Partitioning approach

The technique of this lecture is a very *simple* one:

Break the problem into easily solvable pieces.

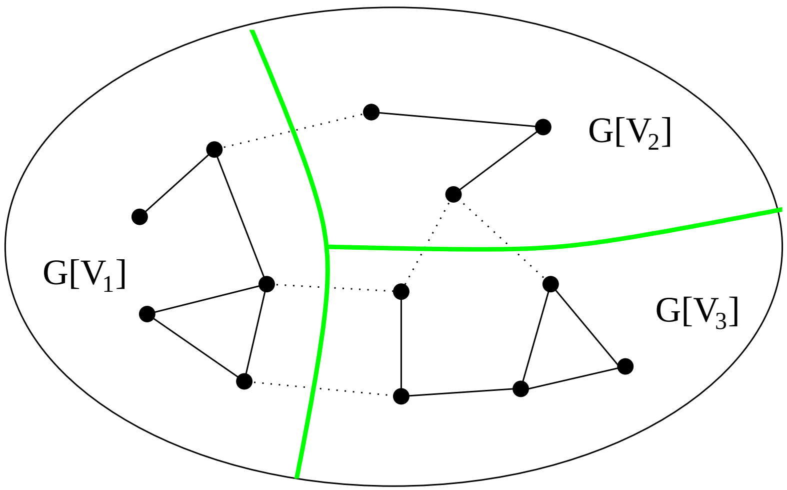
The problem we focus on: Weighted Independent Set.

Partitioning property

Proposition 1 *Let $G = (V, E)$ be a graph. Suppose we can partition V into t subsets $V_1, V_2, \ldots, V_t$ such that* WIS *is polynomial solvable on each induced subgraph $G[V_i]$. Then, the largest of these t solutions is a t-approximation to $WIS(G)$.*

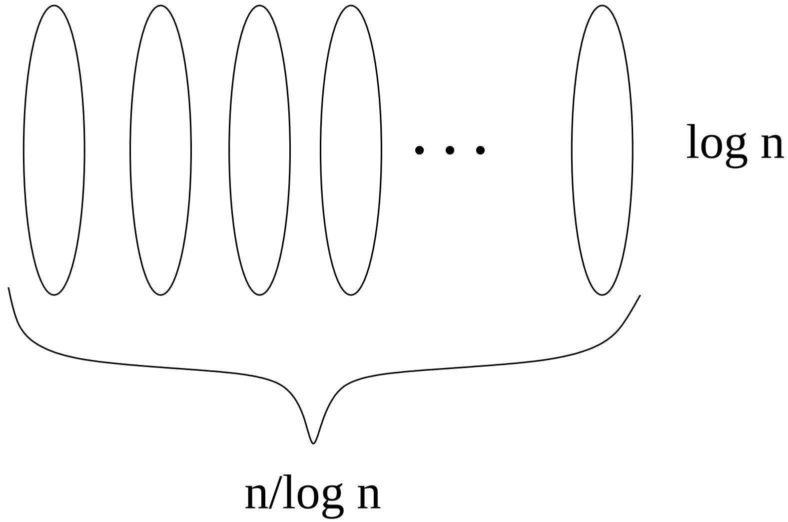
$$
\begin{aligned}
w(OPT) &= w(OPT \cap V_1) + w(OPT \cap V_2) + \cdots + w(OPT \cap V_t) \\
&\leq WIS(G[V_1]) + WIS(G[V_2]) + \cdots + WIS(G[V_t]) \\
&\leq t \cdot (\max_i WIS(G[V_i])) \\
&= t \cdot w(ALG).
\end{aligned}
$$

Therefore, the approximation ratio is at most t.



WIS *in general graphs*

- Break the graph into $n/\log n$ sets of $\log n$ vertices each.
- Solve each subgraph *exhaustively*.



Breaking a graph into $n/\log n$ sets of size $\log n$ each

Complexity:

$$n/\log n \cdot 2^{\log n} \cdot \log^2 n = n^2 \log n.$$

Result: $n/\log n$-approximation ratio

WIS *on bounded-degree graphs*

Lemma 2 (Lovász) *G can be partitioned into $\lceil(\Delta + 1)/3\rceil$ subgraphs, each of maximum degree 2.*

Note: $(\Delta + 1)/3 \cdot 2 < \Delta$!

