\gamma\gamma\gamma$$
$$q\gamma q\gamma q\gamma p\gamma\gamma\gamma\gamma$$
$$q\gamma q\gamma q\gamma q\gamma p\gamma\gamma\gamma\gamma\gamma$$
$$\vdots$$

post*({pγ}) = $\{ (q\gamma)^k p\gamma^{k+1} \mid k \geq 0 \}$ is not regular.

Theorem [Bouajjani, MO, Touili, 2005]

For every DPN, post*(R) is *contextfree* if R is contextfree.

It can be computed in polynomial time.

A Little Bit of Synchronization ...

- CDPNs – Constrained Dynamic Pushdown Networks
- Idea: Threads can observe (stable regular patterns of) their children, but not vice versa
- States are represented by trees in order to mirror father/child relationship
- Use tree automata techniques for
  - representation of state sets and
  - symbolic computation of pre* (under certain conditions)
- See the paper (in the handouts)

Conclusion

- Program analysis very broad topic
- Provides generic analysis techniques
- Tradeoff Precision vs. Efficiency
- Here just one path through the forest
- Many interesting topics not covered

Thank you !