Theorem 3 WIS *can be approximated within a $\lceil(\Delta + 1)/3\rceil$ factor.*

Graphs of maximum degree 2 consist of disjoint paths and cycles. All we need to show is that we can solve WIS optimally on paths and cycles. [Undergraduate exercise in DP]

Lovász' partitioning lemma

Let us first state the lemma more generally.

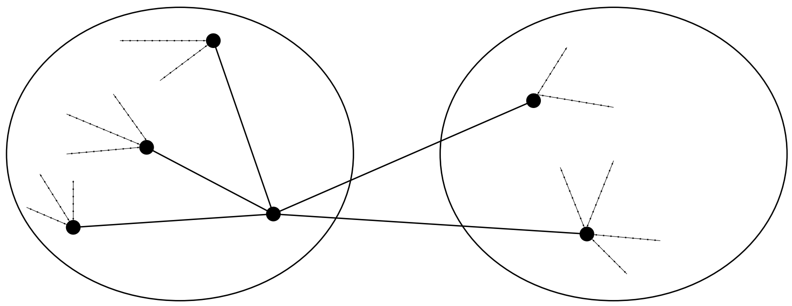
Lemma 4 (Lovász) *Let $G = (V, E)$ be graph of maximum degree Δ, and let k be a positive integer. Then, V can be partitioned into $t = \lceil (\Delta + 1)/(k + 1) \rceil$ subsets $V_1, \ldots, V_t$, such that $\Delta(G[V_i]) \leq k$ for $i = 1, 2, \ldots t$.*

Local search algorithm:

> Start with an arbitrary partition. If there is a vertex v of degree more than k in the current subgraph, move it to a subgraph where it has k or fewer neighbors. Repeat the above operation as often as needed.

Observe: $t \cdot (k + 1) > \Delta$, so v cannot have $k + 1$ neighbors in every subgraph.

Potential function: The number of *edges* crossing subgraphs in the partition.



Degree-3 graphs

Fact: WIS is polynomial solvable on bipartite graphs.
This is obtained via a reduction to maximum flow, which is always integral in a bipartite graph.
The unweighted version holds also via a reduction to maximum matching, by the König-Egerváry theorem.

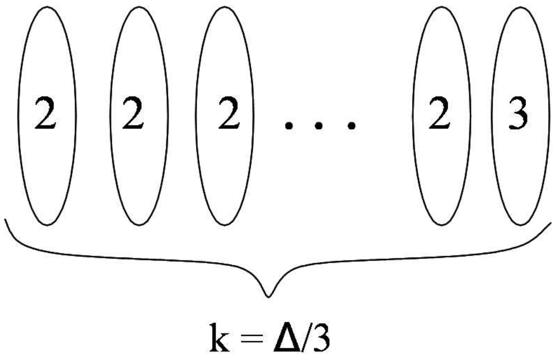
Partitioning into 3 sets of pairs:
By Brooks's theorem, we can color a degree-3 graph using 3 colors, unless G is a 4-clique. If G is a 4-clique, we can solve WIS optimally, so assume otherwise.
Let C_1, C_2, C_3 be the colors. Form partitions $V_1 = C_1 \cup C_2$, $V_2 = C_1 \cup C_3$, and $V_3 = C_2 \cup C_3$.

Claim: At least one of the V_i must contain a $2/3$-fraction of OPT.

Result: $3/2$-approximation of WIS in degree-3 graphs.

Improved ratio of bounded-degree case

We prove now a slightly better ratio of $(\Delta + 2)/3$.
We only need to do better when $\Delta = 3 \cdot q$.
Use a variation of Lovász partitioning lemma, with varying degrees.

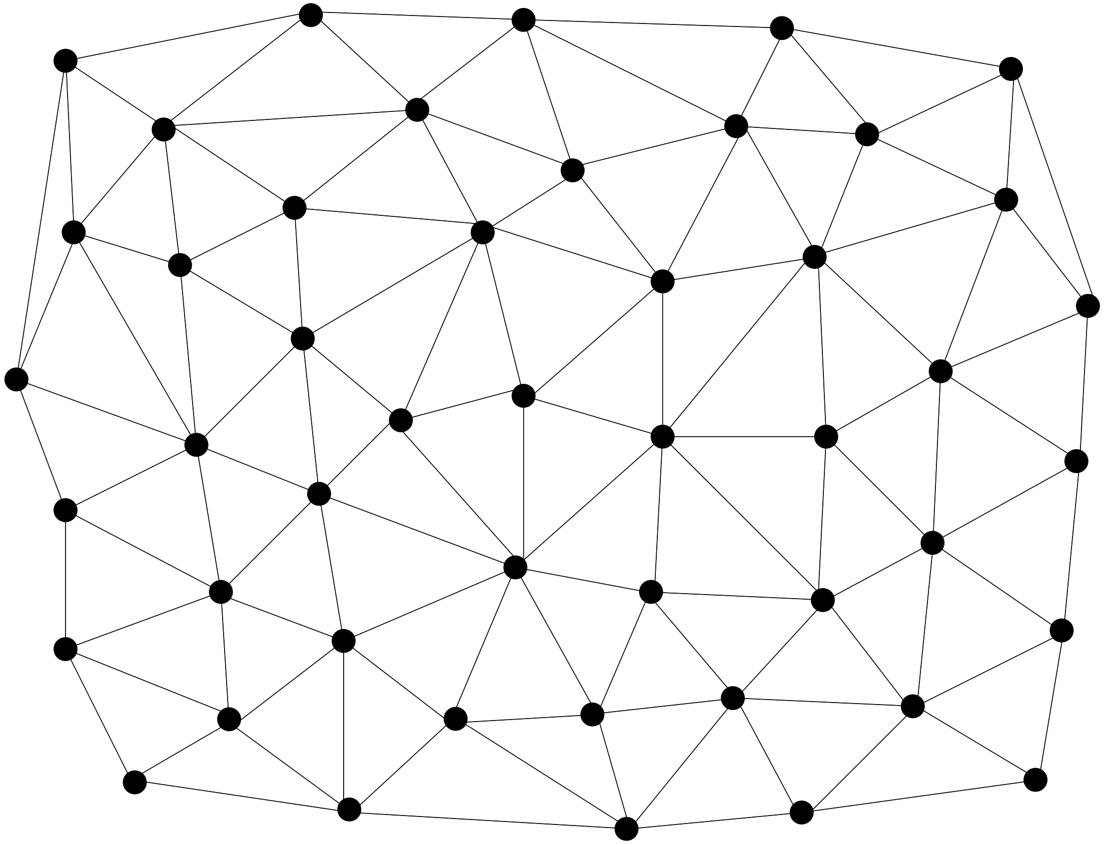


It says that when $\Delta = 3q$, we can partition G into k subgraphs, one of degree 3 and the others of degree 2. Proof left as exercise.

Analysis: OPT contains at most ALG on each subgraph, except $3ALG/2$ on the last. So,

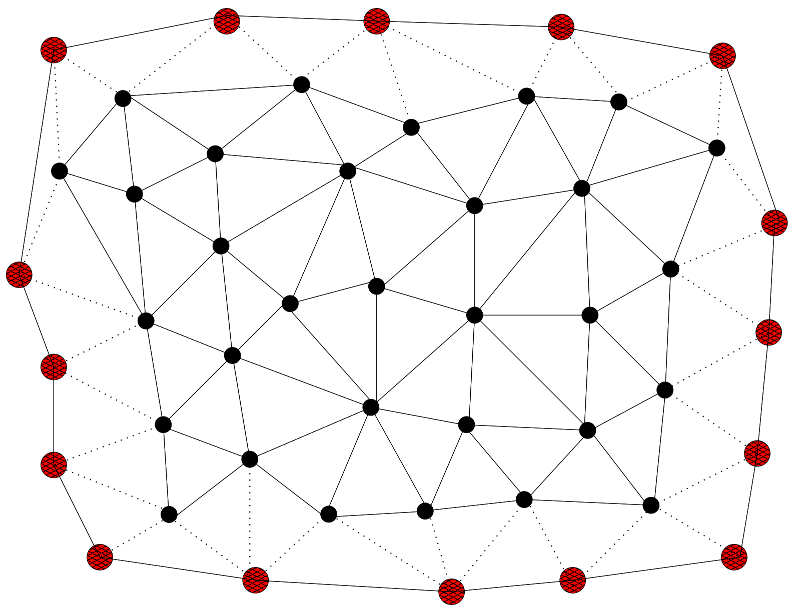
$$OPT \leq (q-1)ALG + 3ALG/2 = (q + 1/2)ALG < (\Delta + 2)/3 \cdot ALG.$$

Planar graphs



Partitioning a planar graph: Step 1

Take the *convex hull* of the plane graph: all the vertices on the infinite face of the graph.

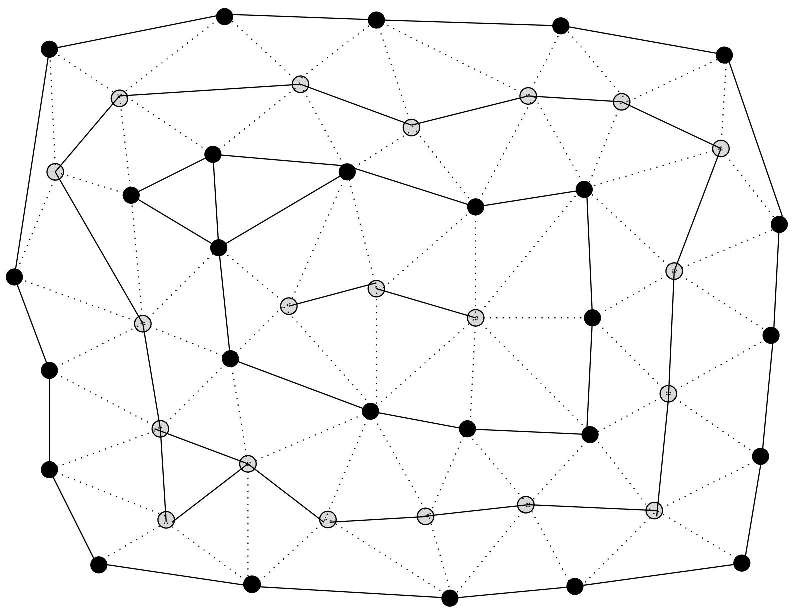


First layer "peeled off"

The selected subgraph is an *outerplanar graph*: No vertex is closed inside a circle.

Planar decomposition

Peel off the convex hulls one by one, obtaining a partition of V into *layers* $V_1, V_2, \ldots, V_t$.



Partition into layers

Separation property: V_i and V_j are disconnected, whenever $i \geq j + 2$.

In other words, vertices in V_i are adjacent only to vertices in V_{i-1}, V_i and V_{i+1}.

Note: $V_o = V_1 \cup V_3 \cup V_5 \cdots$ is also outerplanar. Same with $V_e = V_2 \cup V_4 \cup V_6 \cdots$.

We will soon show that WIS is solvable on outerplanar graphs. This implies:

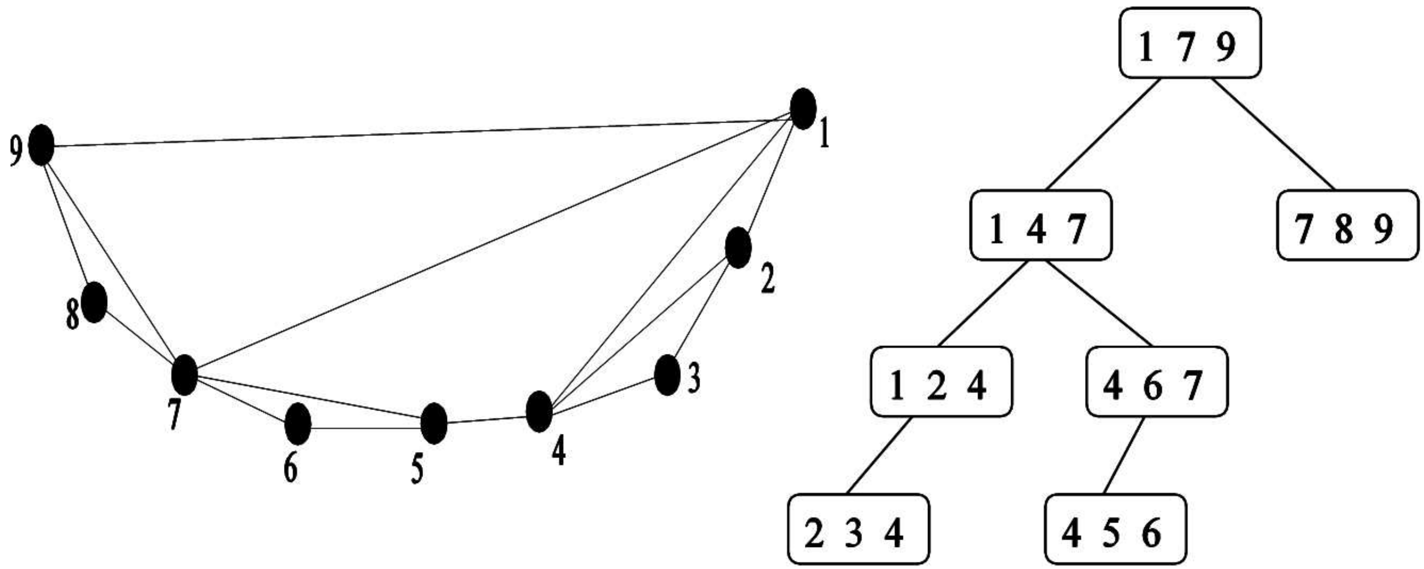
Theorem 5 WIS *on planar graphs can be approximated within a factor 2 in linear time.*

Tree decomposition

A *tree decomposition* of a graph G is a tree $\mathcal{T}$, and a labeling $\mathbf{L} : V(\mathcal{T}) \mapsto 2^{V(G)}$ of the vertices of the tree by sets of vertices in G, satisfying the following two properties:

1. Consider a vertex v of G. The vertices of $\mathcal{T}$ whose labels contain v forms a connected subgraph (subtree) of $\mathcal{T}$.
2. For any edge uv in G, some label of $\mathcal{T}$ contains both u and v.

The *treewidth* of a tree-decomposition is the maximum cardinality of any label of the tree *minus 1*.



An outerplanar graph & its 2-tree-decomposition

Forming a tree decomposition of outerplanar graphs

Tree-decomposition of outerplanar graph G:

- Arbitrarily triangulate G, giving G'
- Form a graph T where
  there is a 1-1 correspondence between nodes in T and triangles in G, and two nodes are connected if the triangles are adjacent.

Claim: T is a tree
There cannot be a cycle of adjacent triangles, because then some vertex will no longer be on the outermost layer.

WIS *on treewidth-k graphs*

Compute a matrix A_v, for each vertex $v \in G$, indexed by subsets of the labels of v.

For each $S \subseteq L(v)$, $A_v[S]$ is the maximum weight of an independent set I in the subtree rooted at v, such that I contains S but does not contain $L_v - S$.

Dynamic-programming strategy:

Recursively compute $A_{u_1}, A_{u_2}, \ldots$, for each child u_i of v

for each $S \subseteq L_v$ do

  Compute $A_v[S]$ using the best compatible solutions among the A_{u_i}.

Nice tree-decomposition

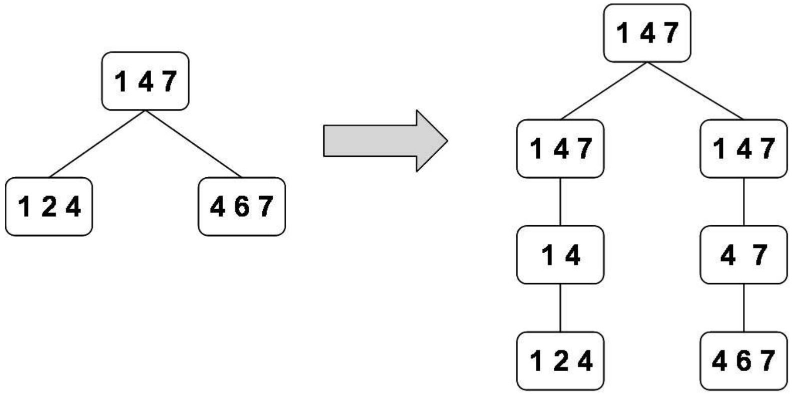
In a nice tree-decomposition, nodes are of only 3 types:

intro : v has 1 child u, and $|L_v| = |L_u| + 1$

forget : v has 1 child u, and $|L_v| = |L_u| - 1$

join : v has 2 children u and w, and $L_v = L_u = L_w$.

A k-tree decomposition of n nodes can be turned into a nice k-tree decomposition on $O(n)$ nodes in linear time.



Transformation into nice decomposition

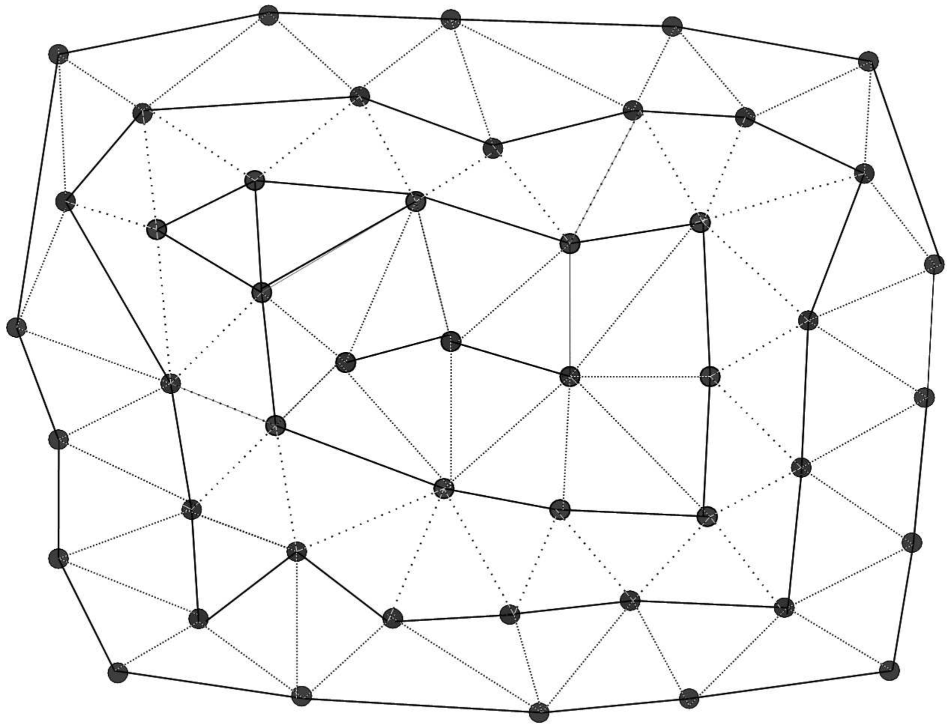
Algorithm for WIS on bounded treewidth graphs

Given: Nice tree-decomposition $\mathcal{T}$, root v
Compute: Matrix A_v

tw-WIS($\mathcal{T}$, v)
if $\mathcal{T} = \emptyset$ then return
$A_u \leftarrow tw - MIS(\mathcal{T}, u)$ (u: first child of v)
if (v is a join node)
  $A_w \leftarrow tw - MIS(\mathcal{T}, w)$
for each $S \subseteq L(v)$ do
  if (S is not independent)
    $A_v[S] \leftarrow 0$
    continue;
  case (type of node v):
    join:
      $A_v[S] \leftarrow A_u[S] + A_w[S] - w(S)$
    forget:
      Let x be the extra element in u
      $A_v[S] \leftarrow \min(A_u[S], A_u[S \cup \{x\}]$
    intro:
      $A_v[S] \leftarrow A_u[S \cap L_u] + w(S - L_u)$

Polynomial-time approximation scheme

PTAS : "Polynomial time approximation scheme" means a sequence of algorithms such that for each $\epsilon > 0$, we can approximate the problem within a ratio $1 + \epsilon$. Complexity is polynomial, dependent on ϵ.



Partition into two 2-outerplanar graphs

Let k be fixed natural number. For $i = 1, 2, \ldots, k$, let

$$U_i = V - \cup_{j=0} V_{jk+i}.$$

Note that each U_i is k-outerplanar.

Proposition 6 ***Generalized partitioning lemma:*** *Let $U_1, \ldots, U_k$ be subsets of V such that each vertex in V appears in at least $k-1$ different U_i. Then, if $ALG = \max_i WIS(G[U_i])$, then $OPT/ALG \leq k/(k-1)$.*

As before,

$$\begin{aligned} w(ALG) &= \max_i WIS(U_i) \\ &\geq (1/k)(WIS(G[U_1]) + WIS(G[U_2]) + \cdots + WIS(G[U_k])) \\ &\geq (1/k)(w(OPT \cap U_1) + w(OPT \cap U_2) + \cdots + w(OPT \cap U_k)) \end{aligned}$$

However, since each vertex appear t times in the U_i's, we have that

$$w(OPT) \leq (1/t)(w(OPT \cap U_1) + w(OPT \cap U_2) + \cdots + w(OPT \cap U_k)). \blacksquare$$

Bodlaender showed that the treewidth of k-outerplanar graphs is at most $3k-2$. (Not covered here)

Theorem 7 WIS *on planar graphs can be approximated within $k/(k-1)$ in time $O(8^k n)$.*

Corollary 8 WIS *on planar graphs can be approximated within $1 + 1/\log n$ in polynomial time.*